—NOTES—

APPLICATION OF ULTRASPHERICAL POLYNOMIALS TO NON-LINEAR OSCILLATIONS—I. FREE OSCILLATION OF THE PENDULUM*

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1. Introduction. In a recent paper [1], Denman has shown that an amplitude-dependent approximation to the frequency of the simple pendulum may be obtained by making a linear Tchebycheff polynomial approximation to $\sin \theta$ in the interval $[-A, A]$, where A is the amplitude of the motion. For this approximation, one finds

$$\sin^* \theta = (2/A)J_1(A)\theta, \tag{1}$$

where * indicates an approximation, and J_1 is the ordinary Bessel function of the first order. The equation of motion for free oscillation of the simple pendulum may be written

$$\frac{d^2\theta}{dt^2} + \omega_o^2 \sin \theta = 0, \tag{2}$$

so that one obtains the approximate solution

$$\theta^* = A \sin (\omega^* t + \phi), \tag{3}$$

where

$$\omega^* = \omega_o[(2/A)J_1(A)]^{1/2} = \omega_o[1 - A^2/16 + \cdots]. \tag{4}$$

The approximate period τ^* is then

$$\tau^* = \tau_o[(2/A)J_1(A)]^{-1/2} = \tau_o(1 + A^2/16 + 10A^4/3072 + \cdots), \tag{5}$$

where $\tau_o = 2\pi/\omega_o$. The exact period τ is given by

$$\tau = 2\tau_o K(k)/\pi = \tau_o(1 + A^2/16 + 11A^4/3072 + \cdots), \tag{6}$$

where $K(k)$ is the complete elliptic integral of the first kind, and $k = \sin (A/2)$.

The above work was motivated by the consideration that, in this approach, one seeks a linear approximation to $\sin \theta$ which gives the greatest accuracy, in some sense, over the *interval* $[-A, A]$. This led to the examination of approximations by orthogonal polynomials. The Tchebycheff polynomial approximation was used because it yields nearly the minimum maximum absolute error over the interval for a given degree polynomial [2]. The question arises whether these results for τ^* might be improved by using some other set of orthogonal polynomials. In this paper, the ultraspherical polynomials, which include the Legendre polynomials, the Tchebycheff polynomials of the first and second kind, and the Taylor series as special cases, will be applied to this problem.

2. Ultraspherical polynomials. The ultraspherical polynomials $P_n^{(\lambda)}(x)$ on the interval $[-1, 1]$ are the sets of polynomials orthogonal on this interval with respect to the weight factor $(1 - x^2)^{\lambda-1/2}$, each set corresponding to a value of $\lambda > -\frac{1}{2}$. They may be obtained from [3]

$$P_n^{(\lambda)}(x) = A_n^{(\lambda)}(1 - x^2)^{-\lambda+1/2}(d/dx)^n(1 - x^2)^{n+\lambda-1/2}, \tag{7}$$

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where $A_n^{(\lambda)}$ is a normalization factor given by

$$A_n^{(\lambda)} = \frac{(-1)^n\Gamma(\lambda + \frac{1}{2})\Gamma(n + 2\lambda)}{2^n n!\Gamma(2\lambda)\Gamma(n + \lambda + \frac{1}{2})}. \tag{8}$$

Then $P_1^{(\lambda)}(x) = 2\lambda x$. This normalization gives $P_n^{(0)}(x) = 0$ for $n \geq 1$. However, one may obtain the Tchebycheff polynomials of the first kind from (7) for $\lambda = 0$ if the normalization factor (8) is redefined for this case as

$$A_n^{(0)} = (-1)^n 2^n n!/(2n)! \tag{8'}$$

One then obtains

$$P_n^{(0)}(x) = T_n(x) = \cos(n \cos^{-1} x), \tag{9}$$

where T_n is the Tchebycheff polynomial of the first kind. Other sets of ultraspherical polynomials are the Legendre polynomials, for which $\lambda = \frac{1}{2}$, and the Tchebycheff polynomials of the second kind, for which $\lambda = 1$. The Taylor series expansion of an analytic function about the origin corresponds to its expansion in the ultraspherical polynomials for which $\lambda \to \infty$ [2].

