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BOOK REVIEWS

Handbook of statistical tables. By D. B. Owen. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962. xii + 580 pp. $12.50.

This book presents an exceptionally complete and up-to-date collection of tables of statistical functions; many are reproduced directly from the output of digital computers and are thus more extensive, in the case of many functions, than any published before. There are twenty sections, including the following tables: normal, student's t, chi-square, F, multivariate normal and t, noncentral t, logistic, Poisson, hypergeometric, bi-and multinomial distributions; Wilcoxon tests; Kolmogorov-Smirnov and Cramér-von Mises statistics; and many others. The explanatory text accompanying each table is brief, to the point and informative. The work will be most helpful to advanced students, practicing statisticians, and research workers.

WALTER FREIBERGER

Information retrieval and machine translation. Allen Kent, Editor. Volume III, Part 2. Interscience Publishers, Inc., New York and London, 1961. ix + 690 pp. $25.00.

This large book, which is the second part of a two-volume report, is based on the "International Conference for Standards on a Common Language for Machine Searching and Translation" which took place, under the sponsorship of Western Reserve University and the Rand Development Corporation, in Cleveland, Ohio, in September of 1959. The volume contains 37 papers as well as the minutes of the discussions which, at times, seem to have been quite energetic. Among those represented by papers are some of the best known researchers in the fields of machine translation and information retrieval. In addition to a large American contingent, papers at the conference were presented by scholars from six foreign countries. Included among these are several contributions by the members of the Soviet delegation which give some rather general information about computer design and machine translation projects in the USSR.

Most of the papers are devoted to various problems of machine translation of languages, the rest to other aspects of information retrieval and related problems. In these fields, this book offers a wealth of information about a great variety of approaches, experiments, projects, and progress claims.

HENRY KUCERA

Direct analysis of diffraction by matter. By R. Hosemann and S. N. Bagchi. North-Holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, 1962. xxi + 734 pp. $21.75.

This book is concerned with what is called "diffraction microscopy" —the analysis of diffraction patterns by direct calculation using Fourier integrals and convolution integrals. The method involves a direct integration of the electromagnetic and quantum mechanical wave equations for the scattered intensity. The fact that under certain conditions at least it is possible in principle to determine a structure uniquely from intensity data alone is apparently a relatively recent realization. The book is particularly concerned with how far one can go with intensity data alone in predicting a unique structure. Errors in measurement inherent in all physical measurements, and the extent to which they affect the observed intensities, are particularly important.

The first five chapters and the appendix are concerned with the mathematical methods involved in calculations of structures from intensity data. The remaining thirteen chapters are concerned somewhat more with the physical details of crystal structures, aggregates, fibrous structures, and fluids.

This work is a highly mathematical discussion of structure determination by methods that should be of interest to applied mathematicians and mathematical physicists as well as those whose research is closely connected with structure analysis.

R. TRUELL

(Continued on p. 132)

BOOK REVIEWS

(Continued from p. 104)

Hypersonic flow research. Edited by Frederick R. Riddell. Academic Press, New York, London, 1962. $x + 758$ pp. $10.50.

This volume represents a compilation of papers presented at a conference on hypersonics sponsored by the American Rocket Society and held at the Massachusetts Institute of Technology in August 1961. The meeting was divided into five technical sessions and the format of the volume follows this breakdown with a section on low Reynolds number flows, one on chemical kinetic effects, one on inviscid analyses, and two on experimental techniques. Included is a short introduction to each section by the session chairman, and a summary of the conference and discussions following the papers.

The papers in this book clearly represent the state of development in hypersonic flow at the time of presentation and, as such, should prove particularly valuable to those individuals desirous of appraising themselves of the current research in this field.

The volume as a whole is an exciting one and in it may be found a large number of papers which this reviewer would recommend to the interested applied mathematician. Of the papers which are mathematical in character we may mention the work of Van Dyke on second-order effects in hypersonic boundary layer theory which elicited much comment. In addition, the paper of Levinsky and Yoshihara on Navier-Stokes solutions at low Reynolds numbers also raises some interesting mathematical and physical questions. A very good example of the understanding which can be achieved of a complicated physical problem through rational mathematical simplification is to be found in the paper of Moore and Rae on dissociated gases. Consideration of some of the seemingly anomalous conclusions which result from the particular mathematical limiting process termed newtonian flow theory and physical interpretations of the results from this rational process may be found in the papers of Cole and Brainerd, Freeman, and Melnik and Scheuing. Finally, the important problem of slightly blunted slender bodies in inviscid flow is treated by a boundary layer approach in the work of Yakura.

