Since by hypothesis $f_0'(u) \not\equiv 0$ and also $f_0'(u) \geq 0$ in the first integral and $f_0'(u) \leq 0$ in the second integral therefore both integrals in the above equation are negative and so the inequality is always satisfied.

Thus, we have shown that all single hump initial velocity distribution functions are stable. This is the same result which is usually obtained by means of a Laplace- or a one-sided Fourier transform with respect to the time variable. A notable feature of the present analysis is that it need not use any of the artificial mathematical assumptions which are, for example, present in regard to the process of analytic continuation when the method of integral transforms is used.

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References

1. L. Landau, Journal of Physics **10**, 25 (1946)
2. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, AMS Colloquim Publications, vol. 19 (1934)

A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING*

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1. Introduction. Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q, p)$. The equations of motion are

$$\dot{q} = H_p , \tag{1.1}$$

$$-\dot{p} = H_q , \qquad 0 \leq t \leq T. \tag{1.2}$$

As boundary conditions we specify

$$q(0) = 0, \tag{1.3}$$

$$p(T) = c. \tag{1.4}$$

If our aim is to solve the system of equations (1.1) and (1.2), subject to the boundary conditions in (1.3) and (1.4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time T, $q(T)$, so that we shall have a complete set of conditions at time T. This would enable us to integrate the system (1.1) and (1.2) numerically, subject to initial values. Toward this end we introduce the function

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$r(c, T)$ = the displacement at time T, the initial displacement being zero and the momentum at time T being c. (1.5)

We wish to derive an equation for $r(c, T)$.

2. Invariant imbedding. In earlier papers [1,2], we showed the following:

If

$$\frac{du}{dz} = F(u, v), \qquad u(0) = 0, \tag{2.1}$$

$$-\frac{dv}{dz} = G(u, v), \qquad v(x) = c, \tag{2.2}$$

for

$$0 \le z \le x, \tag{2.3}$$

and if

$$r(c, x) = u(z)|_{z=x}, \tag{2.4}$$

then the function $r(c, x)$ satisfies the partial differential equation

$$r_x = F(r, c) + G(r, c)r_c \tag{2.5}$$

and the initial condition

$$r(c, 0) = 0. \tag{2.6}$$

Upon applying this result to the system of equations in (1.1) and (1.2), we find that the function $r(c, T)$ satisfies the equation

$$\frac{\partial r}{\partial T} = H_p(r, c) + H_q(r, c)\frac{\partial r}{\partial c}, \tag{2.7}$$

or, more elegantly,

$$\frac{\partial r}{\partial T} = \frac{\partial H(r, c)}{\partial c}. \tag{2.8}$$

This is the desired result, apparently a new equation of mechanics.

3. An example. Consider the case of harmonic oscillations for which

$$H(q, p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2, \tag{3.1}$$

and

$$q(0) = 0, \qquad r(T) = c. \tag{3.2}$$

For the unknown displacement $r(c, T)$ at time T, equation (2.8) becomes

$$\frac{\partial r}{\partial T} = \frac{\partial}{\partial c}\left[\frac{c^2}{2m} + \tfrac{1}{2}kr^2\right] = \frac{c}{m} + kr\frac{\partial r}{\partial c}. \tag{3.3}$$

The solution of this equation, subject to the condition (2.6) is

$$r(c, T) = \frac{c}{\sqrt{km}}\tan\left(\sqrt{\frac{k}{m}}\,T\right). \tag{3.4}$$

This provides the desired displacement at time T and also shows that the function $r(c, T)$ may become infinite for a finite value of T.

4. Discussion. The previous discussion may be generalized to the case of many particles moving in three-dimensional space using the theorem in [1]. Furthermore, various combinations of the q's and p's may be specified at the ends. There are immediate applications to perturbation analysis of results of this nature; see [3].

References

1. R. Bellman, R. Kalaba, and G. M. Wind, *Invariant imbedding and the reduction of two-point boundary-value problems to initial-value problems*, Proc. Nat. Acad. Sci. USA **46** (1960) 1646
2. R. Bellman and R. Kalaba, *The fundamental equations of invariant imbedding*, Proc. Nat. Acad. Sci. USA **47** (1961) 336
3. R. Bellman, *A note on dynamic programming and perturbation theory*, The RAND Corporation, RM—3169—PR, June 1962