The ultraspherical polynomials on the interval $[-A, A]$ are defined as the sets of polynomials orthogonal on this interval with respect to the weight factor $(1 - x^2/A^2)^{\lambda-1/2}$, where $\lambda > -\frac{1}{2}$. The normalization is chosen to give the polynomials $P_n^{(\lambda)}(x/A)$.

3. Application of the ultraspherical polynomials to the pendulum equation. Approximating $\sin\theta$ on $[-A, A]$ with the ultraspherical polynomial linear in θ, one obtains

$$\sin^*\theta = a_1^{(\lambda)}P_1^{(\lambda)}(\theta/A), \tag{10}$$

where

$$a_1^{(\lambda)} = \frac{\int_{-A}^{A}(1 - \theta^2/A^2)^{\lambda-1/2}P_1^{(\lambda)}(\theta/A)\sin\theta\, d\theta}{\int_{-A}^{A}[P_1^{(\lambda)}(\theta/A)]^2(1 - \theta^2/A^2)^{\lambda-1/2}\, d\theta}. \tag{11}$$

Since $P_1^{(\lambda)}(\theta/A) = 2\lambda\theta/A$, and using the transformation $\phi = \cos^{-1}(\theta/A)$, (11) becomes

$$a_1^{(\lambda)} = \frac{\int_0^{\pi}\sin(A\cos\phi)\sin^{2\lambda}\phi\cos\phi\, d\phi}{2\lambda\int_0^{\pi}\cos^2\phi\sin^{2\lambda}\phi\, d\phi}. \tag{12}$$

After some manipulation, one finds

$$a_1^{(\lambda)} = \Gamma(\lambda + 2)J_{\lambda+1}(A)/\lambda(A/2)^{\lambda},$$

so that (10) becomes

$$\sin^*\theta = \frac{\Gamma(\lambda + 2)J_{\lambda+1}(A)}{(A/2)^{\lambda+1}}\theta = \Lambda_{\lambda+1}(A)\theta, \tag{13}$$

where $\Lambda_\lambda(A) = \Gamma(\lambda + 1)J_\lambda(A)/(A/2)^\lambda$. The function Λ_λ is plotted for various integral values of λ in [4]. Equation (13) holds for all values of $\lambda > -\frac{1}{2}$, including 0.

Inserting this approximation in the equation of motion (2), one obtains an approximate solution of the form (3), with the approximate period

$$\tau_\lambda^*(A) = \tau_o[(A/2)^{\lambda+1}/\Gamma(\lambda + 2)J_{\lambda+1}(A)]^{1/2} = \tau_o\Lambda_{\lambda+1}^{-1/2}(A). \tag{14}$$

From the power series representation for $\Lambda_\lambda(A)$, (14) can be written

$$\frac{\tau_\lambda^*}{\tau_o} = \left[1 - \frac{(A/2)^2}{\lambda + 2} + \frac{(A/2)^4}{2!(\lambda + 2)(\lambda + 3)} - \cdots\right]^{-1/2} \tag{15a}$$

$$= 1 + \frac{(A/2)^2}{2(\lambda + 2)} + \frac{(\lambda + 5)(A/2)^4}{8(\lambda + 2)^2(\lambda + 3)} + \cdots . \tag{15b}$$

For $\lambda > -\frac{1}{2}$ and A in the closed interval $[0, \pi]$, τ_λ^*/τ_o is real, finite, and positive. It approaches 1 as A approaches 0 for $\lambda > -\frac{1}{2}$, and becomes infinite as λ approaches $-\frac{1}{2}$ for $A = \pi$ (since $\Lambda_{1/2}(A) = \sin A/A$). Also, it can be shown that

$$d\tau_\lambda^*/dA = \tau_\lambda^* A\Lambda_{\lambda+2}(A)/4(\lambda + 2)\Lambda_{\lambda+1}(A),$$