On the format of the book we would note that the idea of putting out such symposia in a collected form, quickly and at a cheap price by use of photo-offset typewritten manuscripts, is an excellent one. A disturbing feature of the format of this volume is that the recto and verso running heads simply give the volume title. Since there are no indices, and since the page numbers are consecutive, this makes it very difficult for the reader to physically find his way through the volume.

As a collection of symposium papers the book is among the very best this reviewer has seen and can be strongly recommended to those interested in the present state of the art in the hypersonics field.

RONALD F. PROBSTEIN

The wave mechanics of electrons in metals. By Stanley Raimes. North-Holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, 1961. xi + 367 pp. $13.00

This book devotes about half of its attention to the development of wave mechanics and about half of it to specific problems such as metallic cohesion, the interaction of electrons, and plasma oscillations in metals. This is a text book ranging from elementary to intermediate difficulty; it attempts to prepare the student for advanced treatises or theoretical papers.

The chapter headings are: 1. Introduction; 2. Basic Principles of Wave Mechanics; 3. The Variational Method and Perturbation Theory; 4. The Many Electron Problem and the One Electron Approximation; 5. Electron Spin and the Pauli Principle; 6. Metallic Cohesion and the Hartree-Fock Method; 7. The Free-Electron Approximation; 8. Bloch Functions and Brillouin Zones; 9. The Method of Wigner and Seitz; 10. Plasma Oscillations in Metals; 11. Time-Dependence and Transition Probabilities.

The text is for graduate students and for experimental physicists, metallurgists and those who are not primarily students of theoretical physics. It is clear and well written and will certainly have value for many students of solid state physics.

R. TRUELL

(Continued on p. 169)

BOOK REVIEWS

(Continued from p. 132)

Structure of language and its mathematical aspects. Proceedings of Symposia in Applied Mathematics. Volume XII. American Mathematical Society, Providence, R. I., 1961. vi + 279 pp. $7.80.

This volume is a collection of papers which were presented at the "Symposium on the Structure of Language and its Mathematical Aspects" held, under the auspices of the American Mathematical Society, in April of 1960 in New York City. The volume reflects the rapidly growing interest in the field of "mathematical linguistics" and the nature of the papers is perhaps also a good indicator of the direction which scholars interested in this field seem to be following.

The book contains, besides a brief introduction by Roman Jakobson and an index, twenty papers as well as the minutes of the meetings and of the discussions which followed the presentations. The large majority of the papers is concerned, in one way or another, with questions of syntax and the application of methods of mathematical logic in syntactic analysis and grammatical descriptions. W. V. Quine, in the introductory presentation, discusses the general usefulness of such methods in linguistics. N. Chomsky, the main propagator of "transformational grammar," investigates the properties of language grammars and their rules within the framework of his theory, while H. Putnam attempts to outline the conceptual setting for the transformational analysis. H. Hiz examines the notions of congrammaticality, the batteries of transformations and grammatical categories, and H. B. Curry is also concerned with various logical aspects of grammatical structure. R. Abernathy's contribution deals with the problem of linguistic equivalence and that of J. Lambek with the determination of grammaticalness of sentences. C. F. Hockett looks at grammar from the point of view of the hearer and suggests that it might be treated, in this respect, as an infinite-state Markov process, and approximated as accurately as one wishes by a finite-state Markov process. H. G. Herzberger describes a method of analysis of the English sentence by its decomposition into a center and its adjuncts, and N. Goodman discusses the general usefulness of graphs in grammatical descriptions. A lively discussion was occasioned by the paper of V. H. Yngve who presented a brief version of his depth-limitation theory which he considers to be applicable to sentence structure of English and probably other languages as well.

Other levels of language analysis than syntax are represented in this volume only to limited extent. Two papers deal with graphemic problems: M. Eden attempts to determine the distinctive features of cursive English writing for possible mechanical analysis of handwriting, and Y. R. Chao is concerned with the inter-relation between the graphic and phonetic aspects of language and mathematical symbols. G. E. Peterson and F. Harary, in a joint contribution, present a detailed phonetic classification and a quantitatively expressed scheme of sound differentiation as basis of phonemic theory, and M. Halle uses the phonological level of language as the basis of discussion in his advocacy of the necessity of ordered rules in linguistic description. Comparative linguistics is represented by H. A. Gleason who suggests some procedures for the mechanization of lexical comparisons of languages in the determination of their genetic relationship.

