AN APPLICATION OF THE DYNAMIC BETTI-RAYLEIGH RECIPROCAL THEOREM TO MOVING-POINT LOADS IN ELASTIC MEDIA*

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Abstract. Two problems connected with the transient motion of an elastic body acted upon by a moving-point force are solved by an application of the dynamic Betti-Rayleigh reciprocal theorem. This theorem, which is the analogue of Green's theorem for the scalar wave equation, permits the solution to be written as a single expression, irrespective of the value of the (constant) moving-force velocity ν. In particular, the displacement field in an infinite elastic body, due to a transient-point body force moving in a straight line, is given in a simple form. Next the surface motion of an elastic half-space acted upon by a transient pressure spot moving in a straight line is analyzed for a material for which Poisson's ratio is one-fourth. The normal displacement is expressed in a simple manner, but the tangential displacement is quite complicated and is not fully expressible in terms of elementary functions. Singularities of the displacement fields are identified and discussed.

1. Introduction. In recent years, several authors have considered the motion of an elastic solid resulting from a moving load. In most instances, the problems encountered in these studies have been treated by the use of integral transforms, necessitating the separate consideration of the various forms of solutions that arise when the velocity of the moving load ν is greater or less than one of the characteristic velocities of the medium. In this paper, a dynamic version of the reciprocal theorem of Betti and Rayleigh [1]† is employed to treat the problem of moving loads in a general manner for all constant values of ν in the range $0 \leq \nu < +\infty$.

Two specific problems are considered in detail. Firstly, the motion of an infinite elastic body resulting from a moving-point force acting in a direction along the line of motion is presented in Sect. 3. Secondly, the surface motion of an elastic half-space caused by a moving normal-point surface load, is determined in Sect. 4. In both cases, the loads are suddenly applied at the origin at the time $t = 0$, and then move rectilinearly with a constant velocity ν. These problems were solved without any particular application in mind; the half-space problem may be generated by a cyclone or (using superposition) a rolling freight train.

Eason, Fulton, and Sneddon [2] have considered the moving load in an infinite elastic medium for the case of "uniform motion." Thus, if the load moves in the direction x with velocity ν, the Galilean transformation $\xi = x - \nu t$, $y = y$, $z = z$ reduces the number of independent variables from four to three. Mandel and Avramesco [3] use the same method for the half-space problem. These uniform-motion solutions necessarily neglect the transient phase of motion and are only valid long after the load has been applied. The integral-transform methods used by the aforementioned authors will certainly work for the moving load in an infinite elastic body, but how to apply transforms in

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†A discussion of the static case is given in this reference.

the (transient) half-space problem is not clear on account of the asymmetry introduced by the moving load.

Considerably more work has been done on moving-line loads. In particular, Ang [4] treats the transient half-space problem and discusses the uniform-motion version of this problem. Papadopoulos has also worked on related moving-load problems (private communication).

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2. The dynamic Betti-Rayleigh theorem. For a given linear, homogeneous, isotropic elastic body with volume V, surface S, and prescribed surface traction $\mathbf{T}(\mathbf{x}^s, t)$, body force $\mathbf{F}(\mathbf{x}, t)$, initial displacement $\mathbf{u}^0(\mathbf{x})$, and initial velocity $\mathbf{u}_t^0(\mathbf{x})$, there will be a displacement field $\mathbf{u}(\mathbf{x}, t)$ uniquely determined by a solution of the Navier equation,

$$c_2^2 \nabla^2 \mathbf{u} + (c_1^2 - c_2^2) \nabla (\nabla \cdot \mathbf{u}) = \mathbf{u}_{tt} - \frac{1}{\rho} \mathbf{F}.$$

Here c_1 and c_2 are the speeds of propagation of dilatational and equivoluminal waves. Consider now another displacement field $\mathbf{u}'(\mathbf{x}, t)$, corresponding to a primed set of forces and initial conditions which also satisfy the equations of elasticity. The dynamic Betti-Rayleigh theorem then states that,

