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THE FIELD OF A DIPOLE ABOVE AN INFINITE CORRUGATED PLANE*

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Abstract. We investigate the excitation and propagation of the three dimensional electromagnetic field over an infinite corrugated plane which is approximated by an anisotropic impedance boundary condition. Emphasis is placed upon effects of surface anisotropy which are not evident in two dimensional treatments. In particular we consider the excitation by a magnetic point dipole in detail. It turns out that the fields are determined by a scalar wave function which satisfies a mixed boundary condition involving a linear combination of the wave function, its normal derivative and its second order tangential derivative. The exact formal solution is first derived, and then the radiated far field and the surface wave far field are evaluated separately. Both the phase and the amplitude of the excited surface wave are dependent upon the direction of observation. Numerical results are given. The physical significance of this solution is discussed. A comparison is made between this problem and the theory of ship waves.

1. Introduction. In this paper we study the excitation and propagation of the three dimensional electromagnetic field over an infinite corrugated plane which may be approximated by an anisotropic impedance boundary condition. This investigation is motivated by its potential application to scanning surface wave antennas [1]. The fields produced by a magnetic point dipole will be considered in detail. The boundary condition is such that surface waves are generated and it may be described mathematically as:

$$E_x = ZH_z = (R - iX)H_z, \atop E_z = 0, \Bigg\} y = 0,$$

where the time dependence is assumed to be of the form $\exp(-i\omega t)$, and Z, R, and X are the impedance, resistance, and reactance of the surface, respectively. Insofar as surface waves are concerned, the most important case is such that R and X are positive in sign and R is much smaller than X, i.e., Z is almost purely reactive.

The first boundary condition implies that the corrugated plane consists of closely packed very thin metal plates terminating in a perfectly conducting surface. Z corresponds to the input impedance of short-circuited parallel plate waveguides which accommodate the *TEM* mode only (see Collin, *Field theory of guided waves*, McGraw-Hill, pp. 467-468, for discussion). The same condition is used in two dimensional treat-

ments, where, however, the absence of z-dependence eliminates the second order term in Eq. (2.7) of the next section. The second boundary condition is suggested by the vanishing of the field along the tops of the ridges.

It should be noticed that any impedance boundary condition approximating a dielectric coated surface is isotropic. The electromagnetic fields excited by a uniform line source above an infinite reactive plane have been investigated by Cullen [2], and Friedman and Williams [3]. The propagation of oblique surface waves over a corrugated plane has been considered in the literature, but it has not been treated three-dimensionally [1]. The solution of our three dimensional problem will show the effect of the surface anisotropy which is not evident in the two dimensional problems. The problem we treat may be formulated by the use of a double Fourier transform. However, we reduce the problem to one involving only a single Fourier transform by introducing an auxiliary function which is similar to those recently employed for several other problems in diffraction theory [4, 5, 6, 7].

In Sect. 2 we obtain the formal solution for the fields of a magnetic point dipole oriented in the direction of corrugations and located above a corrugated plane which is approximated by an anisotropic impedance boundary condition. The dipole is first decomposed into phased line sources, and the fields of the line sources are then synthesized into the dipole fields. In Sect. 3 the far-zone radiation field is evaluated by an algebraic method involving asymptotic differentiation and the excited surface wave far field is determined by applying the method of stationary phase. All the surface wave characteristics which include the amplitude, the phase, and the exponential decay factor are functions of the direction of observation and the surface reactance. In Sect. 4 we summarize the peculiar properties of the three dimensional surface wave, and present the surface wave power patterns and the surface wave phase patterns. A comparison is also made between this problem and the theory of ship waves.

2. The formal solution. In this section we proceed to study the electromagnetic field which arises from a magnetic point dipole lying above the surface of a plane characterized by an anisotropic impedance boundary condition. Consider a magnetic point dipole located at the point $x = 0$, $y = h$, $z = 0$ (Fig. 1) and oriented in the direction of the z-axis. We are interested in the free space region $y \geq 0$. The impedance boundary condition prescribed on the plane surface $y = 0$ is given by

$$E_z = 0, \qquad E_x = ZH_z = -iXH_z, \qquad (2.1)$$

where the impedance Z is supposed to be purely inductive and will support surface waves. We now wish to solve the time reduced Maxwell's equations subject to the prescribed singularity and boundary conditions.