It is curious to find, in a volume dealing with the mathematical aspects of language, only one paper concerned primarily with the application of statistical methods in language analysis; this is B. Mandelbrot's contribution "On the Theory of Word Frequencies and on Related Markovian Models of Discourse." Two presentations deal with the applicability of communication theory in linguistics: R. Jakobson presents a general review of the problem and R. Wells investigates the notions of subjective information and informativeness of a message. And, finally, one paper is concerned more or less directly with automatic language processing by computers; this is the contribution by A. G. Oettinger, "Automatic Syntactic Analysis and the Pushdown Store."

This volume generates, in this reviewer, a variety of impressions. First of all, the many competent contributions certainly indicate that language can furnish a fruitful focal point for joint researches of linguists, logicians and mathematicians. The collection of papers also shows, however, that "mathematical linguistics" in the United States seems to be predominantly developing towards the analysis of syntactic problems by methods of mathematical logic, while other aspects of linguistic analysis in which mathematical models and procedures could be useful seem to be neglected. Although thus rather one-sided, the volume under review also presents a considerable variety of approaches and some methodo-

logical diffuseness even in these limited areas, something which is probably unavoidable in a field which is still in its early process of development.

It is also unfortunate that, because of the extreme and not always necessary elaborateness of the metalanguage and of the notations used in some of the papers, an "old-fashioned linguist" (*i. e.*, someone trained about a decade ago) would quite likely find many of the presentations incomprehensible.

HENRY KUCERA

Tables of Weber functions. By I. Ye. Kireyeva and K. A. Karpov. Volume I. Translated by Prasenjit Basu. Pergamon Press, New York, Oxford, London, Paris, 1961. xxiv + 340 pp. $20.00.

This volume contains tables of Weber Functions, or parabolic cylinder functions. They are the solutions $D_p(Z)$, $W_p(Z)$ of the equations

$$\frac{d^2y}{dZ^2} + \left(p + \frac{1}{2} \pm \frac{2^2}{4}\right)y = 0$$

which occur in quantum mechanics, radio physics, aero-and hydrodynamics and other fields.

WALTER FREIBERGER

Mathematical tables—tables of the exponential integral. By V. I. Pagurova. Pergamon Press, New York, Oxford, London, Paris, 1961. xviii + 151 pp. $12.50.

This is an English translation of the Russian book published in 1959. The introduction, which gives a useful collection of formulas and an account of interpolation in the tables, is photographically reproduced from typescript for the translated text, the formulas cut from the original, with good success. The tables themselves are very legible.

The tabulated function has the following special cases:

$$E_0(x) = e^{-x}/x$$

$$E_1(x) = -Ei(-x) = \int_x^\infty e^{-u}u^{-1}\,du$$

$$E_{1/2}(x) = (2/x)\,\mathrm{Erfc}\,x = (2/x)\int_x^\infty \exp(-u^2)\,du$$

and

$$E_\nu(x) = \int_x^\infty E_{\nu-1}(u)\,du.$$

There are three tables, as follows. Table I: $E_n(x)$, $x = 0(0.01)2(0.1)10$; $n = 0(1)20$. Table II: $e^xE_n(x)$, $x = 10(0.1)20$; $n = 2(1)10$. Table III: $e^xE\nu(x)$, $x = 0.01(0.01)7(0.05)12(0.1)20$; $\nu = 0(0.1)1$.
Table I is acknowledged to be taken in its entirety from "Tables of functions and zeros of functions", Nat. Bur. Standards, Appl. Math. Ser. No. 37 (1954), pp. 57–111. Entries are generally to 7 figures.

WALTER FREIBERGER

An elementary introduction to the theory of probability. By B. V. Gnedenko and A. Ya. Khinchin. W. H. Freeman & Co., San Francisco, London, 1961. 137 pp. $1.75.

This little book by two of the greatest Russian probabilists is a delightful introduction to the subject, and should be fully comprehensible to bright high-school students. There are numerous examples to

illustrate the basic ideas, since it is the authors' conviction that "probability theory is a mathematical science that is not aloof from practical life and the demands of other sciences; it proceeds apace with the general progress of natural science and technology."