$$\begin{aligned}
&\int_0^t \int_V \mathbf{F}(\mathbf{x}, \tau)\cdot\mathbf{u}'(\mathbf{x}, t - \tau)\, dV\, d\tau + \int_V \rho \mathbf{u}_t^0(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}, t)\, dV \\
&+ \int_V \rho \mathbf{u}^0(\mathbf{x})\cdot\mathbf{u}_t'(\mathbf{x}, t)\, dV + \int_0^t \int_S \mathbf{T}(\mathbf{x}^s, \tau)\cdot\mathbf{u}'(\mathbf{x}^s, t - \tau)\, dS\, d\tau \\
&= \int_0^t \int_V \mathbf{F}'(\mathbf{x}, \tau)\cdot\mathbf{u}(\mathbf{x}, t - \tau)\, dV\, d\tau + \int_V \rho \mathbf{u}_t'^0(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}, t)\, dV \\
&+ \int_V \rho \mathbf{u}'^0(\mathbf{x})\cdot\mathbf{u}_t(\mathbf{x}, t)\, dV + \int_0^t \int_S \mathbf{T}'(\mathbf{x}^s, \tau)\cdot\mathbf{u}(\mathbf{x}^s, t - \tau)\, dS\, d\tau. \qquad (1)
\end{aligned}$$

Equation (1) can readily be established by first taking the Laplace transform of the equations of elasticity, applying the static Betti-Rayleigh theorem, and then using the convolution theorem.

DiMaggio and Bleich [5] have used the dynamic reciprocity theorem to determine the vertical displacement in the interior of an elastic half-space produced by a concentrated vertical surface force. These authors also mention earlier references to the theorem. Knopoff and Gangi [6] give experimental evidence supporting the theorem and also discuss its derivation. Neither of these references treat displacement fields with non-vanishing initial values. Morse and Feshbach [7] establish a vector Green's theorem which should be formally equivalent to (1), but the expressions involved are in a form somewhat unfamiliar to the elastician.

In applying Eq. (1) to the problems at hand, the primed forces will be chosen so as to act at a single point $\mathbf{x}_0$, in V or on S. Consequently, $\mathbf{u}'(\mathbf{x}, \mathbf{x}_0, t)$ will play a role analogous to a Green's function for the scalar wave equation.* It should be stated that Eq. (1) remains valid when the surface displacements are prescribed on S instead of the surface tractions.

3. A moving-point force acting in a direction parallel to its line of motion in an infinite elastic body. Before attacking the problem of a moving-point force acting in a direction parallel to its line of motion in an infinite elastic body, the preliminary problem of the displacement field generated by a stationary (but impulsive) point force must be solved. Upon using this solution and the dynamic Betti-Rayleigh theorem, the solution for the moving force will follow directly by integration. This will be made clear below.

Displacement field produced by a stationary-point force. The preliminary problem to be investigated is,

$$c_2^2 \nabla^2 \mathbf{u}' + (c_1^2 - c_2^2) \nabla (\nabla \cdot \mathbf{u}') = \mathbf{u}'_{tt} - \mathbf{a}_x \delta(\mathbf{x} - \mathbf{x}_0) \delta(t) \tag{2}$$

with the initial conditions $\mathbf{u}'(\mathbf{x}, \mathbf{x}_0, 0) = \mathbf{u}'_t(\mathbf{x}, \mathbf{x}_0, 0) = 0$ in the infinite domain $-\infty < x, y, z < +\infty$. Here, $\mathbf{a}_x$ denotes a unit vector pointing in the positive x direction. The displacement vector $\mathbf{u}'(\mathbf{x}, \mathbf{x}_0, t)$ determined by Eq. (2) is the displacement at a point $\mathbf{x}$ and time t due to a point impulse acting in the x direction at point $\mathbf{x}_0$. This solution can be obtained in a straightforward way. Consequently only the sequence of steps involved, will be listed here. First, move the origin of the $\mathbf{x}$ coordinate system so that it coincides with point $\mathbf{x}_0$. Then change to cylindrical polar coordinates with $r = (y^2 + z^2)^{1/2}$. Next, decompose the nonhomogeneous part of Eq. (2) so that

$$\mathbf{a}_x \frac{\delta(r)}{r} \delta(x) \delta(t) = (\nabla \phi_0 + \nabla \times \mathbf{A}_0) \delta(t).$$