If we assume the time dependence to be of the form $\exp(-i\omega t)$, the monochromatic Maxwell's equations may be reduced to:

$$\text{curl } \mathbf{H} = -i\omega\epsilon \mathbf{E},$$

$$\text{curl } \mathbf{E} = i\omega\mu\mathbf{H} - \delta(x)\delta(y - h)\delta(z)\mathbf{z},$$

where ϵ and μ are the permittivity and magnetic permeability of free space. Because of the boundary condition, the z-component of the electric vector is absent in the fields and hence the field is completely determined by the value of a scalar wave function f.
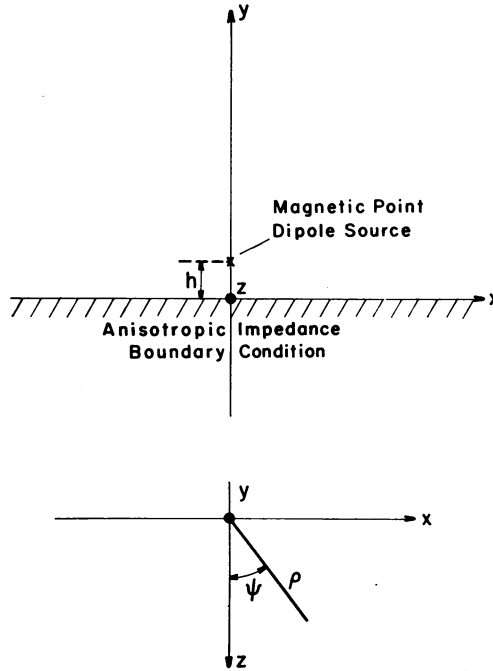
FIG. 1.

We have

$$\mathbf{E} = \operatorname{curl} f\mathbf{z} \quad \text{and} \quad \mathbf{H} = \frac{1}{i\omega\mu} \operatorname{curl} \operatorname{curl} f\mathbf{z}, \tag{2.3}$$

that is,

$$E_z = 0, \qquad E_x = \frac{\partial f}{\partial y}, \qquad E_y = -\frac{\partial f}{\partial x}, \tag{2.4}$$

$$H_x = \frac{1}{i\omega\mu} \frac{\partial^2 f}{\partial x \partial z} \quad H_y = \frac{1}{i\omega\mu} \frac{\partial^2 f}{\partial y \partial z} \quad H_z = \frac{1}{i\omega\mu} \left[\frac{\partial^2 f}{\partial z^2} + k^2 f \right]. \tag{2.5}$$

The function f satisfies the equation

$$(\nabla^2 + k^2)f = \delta(x)\,\delta(y - h)\,\delta(z), \tag{2.6}$$

where ∇^2 is the three-dimensional rectangular Laplacian, k is the propagation constant of free space, and δ is the Dirac Delta function. The appropriate boundary condition may be deduced from (2.1) and (2.3):

$$\frac{\partial f}{\partial y} + \frac{X}{\omega\mu} \left(k^2 f + \frac{\partial^2 f}{\partial z^2} \right) = 0 \quad \text{when } y = 0, \tag{2.7}$$

where $X/\omega\mu$ is a positive number. The mathematical problem has been reduced to that of solving the inhomogeneous wave equation (2.6) subject to the mixed second order boundary condition given by (2.7). Furthermore, the far field should satisfy the radiation condition.

Applying Fourier transform to Eqs. (2.6) and (2.7), we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + K^2\right)F = \delta(x)\delta(y - h), \qquad (2.8)$$

$$\frac{\partial F}{\partial y} + \frac{X}{\omega\mu}K^2F = 0, \qquad (2.9)$$

where the functions $f(x, y, z)$ and $F(x, y, s)$ are related by the following Fourier transform pair:

$$f(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x, y, s)e^{isz}\,ds, \qquad (2.10)$$

$$F(x, y, s) = \int_{-\infty}^{\infty} f(x, y, z)e^{-isz}\,dz, \qquad (2.11)$$

and $K^2 = k^2 - s^2$. Mathematically the transformed equations are a two-dimensional inhomogeneous wave equation and a mixed first-order boundary condition. Physically the point dipole has been decomposed into phased line sources. We are going to first solve the problem of a phased line source and then synthesize the line source fields into the dipole field.

The field of a phased line source may be analyzed either by the Fourier transform method or by using another auxiliary function which is a linear combination of the unknown wave function and its derivative. We choose to employ the latter method because it gives another example of this relatively new technique. Let us introduce the auxiliary function,

$$G = \left(\frac{\partial}{\partial y} + \lambda\right)F, \qquad (2.12)$$

where $\lambda = X/\omega\mu K^2$. Then G satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + K^2\right)G = -\left(\frac{\partial}{\partial h} - \lambda\right)\delta(x)\delta(y - h) \qquad (2.13)$$

and the simple homogeneous boundary condition

$$G = 0 \quad \text{when} \quad y = 0. \qquad (2.14)$$

We first treat the case of positive λ, i.e., $s^2 \leq k^2$. If the function F is denoted by F_1 when λ is positive, then F_1 can be found from G by integrating (2.12) in the following manner.

$$F_1(x, y, s) = e^{-\lambda y}\int_{-\infty}^{y} e^{\lambda\eta}G(x, \eta, s)\,d\eta + e^{-\lambda y}T(x, s), \qquad (2.15)$$

where $T(x, s)$ is an arbitrary function of x and s, which except for certain discontinuities in the derivative must be a homogeneous solution of (2.8). The discontinuities must be adjusted so that $F_1(x, y, s)$ meets the requirements of the problem. This adjustment follows after Eq. (2.18) below.