WALTER FREIBERGER

Fourier transforms and convolutions for the experimentalist. By R. C. Jennison. Pergamon Press, New York, Oxford, London, Paris, 1961. vi + 120 pp. $5.00.

The Fourier Transform is regarded in this book as a transformation from the time domain to the frequency domain, and its principles and uses are presented, from this point of view, with clarity and persuasiveness. The text is addressed to physicists and engineers with little mathematical background and the examples are principally drawn from optics, electronics and antenna design.

WALTER FREIBERGER

Wave mechanics of crystalline solids. By R. A. Smith. John Wiley & Sons, Inc., New York, 1961. xv + 473 pp. $13.00.

This book is concerned with the application of quantum theory to the understanding of electrical, thermal and optical properties of crystalline solids. Particular attention is given to such topics as the motion of charge carriers, the processes of emission and absorption of radiation, lattice vibrations and specific heat theory as well as electrical and thermal transport phenomena. Some attention is also given to the interaction of electrons with photons and phonons.

The chapter headings are as follows: 1. Wave Motion in a Homogeneous Medium; 2. Free Electron Theory of Metals; 3. Lattice Vibrations; 4. Motion of Electrons in a One-Dimensional Periodic Potential; 5. Crystal Lattices; 6. Lattice Vibrations in Three Dimensions; 7. Specific Heat of Crystalline Solids Due to Lattice Vibrations; 8. Motion of Electrons in a Perfect Crystal; 9. Distribution of Electrons Between the Allowed Energy Levels in a Crystal; 10. Electrical and Thermal Transport Phenomena in Crystalline Solids; 11. The Effective-mass Approximation; 12. Magnetic Phenomena Due to Free Electrons and Holes in Crystals; 13. Interaction of Electrons with Photons and Phonons; Appendix 1. Equation of Motion of an Electron in a Crystal in a Magnetic Field; Appendix 2. The f-sum Rule.

This work appears to the reviewer to be an excellent intermediate to advanced text, not as advanced as some of the treatises, but considerably more advanced than most books on solid state physics. The book should be of value both to graduate students and to solid state physicists generally.

R. TRUELL

Stochastic service systems. By John Riordan. John Wiley & Sons, Inc., New York, London, 1962. x + 139 pp. $6.75.

The subjects covered in this book are similar to those in a number of recent books written under the title of queueing theory. The present title removes the authors objection to the inclusion under queueing theory of systems where there is no real queue as in telephone traffic for example. The emphasis throughout the book is on elegant analytical techniques for explicit solution of idealized service models, particularly the single service queues and multiple server telephone system. By avoiding both the tedious and the trivial examples, the author manages to include a wide range of mathematically interesting problems. The discussion of possible probability structures of the traffic, however, is unnecessarily brief with hardly a mention of anything except independent time intervals. Questions such as the existence of equilbrium solutions are also not discussed. Despite such deficiencies, this is a well organized discussion of service models for readers interested in applied mathematics and should appeal to a discriminating audience. The mathematical level is that of a graduate text and assumes an introduction knowledge of probability theory and familiarity with the usual techniques of classical analysis. The reviewer recommends this book as one of the small fraction of books published each year that is really well done.

G. F. NEWELL

Mathematical statistics. By Samuel W. Wilks. John Wiley & Sons, Inc., New York, London, 1962. xvi + 644 pp. $15.00.

This volume is an extensive expansion of the author's earlier book of the same title published in litroprinted form in 1943. The book is a welcome addition to the literature on mathematical statistics. It assumes no previous background in probability and statistics but a rather good undergraduate grounding in mathematics. An outstanding feature of the book is the fact that it touches on and develops many of the recent advances in statistical theory. These include nonparametric statistics, sequential analysis, statistical decision theory and time series analysis.

The following list of chapters gives a good idea of the development and the topics that are discussed: 1. Preliminaries; 2. Distribution functions; 3. Mean values and moments of random variables; 4. Sequences of random variables; 5. Characteristic functions and generating functions; 6. Some special discrete distributions; 7. Some special continuous distributions; 8. Sampling theory; 9. Asymptotic sampling theory for large samples; 10. Linear statistical estimation; 11. Nonparametric statistical estimation; 12. Parametric statistical estimation; 13. Testing parametric statistical hypotheses; 14. Testing nonparametric statistical hypotheses; 15. Sequential statistical analysis; 16. Statistical decision functions; 17. Time series; 18. Multivariate distribution theory.