This decomposition can always be accomplished, e.g., by the formulas given in Sneddon and Berry [8]. The displacement vector $\mathbf{u}'$ is next written in the form $\mathbf{u}' = \nabla \phi + \nabla \times \mathbf{A}$ where, because of symmetry, $\mathbf{A} = \mathbf{a}_\theta A$, $\mathbf{a}_\theta$ being a unit vector in the direction of increasing θ. Equation (2) then separates into two scalar equations which can best be handled by applying Laplace and Hankel transforms. The resulting solution is (after moving the coordinate system back to its initial origin),

$$u^{(1)\prime}(\mathbf{x}, \mathbf{x}_0, t) = \frac{1}{4\pi} \frac{t}{R_0^2} G(x, x_0, R_0, t), \quad v^{(1)\prime}(\mathbf{x}, \mathbf{x}_0, t) = \frac{1}{4\pi} \frac{(y - y_0)(x - x_0)t}{R_0^4} F(R_0, t),$$

$$w^{(1)\prime}(\mathbf{x}, \mathbf{x}_0, t) = \frac{1}{4\pi} \frac{(z - z_0)(x - x_0)t}{R_0^4} F(R_0, t), \tag{3}$$

where

$$F(R_0, t) = \left[\frac{3}{R_0} H(t - R_0/c_1) + \frac{1}{c_1} \delta(t - R_0/c_1) - \frac{3}{R_0} H(t - R_0/c_2) - \frac{1}{c_2} \delta(t - R_0/c_2) \right],$$

$$G(x, x_0, R_0, t) = \left[\left\{ \frac{3(x - x_0)^2}{R_0^2} - 1 \right\} \frac{1}{R_0} H(t - R_0/c_1) + \frac{(x - x_0)^2}{c_1 R_0^2} \delta(t - R_0/c_1) \right.$$

$$\left. - \left\{ \frac{3(x - x_0)^2}{R_0^2} - 1 \right\} \frac{1}{R_0} H(t - R_0/c_2) - \left\{ \frac{(x - x_0)^2}{R_0^2} - 1 \right\} \frac{1}{c_2} \delta(t - R_0/c_2) \right],$$

$$R_0 = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

*The notation used here is that the vector $\mathbf{u}'$ with Cartesian components u', v', and w' is a function of the independent variables x, y, z, and t and also depends on the parameters x_0, y_0, and z_0.

Here H stands for the Heaviside step function and δ for the Dirac delta function. It should be noted that $\mathbf{u}^{(1)\prime}(\mathbf{x}, \mathbf{x}_0, t) = \mathbf{u}^{(1)\prime}(\mathbf{x}_0, \mathbf{x}, t)$, i.e., reciprocity holds in an infinite medium as expected. In Eq. (3), the superscript (1) is to denote the fact that the displacements are due to a point impulse acting in the x direction. The corresponding solutions $\mathbf{u}^{(2)\prime}(\mathbf{x}, \mathbf{x}_0, t)$ and $\mathbf{u}^{(3)\prime}(\mathbf{x}, \mathbf{x}_0, t)$ when the point impulse acts in the y and z directions respectively will not be written out explicitly since these expressions follow directly from Eq. (3) by simply relabeling the axes.

Displacement field produced by a moving-point force. Let a point force be suddenly applied at the origin at time $t = 0$ and then maintained at a constant velocity ν along the positive z axis. No restriction is placed on v other than $\nu \geq 0$. Of course, the expressions for the displacements will differ according to the relation of ν to c_1 and c_2. The problem can now be concisely written as,

$$c_2^2\nabla^2\mathbf{u} + (c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{u}_{tt} - \mathbf{a}_z Q_0 \delta(x)\delta(y)\delta(z - \nu t) \tag{4}$$

with the initial conditions $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_t(\mathbf{x}, 0) = 0$. Here, Q_0 is a constant measuring the strength of the moving force. Equations (4) will now be solved by applying the dynamic Betti-Rayleigh theorem, Eq. (1). Since the initial conditions on $\mathbf{u}$ are zero, the initial conditions for the $\mathbf{u}'$ displacements are chosen to be zero. Also, because the domain is infinite, the integrals containing the surface traction terms are omitted. Equation (1) then reduces as follows,

$$\int_0^t \int_{V_0} \mathbf{F}(\mathbf{x}_0, \tau)\cdot\mathbf{u}'(\mathbf{x}_0, \mathbf{x}, t - \tau)\, dV_0\, d\tau = \int_0^t \int_{V_0} \mathbf{F}'(\mathbf{x}_0, \mathbf{x}, \tau)\cdot\mathbf{u}(\mathbf{x}_0, t - \tau)\, dV_0\, d\tau \tag{5}$$

where the body forces $\mathbf{F}(\mathbf{x}, t)$ and $\mathbf{F}'(\mathbf{x}, t)$ are given by

$$\begin{aligned} \mathbf{F}(\mathbf{x}, t) &= \mathbf{a}_z \rho Q_0 \delta(x)\delta(y)\delta(z - \nu t), \\ \mathbf{F}'(\mathbf{x}, \mathbf{x}_0, t) &= \mathbf{a}_x \rho\, \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t). \end{aligned} \tag{6}$$

