LOMMEL FUNCTIONS WITH IMAGINARY ARGUMENT*

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Abstract. To express the solution to certain physical problems in terms of a real function of a real variable a new function called a modified Lommel Function is defined. Some of its properties are presented along with some of the relations between this new function and modified Bessel functions.

Introduction. In the study of the steady-state temperature distribution in a fin of variable cross-sectional area with internal heat generation it is necessary to determine a particular solution of the nonhomogeneous differential equation

$$u^2 \frac{d^2y}{du^2} + u \frac{dy}{du} - (u^2 + \nu^2)y = -u^{\mu+1}, \tag{1}$$

where ν and μ are constants. Equation (1) is closely related to the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = x^{\mu+1}, \tag{2}$$

which has as a particular solution the Lommel function $S_{\mu,\nu}(x)$ derived in [1]. A summary of Lommel's paper is given by Watson in [2] and further references to Lommel functions appear in Luke [3] and Erdelyi [4].

The substitution $x = iu$ transforms (2) into the equation

$$u^2 \frac{d^2y}{du^2} + u \frac{dy}{dx} - (u^2 + \nu^2)y = i^{\mu+1}u^{\mu+1}, \tag{3}$$

which has as a particular solution

$$y = S_{\mu,\nu}(iu). \tag{4}$$

Thus,

$$y = i^{1-\mu}S_{\mu,\nu}(iu) \tag{5}$$

is a particular solution of (1).

In a physical problem it is usually desirable to present the solution in real form. It is therefore convenient to define a new function, which will be a real function of a real variable. It seems appropriate to call this function a modified Lommel function since it is related to $S_{\mu,\nu}(iu)$ in much the same manner as $I_\nu(u)$ is related to $J_\nu(iu)$, and to denote it by the symbol $R_{\mu,\nu}(u)$ in analogy to the relationship between $J_\nu(iu)$ and $I_\nu(u)$. The modified Lommel function $R_{\mu,\nu}(u)$ will be defined by the relation

$$R_{\mu,\nu}(u) = i^{1-\mu}S_{\mu,\nu}(iu). \tag{6}$$

In the remainder of this paper the series definition of $R_{\mu,\nu}(u)$ will be presented for various conditions on the quantities $\mu + \nu$ or $\mu - \nu$, a recurrence formula for the modified Lommel functions will be presented, and some relations between modified Bessel functions and modified Lommel functions will be developed. The treatment of the modified Lommel function in this paper will closely parallel that of the ordinary Lommel functions given by Watson [2].

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Ascending series—$r_{\mu,\nu}(u)$. When neither $\mu + \nu$ nor $\mu - \nu$ is an odd negative integer, it can be shown that a particular solution of (1), proceeding in ascending powers of u beginning with $u^{\mu+1}$, is

$$y = -\left[\frac{u^{\mu+1}}{(\mu+1)^2 - \nu^2} + \frac{u^{\mu+2}}{[(\mu+1)^2 - \nu^2][(\mu+3)^2 - \nu^2]} + \cdots\right] \tag{7}$$

or equivalently

$$y = -u^{\mu-1} \sum_{m=0}^{\infty} \frac{(u/2)^{2m+2}\Gamma([\mu - \nu + 1]/2)\Gamma([\mu + \nu + 1]/2)}{\Gamma([\mu - \nu + 2m + 3]/2)\Gamma([\mu + \nu + 2m + 3]/2)}. \tag{8}$$

For brevity the expressions on the right are written as $r_{\mu,\nu}(u)$; they will be referred to as the associated function. Thus, $y = r_{\mu,\nu}(u)$ is a particular solution of (1), except when either of the numbers $\mu \pm \nu$ is an odd negative integer.

Descending series—$R_{\mu,\nu}(u)$. The function $R_{\mu,\nu}(u)$ is derived from the consideration of the particular solution of (1) in the form of a descending series. A particular solution of (1) proceeding in descending powers of u beginning with $u^{\mu-1}$ is

$$y = u^{\mu-1}\left[1 + \frac{(\mu-1)^2 - \nu^2}{u^2} + \frac{[(\mu-1)^2 - \nu^2][(\mu-3)^2 - \nu^2]}{u^4} + \cdots\right]. \tag{9}$$

This series converges only if it terminates, but if it terminates it is a solution of (1). The series will be called $R_{\mu,\nu}(u)$.