The first five chapters establish the required background in probability theory. Chapter 6 discusses the hypergeometric, binomial, Poisson and multinomial distributions. Specific continuous distributions like the rectangular, normal, multivariate normal, and Dirichlet distributions as well as those of importance in the analysis of variance are considered in the following chapter. Finite and large sample theory are presented in chapters 8 and 9. There is in particular a discussion of sampling theory for order statistics. Regression analysis, confidence intervals, multiple comparisons and various models of the analysis of variance are treated in the chapter on linear estimation. The nonparametric techniques discussed are confidence intervals and also confidence bounds for distribution functions. Point and interval estimates are considered in chapter 12. The notion of sufficiency is introduced and applied in the treatment of unbiased estimates. Basic properties of maximum likelihood estimates are established. A discussion of testing in the parametric and nonparametric situation follow. The probability ratio sequential test and a fixed interval estimator for the mean of a normal distribution are given in the chapter on sequential analysis. The book closes with two brief chapters on statistical decision theory and time series analysis and a more extended one on multivariate theory. The book treats those topics of interest to the author in greater detail but the specific topics mentioned are a small sampling of the wealth of material in this book. The book should be very suitable for an extended graduate course on mathematical statistics.

M. Rosenblatt

A guide to FORTRAN programming. By Daniel D. McCracken. John Wiley & Sons, Inc., New York, London, 1961. viii + 88 pp. $2.95.

This book is a very readable introduction to programming in FORTRAN. The presentation is based on the version of FORTRAN used with the IBM 704, 709 and 7090 computers, but footnotes and an appendix list variations appropriate to: IBM 1620 FORTRAN and GOTRAN, IBM 650 FORTRAN and FORTRANSIT, IBM 7070 FORTRAN, the Honeywell Algebraic Compiler, the Philco 2000 ALTAC, and the Control Data 1604 FORTRAN. Used in conjunction with the appropriate manual, it should make a very effective teaching aid.

This book is basically another FORTRAN manual serving the same purpose as the manuals produced by the computer manufacturers, but it is superior to all those seen by this reviewer in that greater attention has been paid to explaining the more difficult points, the different types of FORTRAN statements are explained in a logical order with many examples to be worked by the student, and there are eight simple "case studies" chosen to demonstrate the application of FORTRAN to the types of calculations that arise in scientific work. However it also shares one fault common to the manufacturers' manuals; any reader who has no previous programming experience is handicapped by having to wade through many detailed rules which are arbitrary in nature and have no real significance to the technique of preparing a program, before he has a chance to acquire any understanding of the basic concepts; before, in fact, he has any comprehension of what a program is. If the book were prefaced by an additional

chapter that carried the reader briefly through the complete procedure of analysing a problem, preparing the program and the data, compiling and running it, its value would be greatly enhanced. However, the book, as it is, would make an excellent text for a FORTRAN course in which the instructor fills this gap with his first lecture—any one of the eight case studies in Chapter 9 would be suitable for this purpose.

B. A. CHARTRES

Theory of elastic thin shells. By A. L. Gol'denveizer. Translation from the Russian edited by G. Herrmann. Pergamon Press, New York, Oxford, London, Paris 1961. xxi + 658 pp. $15.00.

This impressive volume deals with the theory of thin shells constructed on the hypothesis of preservation of the normal element, the so-called Love-Kirchhoff hypothesis. It is assumed that the shell material is isotropic and obeys Hooke's law, and that the strains, displacements and angles of rotation are so small that second powers of these quantities may be neglected. Within this framework of the linear theory of elastic shells the present volume is the most comprehensive work available on shells of constant thickness.