Substituting (6) into (5) and performing the volume integration, one finds

$$u = \int_0^t Q_0 w^{(1)\prime}(0, 0, \nu\tau; x, y, z; t - \tau)\, d\tau.$$

By employing $\mathbf{F}' = \mathbf{a}_y\rho\ \delta(\mathbf{x} - \mathbf{x}_0)\ \delta(t)$ and $\mathbf{F}' = \mathbf{a}_z\rho\ \delta(\mathbf{x} - \mathbf{x}_0)\ \delta(t)$, the v and w components of the displacement vector satisfying Eq. (4) are similarly determined. With the use of (3), the three displacements are found as

$$\begin{aligned} u(x, y, z, t) &= \frac{Q_0}{4\pi}\int_0^t \frac{x(z - \nu\tau)(t - \tau)}{R^4(\tau)} F[R(\tau), t - \tau]\, d\tau, \\ v(x, y, z, t) &= \frac{Q_0}{4\pi}\int_0^t \frac{y(z - \nu\tau)(t - \tau)}{R^4(\tau)} F[R(\tau), t - \tau]\, d\tau, \\ w(x, y, z, t) &= \frac{Q_0}{4\pi}\int_0^t \frac{(t - \tau)}{R^2(\tau)} G[z, \nu\tau, R(\tau), t - \tau]\, d\tau \end{aligned} \tag{7}$$

with

$$R(\tau) = [x^2 + y^2 + (z - \nu\tau)^2]^{1/2}.$$

Substitution of the expressions for $F[R(\tau), t - \tau]$ and $G[z, \nu\tau, R(\tau), t - \tau]$ into Eq. (7) leads to integrals of the form

$$\int_0^t g(\tau)H[f(\tau)]\,d\tau \quad\text{and}\quad \int_0^t h(\tau)\delta[f(\tau)]\,d\tau,$$

where $f(\tau) = t - \tau - R(\tau)/c$, the symbol c playing the role of either c_1 or c_2. In both of these integrals the integrations are elementary; however care must be exercised in the examination of $f(\tau)$ since its zeros determine the values of both integrals. It is shown in the appendix that $f(\tau)$ may have zero, one, or two roots in the range $0 \leq \tau \leq t$, depending upon the relative values of x, y, z, t, c and ν. The final displacements (in polar cylindrical coordinates (r, θ, z)) are,

$$\begin{aligned} u_r &= \frac{Q_0}{4\pi\nu^2}\left[-\frac{(z-\nu t)}{rR_{c_1}} + \frac{r^2(z-\nu t)+z^3}{rR^3}\right]H(t - R/c_1) + \frac{Q_0}{4\pi\nu^2}\left[-\frac{2(z-\nu t)}{rR_{c_1}}\right]S(C_1) \\ &\quad - \frac{Q_0}{4\pi\nu^2}\left[-\frac{(z-\nu t)}{rR_{c_2}} + \frac{r^2(z-\nu t)+z^3}{rR^3}\right]H(t - R/c_2) - \frac{Q_0}{4\pi\nu^2}\left[-\frac{2(z-\nu t)}{rR_{c_2}}\right]S(C_2), \\ u_\theta &= 0, \\ u_z &= \frac{Q_0}{4\pi\nu^2}\left[\frac{1}{R_{c_1}} - \frac{R^2+z^2-z(z-\nu t)}{R^3}\right]H(t - R/c_1) + \frac{Q_0}{4\pi\nu^2}\left[\frac{2}{R_{c_1}}\right]S(C_1) \\ &\quad - \frac{Q_0}{4\pi\nu^2}\left[\frac{1-\nu^2/c_2^2}{R_{c_2}} - \frac{R^2+z^2-z(z-\nu t)}{R^3}\right]H(t - R/c_2) - \frac{Q_0}{4\pi\nu^2}\left[\frac{2(1-\nu^2/c_2^2)}{R_{c_2}}\right]S(C_2). \end{aligned} \tag{8}$$