The series terminates only if one of the numbers $\mu + \nu$ or $\mu - \nu$ is an odd positive integer. If $\mu - \nu$ is an odd positive integer it is apparent that

$$\mu = \nu + 2p + 1, \tag{10}$$

and that

$$R_{\mu,\nu}(u) = u^{\mu-1} \sum_{m=0}^{p} \frac{\Gamma([\mu - \nu + 1]/2)\Gamma([\mu + \nu + 1]/2)}{(u/2)^{2m}\Gamma([\mu - \nu + 1 - 2m]/2)\Gamma([\mu + \nu + 1 - 2m]/2)} \tag{11}$$

or equivalently

$$R_{\mu,\nu}(u) = u^{\mu-1} \sum_{m=0}^{p} \frac{\Gamma(p+1)\Gamma(p+\nu+1)}{(u/2)^{2m}\Gamma(p+1-m)\Gamma(p+\nu+1-m)}. \tag{12}$$

Letting $n = p - m$, leads to the equation

$$R_{\mu,\nu}(u) = u^{\mu-1} \sum_{n=0}^{p} \frac{(u/2)^{2n-2p}\Gamma(p+1)\Gamma(p+\nu+1)}{\Gamma(n+1)\Gamma(n+\nu+1)} \tag{13}$$

or equivalently

$$R_{\mu,\nu}(u) = 2^{\mu-1}\Gamma(p+1)\Gamma(p+\nu+1)\left[\sum_{n=0}^{\infty} \frac{(u/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)} - \sum_{n=p+1}^{\infty} \frac{(u/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}\right]. \tag{14}$$

Recognizing that the first series is $I_\nu(u)$ and letting $N = n - (p+1)$, one obtains

$$R_{\mu,\nu}(u) = 2^{\mu-1}\Gamma([\mu - \nu + 1]/2)\Gamma([\mu + \nu + 1]/2)I_\nu(u)$$
$$- u^{\mu-1} \sum_{N=0}^{\infty} \frac{(u/2)^{2N+2}\Gamma([\mu - \nu + 1]/2)\Gamma([\mu + \nu + 1]/2)}{\Gamma([\mu - \nu + 2N + 3]/2)\Gamma([\mu + \nu + 2N + 3]/2)}. \tag{15}$$

The series in (15) is identical to that in (8) so that

$$R_{\mu,\nu}(u) = 2^{\mu-1}\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})I_\nu(u) + r_{\mu,\nu}(u) \tag{16}$$

or equivalently

$$R_{\mu,\nu}(u) = -2^{\mu-1}\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})\frac{\sin \pi(\mu + \nu)}{\sin 2\pi\nu} I_\nu(u) + r_{\mu,\nu}(u). \tag{17}$$

When $\mu - \nu = 2p + 1$, the function

$$2^{\mu-1}\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})\frac{\sin (\pi[\mu - \nu])}{\sin 2\pi\nu} I_{-\nu}(u) \tag{18}$$

is zero, and so, when $\mu - \nu$ is an odd positive integer

$$R_{\mu,\nu}(u) = r_{\mu,\nu}(u) + \frac{2^{\mu-1}\Gamma([\mu - \nu + 1]/2)\Gamma([\mu + \nu + 1]/2)}{\sin 2\pi\nu}[[\sin \pi(\mu - \nu)]I_{-\nu}(u) - [\sin \pi(\mu + \nu)]I_\nu(u)]. \tag{19}$$

From (9) it is evident that $R_{\mu,\nu}(u)$ is an even function of ν. It is also evident that the right hand side of (19) is an even function of ν. Thus (19) holds also when $\mu - (-\nu)$ is an odd positive integer, i.e., when $\mu + \nu$ is an odd positive integer. Thus it holds in all cases for which $R_{\mu,\nu}(u)$ has, as yet, been defined, and it will be adopted as the general definition of $R_{\mu,\nu}(u)$ except that when ν is an integer n the equivalent form

$$R_{\mu,n}(u) = r_{\mu,n}(u) + 2^{\mu-1}\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}n + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}n + \tfrac{1}{2}) \cdot\left[\frac{1}{\pi}\sin \pi(\mu - n)K_n(u) - \cos \pi(\mu + n)I_n(u)\right] \tag{20}$$

must be used.