Part I, consisting of four chapters, is devoted to the derivation of the *basic equations* of shell theory. These are formulated in terms of arbitrary curvilinear surface coordinates and specialized for orthogonal coordinates. The required properties of surfaces are briefly discussed in the first chapter. Since tensor notation is completely avoided in the book, the pertinent equations are often rather unwieldy. The equations of equilibrium are first derived in vector form. This derivation has the advantage that it leads in a direct manner to the establishment of four stress functions in terms of which all stress resultants and stress couples may be expressed. The discussion of the deformation of a shell is based on the Love-Kirchhoff assumption and on the introduction of the displacement vector U and the rotation vector Ω. The components of deformation, the angles of rotation and two auxiliary quantities are expressed in terms of derivatives of the displacement vector U and the rotation vector Ω. The integrability conditions then lead to the equations of compatibility. The stress-strain relations are finally formulated in the conventional form in which the effect of the transverse normal stresses is neglected. Various possible simplifications which do not affect the accuracy within the framework of shell theory, are also considered. The last chapter of Part I deals with the equations of compatibility in terms of stress resultants and stress couples, and with the equations of equilibrium in terms of the displacement components. The complete static-geometric analogy, discovered independently by the author and by Lur'e, is treated in detail. A discussion of the boundary conditions at the edge of a shell completes the general theory.

Part II, consisting of five chapters, deals with the *membrane theory* of shells. The static-geometric analogy again plays an important role in the discussion of the general theory of membrane shells. The peculiar properties of a membrane state of stress in shells of zero Gaussian curvature are discussed in a separate chapter. Two chapters are devoted to the membrane theory of spherical shells, giving a complete solution. Some generalizations to other shell forms with positive Gaussian curvature are also considered briefly. A final chapter in this part presents a significant discussion of boundary conditions in the membrane theory.

Part III, consisting of three chapters, is concerned with *circular cylindrical shells.* The equations are reduced to a single eighth-order equation of well-known type, and its solution is assumed in terms of a trigonometric series with respect to one coordinate. The resulting ordinary differential equations are discussed with great care, in particular for the case of closed cylindrical shells. No comparable discussion of the characteristic equation is available elsewhere, as far as the reviewer is aware. The accuracy of various simplifications is examined in full detail and new light is thrown on their range of applicability.

Part IV consists again of three chapters and deals with the state of stress in an *arbitrary shell.* The basis for the discussion is a theory of asymptotic integration of partial differential equations containing a small parameter. This concept is also discussed in the author's addendum to the present English edition, where some results, obtained since the publication of the original Russian (1953) edition, are presented in relation to other asymptotic methods. Many difficulties are as yet unresolved in this field, but the author has contributed significantly to a better understanding of the problems involved. His consistent attempts to obtain estimates for the errors incurred in the first steps of an asymptotic integration are particularly noteworthy. The reviewer has little doubt that this part of the present treatise will prove most stimulating for further research.

Part V, consisting of three chapters, deals with the application of the general theory to obtain various *approximate theories of shells*. The major distinction of this part from other treatments of approximate solution in shell theory is the unifying point of view provided by the preceding discussion of asymptotic integration theory. Particular emphasis is put here on the various types of edge effect induced by the boundary conditions.

In summary, the present treatise is an outstanding achievement. It will serve as a standard reference in shell theory for many years to come, and is an indispensable aid for all workers in this field. The translation reads very well. It is only to be regretted that the printing of the formulae is no better than reviewers and readers have come to associate with translations from the Russian published by the present firm. Moreover, their standard notice, explaining the high price of the book, strikes this reviewer as definitely odd in this case, in view of the acknowledgment on the title page of a grant-in-aid from the National Science Foundation.

W. T. KOITER

Proceedings of the International Symposium on Linear Spaces. Jerusalem Academic Press, Jerusalem, and Pergamon Press, New York, Oxford, London, Paris, 1961. xi + 452 pp. $14.00.

This book presents the papers given at the International Symposium on Linear Spaces, which was held at the Hebrew University of Jerusalem in July 1960 under the auspices of the International Mathematical Union. It is clearly impossible to summarize, in a brief review, the papers of the 32 distinguished contributors. Of greatest interest to applied mathematicians are, perhaps, those papers concerned with the modern theory of differential equations. S. Agmon is concerned with a general class of self-adjoint elliptic boundary value problems which possess a discrete set of real eigenvalues and aims to determine in this class those problems having their eigenvalues contained in a half-line. L. Ehrenpreis summarizes certain recent methods and results on systems of linear partial differential equations with constant coefficients and applies them to Cauchy and initial value problems. E. Hille discusses linear ordinary differential equations in Banach algebras, generalizing certain classical problems. J. L. Massera applies to linear differential equations the techniques of function spaces with translations, to discover relations existing between certain properties of the three differential equations $x + A(t)x = (i)\,0$, $(ii)\,f(t)$, $(iii)\,h(x-t)$, $A(t)$ being an nxn matrix. L. Nirenberg and G. Stampacchia are concerned with some of the striking developments in elliptic partial differential equations of the last few years; in particular, Nirenberg's paper is an excellent summary of recent work designed to prove the continuous dependence of the solution in terms of the given data, in various topologies. He discusses estimates for the solutions of $Lu = f$ and proposes certain open problems, such as whether the spectrum of L, under Dirichlet boundary conditions, can be the whole plane.