These displacements constitute the complete solution of the equation of linear elasticity for a suddenly applied point force, acting along the z axis whose motion is then maintained at a constant velocity ν along the z axis. Some explanations of the symbols used in Eq. (8) are necessary. For example, $r = (x^2 + y^2)^{1/2} \geq 0$ is the polar coordinate radius measured from the z axis, while $R = (x^2 + y^2 + z^2)^{1/2} \geq 0$ is the spherical coordinate radius measured from the origin. The quantity R_α (where α may be, c_1 or c_2) is defined by $R_\alpha = [(z - \nu t)^2 + (1 - \nu^2/\alpha^2)y^2]^{1/2}$. The term $H(t - R/c_i)$ is the Heaviside unit step function of argument $t - R/c_i$ and signifies that the bracket which it multiplies is included in the solution, if R is inside the sphere of radius $c_i t$, and excluded from the solution, if R is outside this sphere. The term $S(C_i)$ is an abbreviation for a function which has the value 1 inside the conical region

$$R < c_i t,\ t - z/\nu - [(\nu/c_i)^2 - 1]^{1/2}r/\nu > 0, \quad\text{and}\quad z > r[(\nu/c_i)^2 - 1]^{-1/2},$$

and zero outside this region. For example when $c_2 < \nu < c_1$, the function $S(C_2)$ is 1 inside the conical region ABC (Fig. 1) and 0 outside this region. Of course, a conical wave front is present only if the velocity of the moving force is greater than that of the shear-wave velocity c_2. If $\nu > c_1$, then there will be two propagating conical wave fronts. These conical fronts are similar to the familiar Mach cones encountered in high-speed aerodynamics.

Diagrams of the wave-front patterns are shown in Fig. 1 for the three cases $\nu < c_2$, $c_2 < \nu < c_1$, and $\nu > c_1$. The special cases of ν equal to c_1 or c_2 are also included in the solution given by (8). The displacements have singularities and become unbounded when $R_{c_i} = 0$ for $i = 1, 2$. If $0 < \nu \leq c_i$, this singularity will be at the point $z = \nu t$, $r = 0$; that is, at the moving force singularity. However if $\nu > c_i$, then $R_{c_i} = 0$ on the cone C_i, so that the displacements are unbounded everywhere on the conical surfaces.

The terms $[r^2(z - \nu t) + z^3]/rR^3$ and $[R^2 + z^2 - z(z - \nu t)]/R^3$ are, in effect, transient terms which disappear when a solution valid for uniform motion is obtained. The uniform-

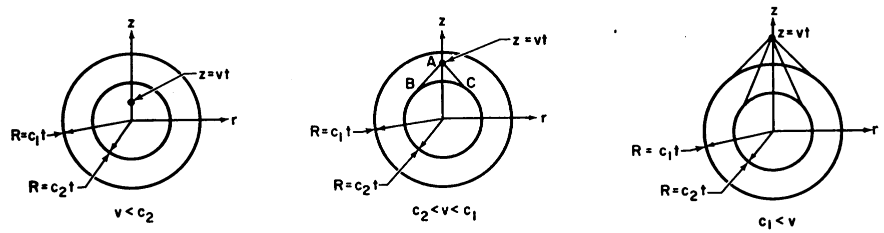

FIG. 1. Wave-front patterns at time t for a moving-point force in an infinite elastic medium (ν = velocity of moving point force, c_2 = velocity of shear waves, and c_1 = velocity of dilatational waves).

motion solution corresponds to applying the point force at time $t = -\infty$ and then considering the resulting displacement field at a much later time. This can be obtained from Eq. (8) by replacing z by $z + vt'$ and t by $t + t'$ and then requiring that $t' \to +\infty$. The solution then agrees with that of Eason, Fulton, and Sneddon [2].

When the moving-point force vector makes an arbitrary (but constant) angle with the line of motion, its effect can be handled by considering separately the displacements from a moving-point force acting in the same direction as the motion and the displacements resulting from a moving source which acts normal to its line of motion. Eason, Fulton, and Sneddon [2] consider both cases for uniform motion. However, since nothing new is involved in these cases, a more interesting problem will next be considered.

4. Surface motion of an elastic half-space produced by a moving pressure spot. In Sect. 3, the response of an infinite elastic space to a moving-point force was determined. The solution to this problem was relatively easy since no boundaries were present. This suggests that the problem with the next order of difficulty will be the response of an elastic half-space to a moving-point force below the surface. Unfortunately, an explicit solution for the stationary buried point-source problem in a half-space cannot be obtained in terms of known functions [9]. However, Pekeris [10] has given an elegant solution for the surface response of an elastic half-space due to a suddenly applied (and stationary) point load. For this reason, only the surface motion of an elastic half-space excited by a moving pressure spot on the surface will be considered herein. As will be made clear below, even this solution becomes rather involved for the inplane displacements.