By applying the operator $-(\partial/\partial h - \lambda)$ to the unique Green's function satisfying Eq. (2.8) and vanishing at $y = 0$, we obtain the solution for the auxiliary function:

$$G = \frac{i}{4}\left(\frac{\partial}{\partial h} - \lambda\right)\{H_0^{(1)}[K\{x^2 + (y - h)^2\}^{1/2}] - H_0^{(1)}[K\{x^2 + (y + h)^2\}^{1/2}]\}. \qquad (2.16)$$

We substitute (2.16) into (2.15) and introduce the excited surface wave terms to play the role of $e^{-\lambda y} T(x, s)$

$$F_1(x, y, s) = -\frac{i}{4} e^{-\lambda y} \int_{-\infty}^{y} e^{\lambda \eta} \left\{ \frac{\partial}{\partial \eta} H_0^{(1)} [K\{x^2 + (\eta - h)^2\}^{1/2}] \right.$$

$$+ \frac{\partial}{\partial \eta} H_0^{(1)} [K\{x^2 + (\eta + h)^2\}^{1/2}] \right\} d\eta - \frac{i}{4} \lambda e^{-\lambda y} \int_{-\infty}^{y} e^{\lambda \eta} \{ H_0^{(1)} [K\{x^2 + (\eta - h)^2\}^{1/2}]$$

$$- H_0^{(1)} [K\{x^2 + (\eta + h)^2\}^{1/2}] \} d\eta + C \exp(-\lambda y + i[K^2 + \lambda^2]^{1/2} | x |). \qquad (2.17)$$

Integrating by parts, we have

$$F_1(x, y, s) = -\frac{i}{4} H_0^{(1)} [K\{x^2 + (y - h)^2\}^{1/2}] - \frac{i}{4} H_0^{(1)} [K\{x^2 + (y + h)^2\}^{1/2}]$$

$$+ \frac{i}{2} \lambda e^{-\lambda y} \int_{-\infty}^{y} e^{\lambda \eta} H_0^{(1)} [K\{x^2 + (\eta + h)^2\}^{1/2}] d\eta$$

$$+ C \exp(-\lambda y + i[K^2 + \lambda^2]^{1/2} | x |). \qquad (2.18)$$

Now the surface wave amplitude C may be determined by requiring the continuity of $\partial F_1/\partial x$ across the line $x = 0$, i.e.,

$$\left[\frac{\partial F_1}{\partial x} \right] = \lim_{x \to 0} \left\{ \frac{\partial}{\partial x} F_1(x > 0, y) - \frac{\partial}{\partial x} F_1(x < 0, y) \right\} = 0. \qquad (2.19)$$

Substituting (2.18) into the jump condition (2.19) yields

$$\lim_{x \to 0} \left\{ -i\lambda K e^{-\lambda y} \int_{-\infty}^{y} e^{\lambda \eta} H_1^{(1)} [K\{x^2 + (\eta + h)^2\}^{1/2}] \frac{x}{\{x^2 + (\eta + h)^2\}^{1/2}} d\eta \right.$$

$$+ i2C\{K^2 + \lambda^2\}^{1/2} \exp(-\lambda y + i[K^2 + \lambda^2]^{1/2} | x |) \right\} = 0. \qquad (2.20)$$

The limit of the integral in the above equation contributes only in the neighborhood of $\eta = -h$. Hence

$$C = \frac{\lambda K e^{-\lambda h}}{2\{K^2 + \lambda^2\}^{1/2}} \lim_{x \to 0} \int_{-h-\epsilon}^{-h+\epsilon} \frac{H_1^{(1)} [K\{x^2 + (\eta + h)^2\}^{1/2}] x}{\{x^2 + (\eta + h)^2\}^{1/2}} d\eta. \qquad (2.21)$$

Using the asymptotic formula of Hankel function for small argument, we find that

$$C = \frac{\lambda K e^{-\lambda h}}{2\{K^2 + \lambda^2\}^{1/2}} \lim_{x \to 0} \int_{-h-\epsilon}^{-h+\epsilon} \left(\frac{-2i}{\pi K\{x^2 + (\eta + h)^2\}^{1/2}} \right) \frac{x \, d\eta}{\{x^2 + (\eta + h)^2\}^{1/2}}. \qquad (2.22)$$

Carrying out the integration gives

$$C = -\frac{i\lambda e^{-\lambda h}}{\{K^2 + \lambda^2\}^{1/2}}. \qquad (2.23)$$