The book presents an excellent survey of the field of the Symposium and is exceptionally well produced and printed.

WALTER FREIBERGER

Applied dynamic programming. By Richard E. Bellman and Stuart E. Dreyfus. Princeton University Press, New Jersey, 1962. xxii + 363 pp. $8.50.

This is a most valuable supplement to the senior author's Dynamic Programming (by R. Bellman, Princeton University Press, 1957). While there the emphasis has been on the general theory, much of the present volume is concerned with the computational feasibility of dynamic programming.

Chapter 1 (One-dimensional allocation problems) deals with processes that lead to sequences of functions of a single variable and introduces the reader to the basic ideas of both, dynamic programming and the computational techniques that are to be developed in the remainder of the book. Chapter 2 (Multidimensional allocation processes) extends these ideas to problems involving sequences of functions of two variables, stressing the use of Lagrange's multipliers and successive approximations in policy space. Whereas these chapters treat static allocation processes, smoothing and scheduling processes, which naturally present themselves in dynamic form, are considered in Chapter 3 (One-dimensional

smoothing and scheduling processes). Problems of dynamic programming that arise in the numerical solutions of dynamic-programming problems are discussed in Chapter IV (Optimal search techniques). In Chapter V (Dynamic programming and calculus of variations) dynamic programming is used to derive a nonlinear partial differential equation from which the principal results of the calculus of variations are readily obtained. Chapters 6 (Optimal trajectories), 7 (Multistage production processes utilizing complexes of industries), and 8 (Feedback control processes) exploit this approach. Numerical solutions of typical feedback control processes are discussed in Chapter 9 (Computational results for feedback control processes). Chapter 10 (Linear equations and quadratic criteria) is concerned with computational procedures that apply when the equations describing a process are linear but the criterion functions are quadratic. Other ways of taking advantage of the analytical structure of a specific problem to improve the computational procedure are discussed in Chapter 11 (Markovian decision processes). Chapter 12, finally, presents preliminary results concerning questions of accuracy, reduction of computing time, and reduction of dimensionality. Five appendices contain supplementary results (I. On a transcendental curve, by O. Gross—II. A new approach to the duality theory of mathematical programming, by S. Dreyfus and M. Freimer—III. A computational technique based on successive approximations in policy space, by S. Dreyfus—IV. A new functional transform in analysis: the maximum transform, by R. Bellman and S. Dreyfus) and a description of the computer used by the authors (V. The Rand Johnniac computer, by S. Dreyfus).

As far as possible, the authors have tried to make this stimulating work independent of the volume mentioned at the beginning of this review.

W. PRAGER

Studies in statistical mechanics. Editors J. de Boer and G. E. Uhlenbeck. Volume I. North-holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, 1962. x + 350 pp. $13.75.

This is the first in what will presumably be a series of compilations of articles reviewing various areas of statistical mechanics. It consists of four contributions, the first of which is a most welcome translation of N. N. Bogoliubov's monograph on "Dynamical Theory in Statistical Physics." Although this work first appeared some sixteen years ago, it does not seem in any way dated; in particular, Bogoliubov's use of generating functionals in the expansion of distribution functions is an approach which this reviewer, at least, had not seen before. Bogoliubov's outstanding contribution is of course his treatment of the relation between the Liouville equation and equations of the Boltzmann type; even today, his discussion of this difficult problem is probably the best to be found anywhere. Some fifty references containing applications and discussion of Bogoliubov's theory have been added by the translator, E. K. Gora.

Next in sequence is a rather formal discussion, by G. E. Uhlenbeck and G. W. Ford, of the theory of linear graphs as applied to virial expansions of the thermodynamic properties and equilibrium distribution functions of gases. Of particular interest here is the chapter on the so-called Gaussian model, in which the assumption of a Gaussian form for the Mayer function greatly simplifies the evaluation of various cluster integrals.