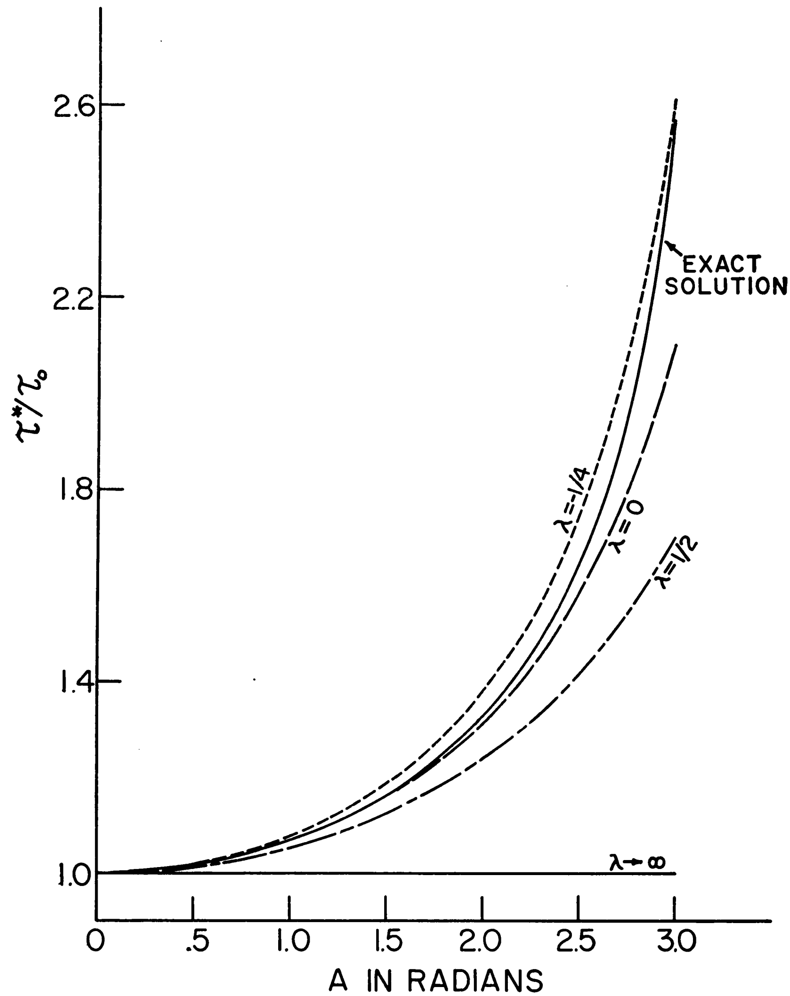

FIG. 1. Dimensionless period as a function of amplitude A for the exact solution and for $\lambda = -\frac{1}{4}$, 0, $\frac{1}{2}$, and ∞

which is positive for $\lambda > -\frac{1}{2}$ and A in $[0, \pi]$, so that τ_λ^* is monotone increasing with A in this region. It can be further shown that for A fixed in $[0, \pi]$, τ_λ^* increases as λ decreases from ∞ toward $-\frac{1}{2}$. For $\lambda < 0$, comparison of (6) and (15b) indicates that τ^* is greater than the exact period for sufficiently small A, but at $A = \pi$, τ is infinite while τ_λ^* is finite. Thus the approximate period must equal the exact period for at least one intermediate value of A for $-1/2 < \lambda < 0$.

4. Numerical results. In Fig. 1, τ/τ_0 is plotted as a function of A for the exact solution (6), and τ^*/τ_0 for λ values $\frac{1}{2}$, 0, $-\frac{1}{4}$ and $\lambda \to \infty$. Since τ is infinite at $A = \pi$, and no oscillatory solution exists for $A > \pi$, an upper bound on A was chosen *arbitrarily* at 3 radians. For $\lambda = 0$ (Tchebycheff approximation), (14) becomes