It will be shown in a later section that $R_{\mu,\nu}(u)$ has a limit when $\mu \pm \nu$ is an odd negative integer, that is when $r_{\mu,\nu}(u)$ is undefined.

Recurrence formulas for $R_{\mu,\nu}(u)$. From (9) it can be shown that

$$R_{\mu+2,\nu}(u) = u^{\mu+1} + [(\mu + 1)^2 - \nu^2]R_{\mu,\nu}(u). \tag{21}$$

Also from (9) it can be shown that

$$\frac{d}{du}[u^\nu R_{\mu,\nu}(u)] = (\mu + \nu - 1)u^\nu R_{\mu-1,\nu-1}(u), \tag{22}$$

so that

$$R'_{\mu,\nu}(u) + \left(\frac{\nu}{u}\right)R_{\mu,\nu}(u) = (\mu + \nu - 1)u^\nu R_{\mu-1,\nu-1}(u). \tag{23}$$

From (23) using the fact that $R_{\mu,\nu}(u)$ is an even function of ν it is easy to obtain

$$R'_{\mu,\nu}(u) - \left(\frac{\nu}{u}\right)R_{\mu,\nu}(u) = (\mu - \nu - 1)u^\nu R_{\mu-1,\nu+1}(u). \tag{24}$$

Adding and subtracting (23) and (24) one obtains

$$\left(\frac{2\nu}{u}\right)R_{\mu,\nu}(u) = (\mu + \nu - 1)R_{\mu-1,\nu-1}(u) - (\mu - \nu - 1)R_{\mu-1,\nu+1}(u) \tag{25}$$

and

$$2R'_{\mu,\nu}(u) = (\mu + \nu - 1)R_{\mu-1,\nu-1}(u) + (\mu - \nu - 1)R_{\mu-1,\nu+1}(u). \tag{26}$$

The functions $R_{\mu,\nu}(u)$ can be replaced throughout these formulas by the associated functions $r_{\mu,\nu}(u)$.

$R_{\mu,\nu}(u)$ when $\mu \pm \nu$ is an odd negative integer. It is necessary to consider two distinct cases which are defined by

Case I: Either $\mu + \nu$ or $\mu - \nu$, but not both, is an odd negative integer.
Case II: Both $\mu + \nu$ and $\mu - \nu$ are simultaneously odd negative integers.

Case I. When $\mu + \nu$ or $\mu - \nu$ is an odd negative integer equation (19) assumes an undetermined form. However, if $\mu - \nu$ is an odd negative integer*, one may write

$$\mu = \nu - 2p - 1 \tag{27}$$

so that

$$R_{\mu,\nu}(u) = R_{\nu-2p-1,\nu}(u). \tag{28}$$

Then $R_{\nu-2p-1,\nu}(u)$ can be written in terms of $R_{\nu-1,\nu}(u)$ by repeated use of (21) which gives

$$R_{\nu-2p-1,\nu}(u) = -\sum_{m=0}^{p-1} \frac{u^{\nu-2p+2m}}{2^{2m+2}(-p)_{m+1}(\nu - p)_{m+1}} + \frac{R_{\nu-1,\nu}(u)}{2^{2p}p!(1 - \nu)_p}. \tag{29}$$

$R_{\nu-1,\nu}(u)$ is defined by the limiting form of (21), namely

$$R_{\nu-1,\nu}(u) = \lim_{\mu\to\nu-1} \left[\frac{-u^{\mu+1} + R_{\mu+2,\nu}(u)}{(\mu - \nu + 1)(\mu + \nu + 1)}\right]. \tag{30}$$

Since both the numerator and the denominator vanish at $\mu = \nu - 1$, it is convenient to use L'Hospital's theorem. Thus

$$R_{\nu-1,\nu}(u) = \frac{1}{2^\nu}\left[-u^\nu \ln u + \frac{\partial}{\partial\mu} R_{\mu+2,\nu}(u)\right]_{\mu=\nu-1}. \tag{31}$$