Now we consider the case of negative λ, i.e., $s^2 > k^2$. If the function F is denoted by F_2 when λ is negative, then F_2 may be found from G by integrating (2.12) as follows:

$$F_2(x, y, s) = -e^{-\lambda y} \int_{y}^{\infty} e^{\lambda \eta} G(x, \eta, s) \, d\eta. \qquad (2.24)$$

Substituting (2.16) into (2.24) and integrating by parts, we have

$$F_2(x, y, s) = -\frac{i}{4} H_0^{(1)}[K\{x^2 + (y - h)^2\}^{1/2}] - \frac{i}{4} H_0^{(1)}[K\{x^2 + (y + h)^2\}^{1/2}]$$

$$- \frac{i}{2} \lambda e^{-\lambda y} \int_y^\infty e^{\lambda \eta} H_0^{(1)}[K\{x^2 + (\eta + h)^2\}^{1/2}] \, d\eta. \tag{2.25}$$

The limits of integration in (2.24) have been so chosen that they insure the convergence of the integral. Because $\partial F_2/\partial x$ is continuous across the line $x = 0$, no surface waves are excited in this case. Since the arguments of Hankel Functions in (2.25) are imaginary, F_2 will not give rise to a radiation field.

The substitution of (2.18) and (2.25) into (2.10) completes the formal solution of this problem. It is difficult to evaluate the integrals exactly; however, we shall obtain the far field expressions in the next section.

3. Determination of the far fields. In this section we determine the radiated far field and the surface wave far field. We first treat the radiated far field by an algebraic method involving asymptotic differentiation. This method has proved useful in two dimensional problems [4, 5, 6, 7]. It gives the radiated far field directly, without intermediary use of the previously obtained complete solution. Let us now introduce the following function

$$g = \frac{\partial f}{\partial y} + \frac{X}{\omega \mu} \left(k^2 f + \frac{\partial^2 f}{\partial z^2} \right). \tag{3.1}$$

Then g satisfies the equation

$$(\nabla^2 + k^2)g = \left[-\frac{\partial}{\partial h} + \frac{X}{\omega \mu} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \right] \delta(x) \, \delta(y - h) \, \delta(z) \tag{3.2}$$

and vanishes when $\dot{y} = 0$. The solution for g may be written down immediately:

$$g = \frac{1}{4\pi} \left[\frac{\partial}{\partial h} - \frac{X}{\omega \mu} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \right] \left[\frac{e^{ikR'}}{R'} - \frac{e^{ikR''}}{R''} \right], \tag{3.3}$$

where

$$R' = \{x^2 + (y - h)^2 + z^2\}^{1/2} \approx R - h \sin \theta \sin \phi,$$

$$R'' = \{x^2 + (y + h)^2 + z^2\}^{1/2} \approx R + h \sin \theta \sin \phi,$$

$$R = \{x^2 + y^2 + z^2\}^{1/2}, \quad \theta = \tan^{-1} \frac{z}{\{x^2 + y^2\}^{1/2}}, \quad \text{and} \quad \phi = \tan^{-1} \frac{y}{x}.$$

We may expect the far-zone radiation field of f to be of the form

$$f = \frac{m(\theta, \phi)}{R} e^{ikR}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1), introducing polar coordinates, and keeping terms of order $1/R$, we obtain

$$\left\{[-ik \sin \theta \sin \phi - \frac{X}{\omega\mu}(k^2 - k^2 \cos^2 \theta)] \exp(-ikh \sin \theta \sin \phi)\right.$$

$$+ \left[-ik \sin \theta \sin \phi + \frac{X}{\omega\mu}(k^2 - k^2 \cos^2 \theta)\right] \exp(ikh \sin \theta \sin \phi)\left.\right\} \frac{e^{ikR}}{4\pi R}$$

$$= \left[ik \sin \theta \sin \phi + \frac{X}{\omega\mu}(k^2 - k^2 \cos^2 \theta)\right] f, \tag{3.5}$$

$$f = -\frac{e^{ikR}}{4\pi R}\left[\exp(-ikh \sin \theta \sin \phi) + \frac{i \sin \phi - n \sin \theta}{i \sin \phi + n \sin \theta} \exp(ikh \sin \theta \sin \phi)\right], \tag{3.6}$$

where $n = kX/(\omega\mu) = X(\mu/\epsilon)^{-1/2}$ may be interpreted as normalized surface reactance.