$$\tau^*/\tau_0 = [A/2J_1(A)]^{1/2},$$

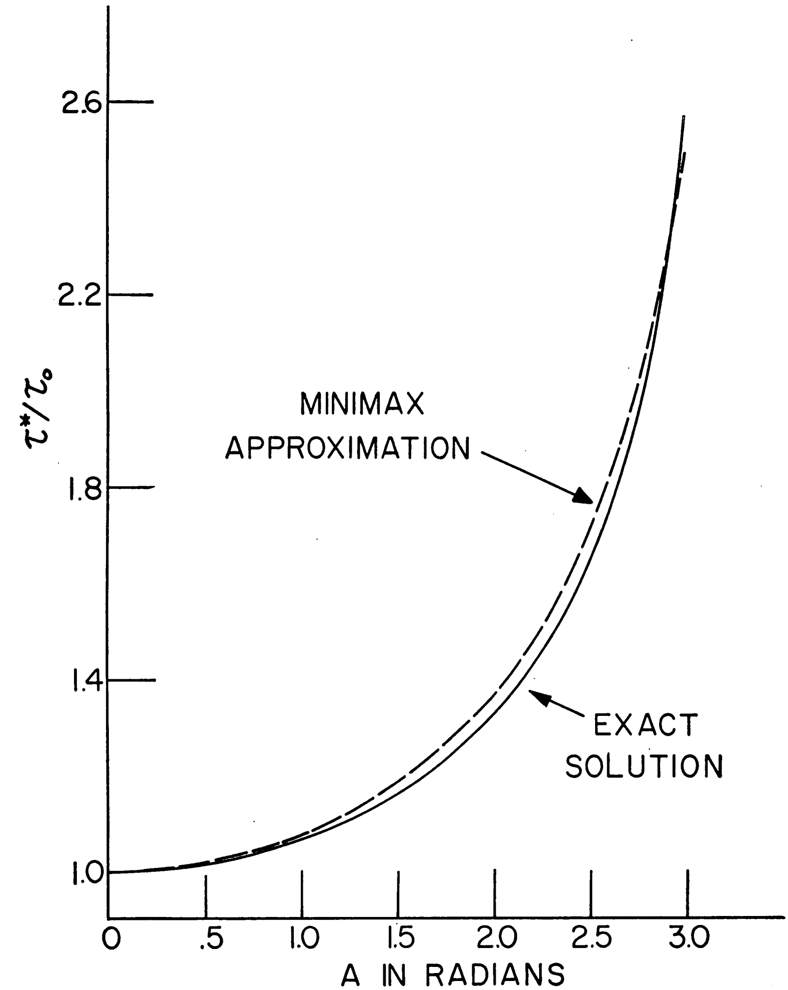

FIG. 2. Dimensionless period as a function of amplitude A for the exact solution and for $\lambda = -0.21$ (minimax approximation).

and tabulated values [5] of J_1 were used. For $\lambda = \frac{1}{2}$ (Legendre approximation), (14) becomes

$$\tau^*/\tau_o = (2/9\pi)^{1/4}[A^{3/2}/J_{3/2}(A)]^{1/2},$$

and tabulated values [6] of the spherical Bessel function $j_1(A)$ were used. As $\lambda \to \infty$, the Taylor series approximation $\sin^* \theta = \theta$ is obtained, which, in (2), yields $\tau^* = \tau_o$. This is verified by taking the limit as $\lambda \to \infty$ in (15b). For $\lambda = -\frac{1}{4}$, (14) gives

$$\tau^*/\tau_o = [(A/2)^{3/4}/\Gamma(7/4)J_{3/4}(A)]^{1/2},$$

and tabulated values [7] of $J_{3/4}$ were used. For $\lambda = -0.21$, (15a) was used, and the result plotted with τ/τ_o in Fig. 2. This value of λ was chosen for the reasons indicated below. In Fig. 3, dimensionless error curves were plotted for the λ values given above, where this error is defined as $(\tau^* - \tau)/\tau_o$.