Performing the indicated differentiation and evaluating, one obtains

$$R_{\nu-1,\nu}(u) = -\frac{u^\nu}{4}\Gamma(\nu)\sum_{m=0}^{\infty}\frac{(u/2)^{2m}}{m!\Gamma(\nu + m + 1)}\left[2\ln\frac{u}{2} - \psi(m + 1) - \psi(m + \nu + 1)\right] - \frac{2^{\nu-1}\pi\Gamma(\nu)}{\sin 2\pi\nu}[I_{-\nu}(u) - [\cos 2\pi\nu]I_\nu(u)]. \tag{32}$$

It might be well to point out that Case I can never occur when ν is an integer. For if ν is an integer and $\mu - \nu$ is an odd negative integer then μ must also be an integer. It is easy to show that under these hypotheses $\mu + \nu$ must be either an odd negative integer in which case both $\mu - \nu$ and $\mu + \nu$ are odd negative integers and one has Case II rather than Case I, or $\mu + \nu$ is an odd positive integer in which case formula (9) can be used.

A special case of (32) which is of interest occurs if $\nu = n/2$ where n is an odd integer. In this case (32) reduces to

*Since $R_{\mu,\nu}(u)$ is an even function of ν, it is sufficient to consider the case in which $\mu - \nu$ is an odd negative integer.

$$R_{(n/2)-1,(n/2)}(u) = -\tfrac{1}{4}u^{n/2}\Gamma\left(\frac{n}{2}\right)\sum_{m=0}^{\infty}\frac{(u/2)^{2m}}{m!\Gamma([2m+n+2]/2)} \cdot\left[2\ln\frac{u}{2} - \psi(m+1) - \psi\left(m+\frac{n}{2}+1\right)\right]. \tag{33}$$

Case II. If $\mu + \nu$ and $\mu - \nu$ are both odd negative integers it is necessary but not sufficient that μ be a negative integer and that ν be an integer. Thus in this case one may write

$$\mu = -n - 2p - 1 \tag{34}$$

where n and p are positive integers. Repeated use of (21) gives

$$R_{-n-2p-1,n}(u) = -\sum_{m=0}^{p-1}\frac{u^{-n-2p+2m}}{2^{2m+2}(-p)_{m+1}(-n-p)_{m+1}} + \frac{R_{-n-1,n}(u)}{2^{2p}p!(1+n)_p}. \tag{35}$$

In order to evaluate $R_{-n-1,n}(u)$ one may use the derivative formula (22)

$$\frac{d}{du}[u^{\nu}R_{\mu,\nu}(u)] = (\mu+\nu-1)u^{\nu}R_{\mu-1,\nu-1}(u). \tag{22}$$

Using the chain rule it follows that

$$\frac{d}{du}[u^{\nu/2}R_{\mu,\nu}(u^{1/2})] = \tfrac{1}{2}(\mu+\nu-1)u^{(\nu-1)/2}R_{\mu-1,\nu-1}(u^{1/2}), \tag{36}$$

and

$$\frac{d^2}{du^2}[u^{\nu/2}R_{\mu,\nu}(u^{1/2})] = \tfrac{1}{2}\cdot\tfrac{1}{2}(\mu+\nu-1)(\mu+\nu-3)u^{(\nu-2)/2}R_{\mu-1,\nu-1}(u^{1/2}). \tag{37}$$

Thus one is led to the result

$$\frac{d^n}{du^n}(u^{\nu/2}R_{\mu,\nu}(u^{1/2})) = \frac{u^{(\nu-n)/2}}{2^n}\prod_{k=1}^{n}(\mu+\nu+1-2k)R_{\mu-n,\nu-n}(u^{1/2}). \tag{38}$$

Letting $\nu = 0$ and $\mu = -1$ in (38) there results

$$\frac{d^n}{du^n}R_{-1,0}(u^{1/2}) = \frac{u^{-n/2}}{2^n}\prod_{k=1}^{n}(-2k)R_{-n-1,n}(u^{1/2}) \tag{39}$$

or equivalently

$$R_{-n-1,n}(u) = \frac{2^n u^n}{\prod_{k=1}^{n}(-2k)}\frac{d^n}{d(u^2)^n}R_{-1,0}(u). \tag{40}$$

Thus $R_{-n-1,n}(u)$ can be expressed in terms of the n-th derivative of $R_{-1,0}(u)$ with respect to the square of its argument.