Now we turn our attention to the surface wave far field. Substituting the last term of (2.18) into (2.10) and using (2.23), we obtain the surface wave field:

$$f_s = \frac{-i}{2\pi}\int_{-k}^{k} \frac{\lambda e^{-\lambda h}}{\{K^2 + \lambda^2\}^{1/2}} \exp(-\lambda y + i[\{K^2 + \lambda^2\}^{1/2}|x| + sz]) ds. \tag{3.7}$$

It should be remembered that $\lambda = (X/\omega\mu)K^2$ and $K^2 = k^2 - s^2$. In order to evaluate the integral by the method of stationary phase it is convenient to make the following substitutions

$$z = \rho \cos \psi, \qquad |x| = \rho \sin \psi, \qquad \frac{s}{k} = \alpha. \tag{3.8}$$

Equation (3.7) now becomes

$$f_s = \int_{-1}^{1} A(\psi, n, \alpha) \exp[ik\rho P(\psi, n, \alpha)] d\alpha, \tag{3.9}$$

where

$$A(\psi, n, \alpha) = -\frac{ink}{2\pi} \frac{1 - \alpha^2}{\{1 - \alpha^2 + n^2(1 - \alpha^2)^2\}^{1/2}} \exp[-nk(1 - \alpha^2)(y + h)]. \tag{3.10}$$

$$P(\psi, n, \alpha) = \alpha \cos \psi + [1 - \alpha^2 + n^2(1 - \alpha^2)^2]^{1/2} \sin \psi; \tag{3.11}$$

when $k\rho$ is sufficiently large

$$f_s = A(\alpha_s)\left[\frac{2\pi}{k\rho |\partial^2 P(\psi, n, \alpha_s)/\partial\alpha^2|}\right] \exp i\left[k\rho P(\psi, n, \alpha_s) - \frac{\pi}{4}\right]. \tag{3.12}$$

It can be easily shown that $\partial^2 P(\psi, n, \alpha_s)/\partial\alpha^2$ is always negative. The stationary point α_s must satisfy the following equation:

$$\frac{dP}{d\alpha} = \cos \psi - \sin \psi \frac{\alpha_s[1 + 2n^2(1 - \alpha_s^2)]}{[1 - \alpha_s^2 + n^2(1 - \alpha_s^2)^2]^{1/2}} = 0. \tag{3.13}$$

We may easily express ψ in terms of n and α_s

$$\tan \psi = \frac{[1 - \alpha_s^2 + n^2(1 - \alpha_s^2)^2]^{1/2}}{\alpha_s[1 + 2n^2(1 - \alpha_s^2)]}. \tag{3.14}$$

The right hand side of equation (3.14) is a monotonically decreasing function in the interval $0 \leq \alpha \leq 1$ for all n. The truth of this statement is suggested by Fig. 2 and can
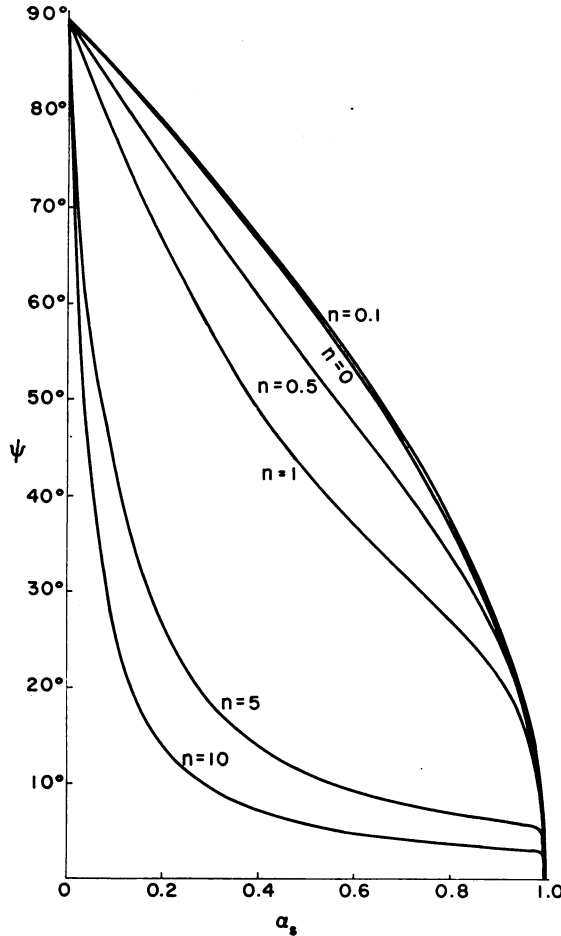
FIG. 2.