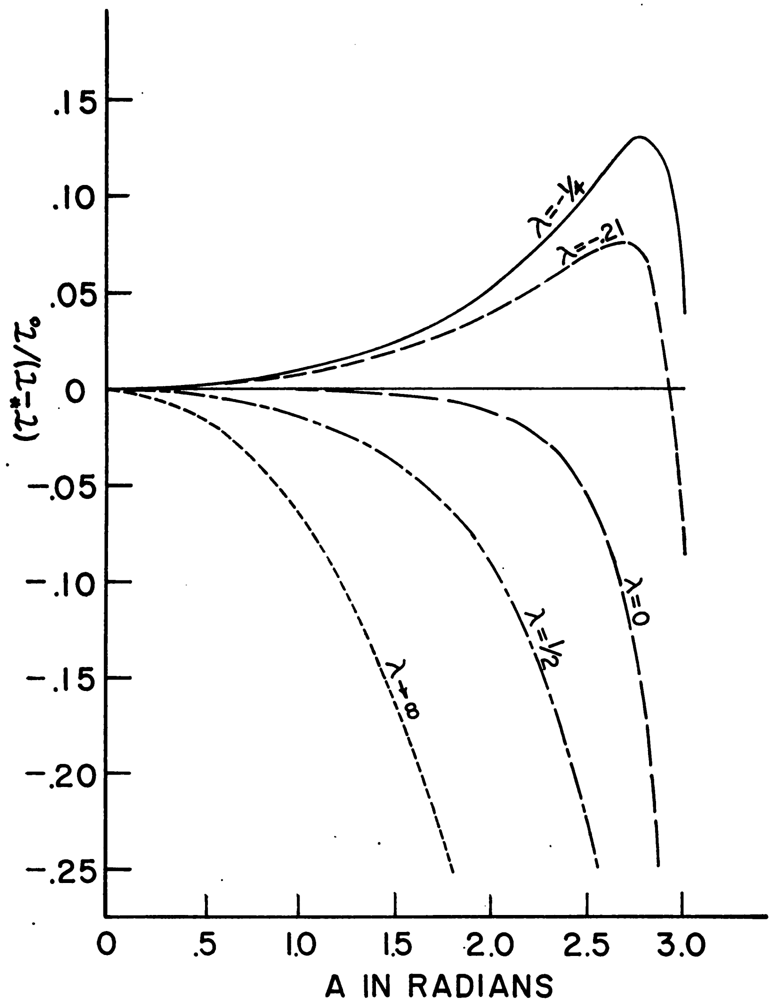

FIG. 3. Dimensionless error curves.

5. Discussion. As indicated in Figs. 1 and 3, the errors in the approximate periods increase monotonically with A for all *positive* values of λ. Of all values of $\lambda \geq 0$, the value 0, corresponding to the linear Tchebycheff polynomial (first kind) approximation, gives the smallest errors, while $\lambda \to \infty$, corresponding to the Taylor series approximation, gives the largest errors. Since all values $\tau_\lambda^*(\pi)$ are finite, the error curves become infinite at this value. For $0 > \lambda > -\frac{1}{2}$, it was shown that $\tau^* = \tau$ at some value of A between 0 and π; this was illustrated in Fig. 2 for $\lambda = -0.21$. However, for fixed values of $\lambda > -\frac{1}{2}$, comparison of (6) and (15b) shows that there exists a range of values of A around 0 for which the Tchebycheff approximation ($\lambda = 0$) gives the best τ_λ^*.

If a maximum value of $A < \pi$ is chosen, one can find a value of $\lambda < 0$ such that the maximum error in τ_λ^* is minimized in this range. Thus, for $0 < A \leq 3$ radians, the value of λ corresponding to the minimum maximum (minimax) error was found by trial and error to be approximately -0.21. The corresponding curve for τ^*/τ_o is plotted in Fig. 2, and the error curve in Fig. 3.

6. Conclusions. In this paper the linearization of the non-linear ordinary differential equation governing the free oscillation of a pendulum has been accomplished by approximation of the non-linear restoring torque with a linear ultraspherical polynomial. This yields an expression for the approximate period which is dependent on the amplitude of the oscillation. Graphical comparison with the exact period indicates that for all *positive* values of the parameter λ, the best results are obtained for the choice $\lambda = 0$, which corresponds to the linear Tchebycheff polynomial approximation. If negative values of $\lambda > -\frac{1}{2}$ are considered, the approximate period equals the exact period for some value of A between 0 and π.

In future papers, the treatment of other non-linear oscillatory systems by this technique will be considered.

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