The third section, by H. Mori, I. Oppenheim, and J. Ross, is concerned with the Wigner distribution function in quantum statistical mechanics. This function permits the derivation of an equation closely resembling the Liouville equation of classical statistical mechanics, and is particularly useful in approaching the classical limit; the authors show its application to both equilibrium thermodynamics and transport processes. In addition, this section contains a description of the correlation function approach to irreversible phenomena in quantum mechanical systems.

The final contribution is a relatively short chapter by M. Dresden, presenting a number of "ring" models devised by M. Kac to investigate the old and still unresolved problem of just how deterministic systems can give rise to irreversible processes. The models consist essentially of a large number of equidistant particles moving on a circle, and changing color, from black to white or vice versa, whenever they pass certain randomly distributed fixed points on the circle. Several sets of detailed rules are discussed, and it is really quite surprising how much insight can be gained by looking at these relatively simple (and highly artificial) systems.

All four sections in this volume are written for experts in the field; none make easy reading, which

is hardly surprising in view of the difficulty of the problems tackled. The emphasis is heavily on general procedures rather than on actual solutions. Again this is mainly a reflection of the current state of statistical mechanics, particularly as applied to irreversible phenomena. Nevertheless, it may perhaps be appropriate to express here the hope that future volumes in the series will devote some space to the derivation of numerical results for systems other than dilute gases, even if this means sacrificing the high standards of rigor which the authors of the present volume have consistently maintained.

STEPHEN PRAGER

Management models and industrial applications of linear programming. By A. Charnes and W. W. Cooper. Volume II. John Wiley & Sons, Inc., New York and London, 1961. xxi + 859 pp. $11.75.

The publication of "Management Models and Industrial Applications of Linear Programming," a two volume work by the famous team of A. Charnes and W. W. Cooper, is a major event in the brief but lively history of linear programming. A comparison with Charnes, Cooper, Henderson "An Introduction to Linear Programming" (1953) indicates perhaps better than any other yardstick the impressive growth of the field since then, in no small part due to the productive efforts of Charnes and Cooper. While this new book is intended mainly for "persons, who are interested in managerial applications of linear programming" both as a textbook and reference work, it is of at least equal interest to the applied mathematician for the original, comprehensive, and authoritative treatment of the mathematical theory of linear programming with special reference to computation methods.

The first seven chapters, with appendices, lay the theoretical groundwork and analyse the general model both from an algebraic and geometric point of view. This is followed by an examination of those structures of the (linear or piecewise linear) objective function and of the matrix of (linear) constraints which are mathematically significant and practically relevant. In this way the authors cover a considerable portion of the industrial and "managerial" applications of linear programming which have been undertaken so far, a field in which the authors themselves have been the leading contributors and often the pioneers. Less emphasis is placed on military management problems of the type that is investigated at the RAND corporation. (This gap may some day be filled by the long-awaited publication of Dantzig's book on linear programming, chapters of which have so far been available only in dittoed form, but have had a great influence on current work in linear programming). The bulk of the present work is devoted to linear programming proper, but chapters are included on input-output analysis, game theory, convex and quadratic programming, and the recent integer programming. In convex programming particular—but not exclusive—attention is paid to those methods, by which a piecewise linear programming problem is obtained. It is, of course, not possible in the limited space of a review to do justice to the wealth of ideas and models contained in the two volumes of this book.

In the analysis of the economic aspects of linear programming the authors have gone their own way, too, both in the selection of problems and in the distribution of emphasis. Thus there is rather little overlap in this area with the Dorfman, Samuelson, Solow book. A new twist is given to the fundamental notion of an efficient point (any admissible point whose coordinates are greater or equal to those of an efficient point coincides with the efficient point): The space need not be that of commodities, but may be defined in terms of any monotone functions of activity levels. The terminology for this (delegation models, functional efficiency) and many other neologisms are distinctly the authors' own.

There are perhaps no great surprises in this book since earlier versions of many sections were published as articles. While almost every subject in linear programming has been covered here at one place or another, the authors have, whenever possible, used their own approach and created their own system. An index of applications might have facilitated one's orientation through the 859 pages. As a reference book it also suffers somewhat by the almost complete absence of references to publications not in the English language. (By now a significant body of managerial applications has come into existence in both France and Italy.) The emphasis on originality and priority and the accounts given to establish claims of priority must be irritating at times and to some persons. But these are minor points. There is no doubt that this book will become a standard reference for anyone, who is working in the still growing field of linear programming and its offshoots.

MARTIN BECKMANN