be proved by differentiating (3.14) with respect to α_s . Therefore α_s can be found numerically for the directions of observation $0 \leq \psi \leq 90°$. Those values corresponding to the interval $90° \leq \psi \leq 180°$ and $-1 \leq \alpha_s \leq 0$ are obtained by the symmetry requirement. Now the surface wave far field may be written as

$$f_s = -\frac{ink}{\{2\pi k\rho\}^{1/2}} \frac{(1 - \alpha_s^2)\{\alpha_s^2[1 + 2n^2(1 - \alpha_s^2)]^2 + [1 - \alpha_s^2 + n^2(1 - \alpha_s^2)^2]\}^{1/4}}{\{\alpha_s^2 + [1 + 2n^2(1 - \alpha_s^2)][1 - \alpha_s^2 + n^2(1 - \alpha_s^2)^2]\}^{1/2}}$$

$$\exp\left(i\left[k\rho P(\psi, n, \alpha_s) - \frac{\pi}{4}\right]\right) \exp\left(-nk[1 - \alpha_s^2][y + h]\right), \qquad (3.15)$$

where α_s is a function of ψ as indicated by (3.14). The electric and magnetic field components of the far-zone surface wave may be obtained by substituting (3.15) into (2.4) and (2.5). However, in order to understand the power flow of the surface wave far field, it is desirable to use the cylindrical coordinates (ρ, ψ, y). We first write down the components of f_s :

$$f_s \mathbf{z} = f_s \cos \psi \boldsymbol{\varrho} - f_s \sin \psi \boldsymbol{\psi} \tag{3.16}$$

Substitution of (3.16) into (2.3) yields

$$\mathbf{E} = \boldsymbol{\varrho}[-nk(1 - \alpha_s^2) \sin \psi f_s] + \boldsymbol{\psi}[-nk(1 - \alpha_s^2) \cos \psi f_s]$$
$$+ \mathbf{y}\left[-ikP \sin \psi f_s - ik \frac{\partial P}{\partial \psi} \cos \psi f_s \right] \tag{3.17}$$

$$\mathbf{H} = \frac{1}{i\omega\mu} \left\{ \boldsymbol{\varrho}\left[k^2 P \frac{\partial P}{\partial \psi} \sin \psi f_s + k^2 \left(\frac{\partial P}{\partial \psi}\right)^2 \cos \psi f_s - n^2 k^2 (1 - \alpha_s^2)^2 \cos \psi f_s \right]\right.$$
$$+ \boldsymbol{\psi}\left[n^2 k^2 (1 - \alpha_s^2)^2 \sin \psi f_s - k^2 P^2 \sin \psi f_s - k^2 P \frac{\partial P}{\partial \psi} \cos \psi f_s \right] \tag{3.18}$$
$$\left. + \mathbf{y}\left[-ink^2(1 - \alpha_s^2)P \cos \psi f_s + ink^2(1 - \alpha_s^2) \frac{\partial P}{\partial \psi} \sin \psi f_s \right] \right\},$$

where all terms of higher order than $\rho^{-1/2}$ have been neglected. The average Poynting vector is

$$\tfrac{1}{2}Re(\mathbf{E} \times \mathbf{H}^*) = \tfrac{1}{2}Re(E_\psi H_y^* - E_y H_\psi^*)\boldsymbol{\varrho}$$
$$+ \tfrac{1}{2}Re(E_y H_\rho^* - E_\rho H_y^*)\boldsymbol{\psi} + \tfrac{1}{2}Re(E_\rho H_\psi^* - E_\psi H_\rho^*)\mathbf{y}. \tag{3.19}$$

Lengthy but straightforward substitutions into the above equation show that the ψ and y components vanish and give the expression for the ρ component

$$(P_{av})_\rho = \tfrac{1}{2}\frac{|f_s|^2}{\omega\mu}(1 - \alpha_s^2)\{\alpha_s^2[1 + 2n^2(1 - \alpha_s^2)]^2 + [1 - \alpha_s^2 + n^2(1 - \alpha_s^2)^2]\}^{1/2} \tag{3.20}$$
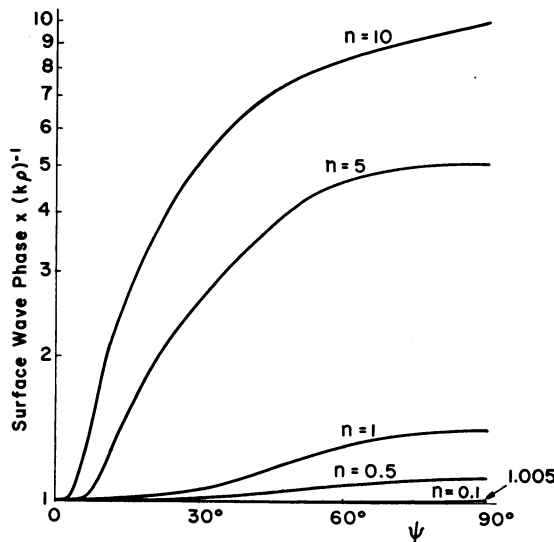
where f_s is given in (3.15).

Fig. 3.