It remains to determine $R_{-1,0}(u)$. When $\nu = 0$ the recurrence formula (21) becomes

$$R_{\mu,0}(u) = \frac{-u^{\mu+1} + R_{\mu+2,0}(u)}{(\mu+1)^2}. \tag{41}$$

If $\mu = -1$ the right side of (41) is an indeterminate form so by L'Hospital's theorem

$$R_{-1,0}(u) = \frac{1}{2}\left[\frac{\partial^2}{\partial\mu^2}[-u^{\mu+1} + R_{\mu+2,0}(u)]\right]_{\mu=-1}. \tag{42}$$

From (20)

$$R_{\mu+2,0}(u) = r_{\mu+2,0}(u) + 2^{\mu+1}[\Gamma(\tfrac{1}{2}\mu + \tfrac{3}{2})]^2\left\{\frac{1}{\pi}[\sin \mu\pi]K_0(u) - [\cos \mu\pi]I_0(u)\right\}. \tag{43}$$

Performing the indicated differentiation in (42) and evaluating the result at $\mu = -1$ gives the final result

$$R_{-1,0}(u) = -\frac{1}{2}\sum_{m=0}^{\infty}\frac{(u/2)^{2m}}{(m!)^2}\left\{\left[\ln\frac{u}{2} - \psi(m+1)\right]^2 - \tfrac{1}{2}\psi'(m+1) + \pi^2\right\}. \tag{44}$$

Thus by combining equations (35), (40) and (44) it is possible to evaluate $R_{\mu,\nu}(u)$ when both $\mu + \nu$ and $\mu - \nu$ are odd negative integers.

Relation of modified Lommel functions to modified Bessel functions. It is possible to express the important integrals

$$\int^u u^\mu I_\nu(u)\,du \quad \text{and} \quad \int^u u^\mu K_\nu(u)\,du$$

in terms of modified Lommel functions. Consider the following derivatives

$$\frac{d}{du}[u^\nu I_\nu(u)\cdot u^{1-\nu}R_{\mu-1,\nu-1}(u)] = uI_{\nu-1}(u)R_{\mu-1,\nu-1}(u) + uI_\nu(u)(\mu - \nu - 1)R_{\mu-2,\nu}(u) \tag{45}$$

and

$$\frac{d}{du}[u^{1-\nu}I_{\nu-1}(u)\cdot u^\nu R_{\mu,\nu}(u)] = uI_\nu(u)R_{\mu,\nu}(u) + (\mu + \nu - 1)uI_{\nu-1}(u)R_{\mu-1,\nu-1}(u). \tag{46}$$

Multiplying (45) by the quantity $(\mu + \nu - 1)$ and subtracting (46) gives

$$\frac{d}{du}[(\mu + \nu - 1)uI_\nu(u)R_{\mu-1,\nu-1}(u) - uI_{\nu-1}(u)R_{\mu,\nu}(u)]$$
$$= uI_\nu(u)[[(\mu - 1)^2 - \nu^2]R_{\mu-2,\nu}(u) - R_{\mu,\nu}(u)], \tag{47}$$

but from (21)

$$[(\mu - 1)^2 - \nu^2]R_{\mu-2,\nu}(u) = R_{\mu,\nu}(u) - u^{\mu-1}. \tag{48}$$

Combining (47) and (48) and integrating the result gives

$$\int u^\mu I_\nu(u)\,du = uI_{\nu-1}(u)R_{\mu,\nu}(u) - (\mu + \nu - 1)uI_\nu(u)R_{\mu-1,\nu-1}(u). \tag{49}$$

By a similar procedure it can be shown that

$$\int u^\mu K_\nu(u)\,du = -(\mu + \nu - 1)uK_\nu(u)R_{\mu-1,\nu-1}(u) - uK_{\nu-1}(u)R_{\mu,\nu}(u). \tag{50}$$

Concluding remarks. In the preceding sections the modified Lommel function has been defined and some of its properties and relations to other functions have been presented paralleling Watson's treatment of Lommel functions. This modified function appears to be useful for expressing the solutions of certain physical problems in terms of a real function of a real variable and for evaluating certain integrals involving modified Bessel functions.

At the present time a project is underway to compute extensive tables of $R_{\mu,\nu}(u)$. It is hoped that these tables will be published in the near future.

References

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