Integrating (3.20) with respect to y gives the pattern function for the total surface wave power.

$$\int_0^\infty (P_{av})_\rho \, dy = \frac{1}{8\pi\omega\mu\rho} \frac{n(1-\alpha_s^2)^2\{\alpha_s^2[1+2n^2(1-\alpha_s^2)]^2 + [1-\alpha_s^2+n^2(1-\alpha_s^2)^2]\}}{\alpha_s^2 + [1+2n^2(1-\alpha_s^2)][1-\alpha_s^2+n^2(1-\alpha_s^2)^2]}$$

$$\times \exp\left[-2nk(1-\alpha_s^2)h\right]. \qquad (3.21)$$

This is a function of ψ through equation (3.14).

4. Discussion.   Both the phase and amplitude of the excited surface wave found in the last section are dependent upon the direction of observation. The surface wave amplitude shows the typical exponential decay away from the reactive surface. Notice, however, that the exponential decay factor also depends upon the direction of observation. Because the direction of observation is approximately constant in any fixed small region far away from the source, the three dimensional surface wave may be regarded
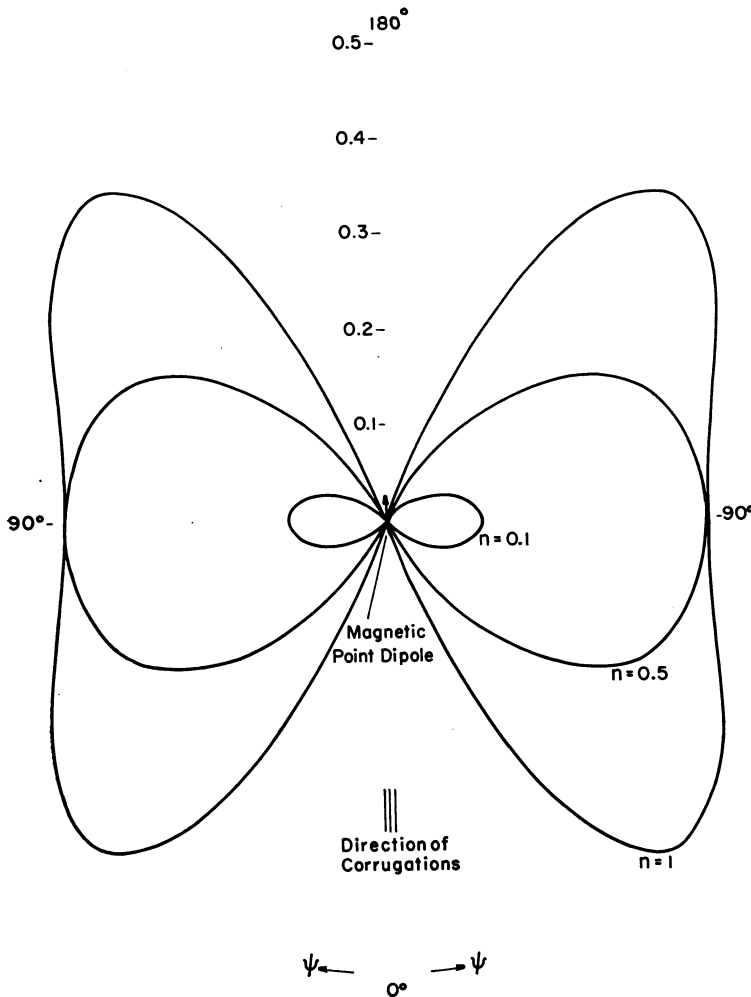

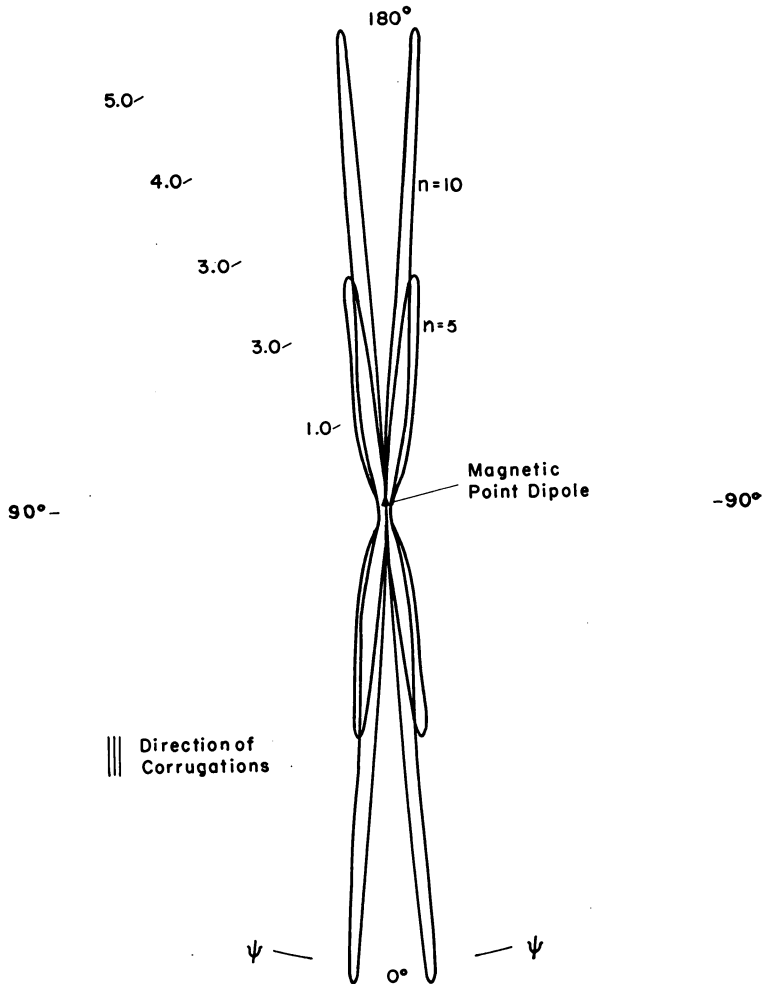Fig. 4a. Surface wave power angular density $\times$ $(8\pi\omega\mu)$.

FIG. 4b. Surface wave power angular density $\times(8\pi\omega\mu)$.

locally there as an oblique plane surface wave. It is recalled that none of the anisotropic characteristics of the dipole-excited surface wave are present in the two dimensional treatments where an infinite line source is assumed [2, 3] or where the propagation of an oblique plane surface wave is considered [1]. In the case of a finite line source we may expect to observe the two dimensional behavior at points close to the line source; however, the three dimensional behavior should appear when the distance from the source is large compared with the length of the source. There must be a transition between the two dimensional and three dimensional characteristics.

The surface wave phase pattern and the surface wave power pattern have been computed numerically for different values of normalized surface reactance n and $h = 0$. The phase curves using rectangular coordinates are plotted in Fig. 3 for $0 \le \psi \le 90°$. Those corresponding to $90° \le \psi \le 180°$ may be obtained by the symmetry requirement. The phase curve for $n = 0.1$ is very close to unity. The surface wave power patterns using polar coordinates are plotted in Figs. 4a and 4b for a magnetic point dipole of

unit strength. The surface wave power density vanishes at $\psi = 0$ and 180° for all values of n. It is interesting to notice that the surface wave power patterns for $n = 0.1$ and 0.5 are broadly directional with the maximum at $\psi = 90°$, while those for $n = 5$ and 10 are sharply directional with the maximum at an angle close to $\psi = 0°$ and 180°. This observation may lead to the conclusion that the surface wave power pattern of a finite line source resembles that of a continuous array of isotropic point sources for small values of n. However, a similar statement is not valid in the case of large values of n.

If frequency is very low, the limiting form of the wave equation and boundary condition in our problem will be reduced to the same as those in Kelvin's ship wave problem [8, 9]. But we are interested in the fields above the reactive plane, while the ship wave solution deals with the space below the water surface. Although the wave amplitude and phase in both problems depend upon the angle of observation, we find one wave in all directions but vanishing at $\psi = 0°$ and 180°, while the ship wave consists of two orthogonal waves and is confined within a sector behind the ship.

BIBLIOGRAPHY

1. R. W. Hougardy and R. C. Hansen, *Scanning surface wave antennas—oblique surface waves over a corrugated conductor*, Trans. I. R. E. **AP-6**, 370 (1958). Comments by L. O. Goldstone and A. A. Oliner, *ibid.*, **AP-7**, 274 (1959). Comments by R. E. Collins, **AP-7**, 276 (1959).
2. A. L. Cullen, *The excitation of plane surface waves*, Proc. I. E. E. (4) **101**, 225 (1954).
3. B. Friedman and W. E. Williams, *Excitation of surface waves*, Proc. I. E. E. **105C**, 252 (1958).
4. S. N. Karp and F. C. Karal, *Surface waves on a right-angled wedge*, Comm. Pure Appl. Math. **12**, 435 (1959) [see also Wescon 1958].
5. S. N. Karp, *Two-dimensional Green's function for a right-angled wedge under an impedance boundary condition*, Comm. Pure Appl. Math. **13**, 203 (1960).
6. S. N. Karp and F. C. Karal, *A new method for the determination of far fields with application to the problem of radiation of a line source at the tip of an absorbing wedge*, Trans. I. R. E. **AP-7**, S91 (1959).
7. F. C. Karal, S. N. Karp, T. S. Chu and R. G. Kouyoumjian, *Scattering of a surface wave by a discontinuity in the surface reactance on a right-angled wedge*, Comm. Pure Appl. Math. **14**, 35 (1961).
8. A. S. Peters, *A new treatment of the ship wave problem*, Comm. Pure Appl. Math. **2**, 123 (1949).
9. J. K. Lunde, *On the linearized theory of wave resistance for displacement ships in steady and accelerated motion*, A technical report published by the Society of Naval Architects and Marine Engineers in 1951.