Surface-displacement field produced by a stationary surface impulse. Before considering the problem posed by the moving pressure spot, the appropriate solution for a stationary load must be determined. In particular, the normal component of the surface displacement vector for a surface traction vector which acts normal to the (otherwise) stress-free surface of the elastic half-space is needed. The normal surface displacement for the moving pressure spot can then be found from (1). To find the tangential components of the surface displacement vector corresponding to a moving pressure spot, the normal surface displacements resulting from a stationary surface traction vector acting in both the x and y directions (where the half-space is represented by $-\infty < x, y < +\infty$ and $z \leq 0$) are needed. These, however, can be determined from the tangential displacements due to a normal load by the application of Eq. (1). For the stationary surface impulse, the preliminary problem is,

$$c_2^2\nabla^2\mathbf{u}' + (c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}') = \mathbf{u}'_{tt} \tag{9}$$

for the half-space $z \leq 0$ with the boundary conditions $\mathbf{T} = \mathbf{a}_z\, \delta(\mathbf{x}^s - \mathbf{x}_0^s)\, \delta(t)$ and zero initial conditions on the displacements and velocities. Pekeris [10] has considered this problem with the exception that his load is applied at the origin ($x = y = 0$) and has a step-function time dependence, whereas a delta function for the time part of the surface load is used here. Hence, after translating the origin of the coordinate system so as to coincide with the load, and differentiating the displacements once with respect to the time variable, the solution of Eq. (9) follows from Pekeris' results.

Let the normal component of the surface displacement vector which satisfies Eq. (9) for a surface traction vector of magnitude $\delta(\mathbf{x}^s - \mathbf{x}_0^s)\, \delta(t)$ acting in the direction x^i (where $x^1 = x$ etc.) be denoted by $w^{(i)\prime}$. Then,

$$w^{(1)\prime}(\mathbf{x}^s, \mathbf{x}_0^s, t) = -\frac{x - x_0}{r_0} A(r_0, t),$$

$$w^{(2)\prime}(\mathbf{x}^s, \mathbf{x}_0^s, t) = -\frac{y - y_0}{r_0} A(r_0, t),$$

$$\begin{aligned} w^{(3)\prime}(\mathbf{x}^s, \mathbf{x}_0^s, t) = -\frac{c_2}{16\mu}\Bigg\{\Bigg[&-\frac{3^{1/2}c_2 t}{(c_2^2t^2 - r_0^2/4)^{3/2}} + \frac{k_1 c_2 t}{(c_2^2t^2 - r_0^2\beta^2)^{3/2}} \\ &+ \frac{k_2 c_2 t}{(\gamma^2 r_0^2 - c_2^2t^2)^{3/2}} H\left(\gamma - \frac{c_2 t}{r_0}\right) - \frac{k_2}{(\gamma^2 r_0^2 - c_2^2 t^2)^{1/2}}\, \delta(\gamma r_0 - c_2 t)\Bigg] H\left(t - \frac{r_0}{c_1}\right) \\ &+ \Bigg[\frac{3^{1/2}c_2 t}{(c_2^2t^2 - r_0^2/4)^{3/2}} - \frac{k_1 c_2 t}{(c_2^2t^2 - r_0^2\beta^2)^{3/2}} + \frac{k_2 c_2 t}{(\gamma^2 r_0^2 - c_2^2t^2)^{3/2}} H\left(\gamma - \frac{c_2 t}{r_0}\right) \\ &- \frac{k_2}{(\gamma^2 r_0^2 - c_2^2 t^2)^{1/2}}\, \delta(\gamma r_0 - c_2 t)\Bigg] H\left(t - \frac{r_0}{c_2}\right)\Bigg\}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} A(r_0, t) = -\frac{c_2}{\mu r_0^2}\Bigg[&\frac{1}{\pi}\frac{\partial}{\partial \xi}(\xi B_1(\xi))H(\xi - 3^{-1/2})H(1 - \xi) - \frac{1}{\pi}\xi B_1(\xi) H(\xi - 3^{1/2})\delta(\xi - 1) \\ &+ \frac{1}{\pi}\frac{\partial}{\partial \xi}(\xi B_2(\xi))H(\xi - 1) + \frac{1}{\pi}\xi B_2(\xi)\delta(\xi - 1) - \frac{1}{4}\frac{\partial}{\partial \xi}\left(\frac{\xi}{(\xi^2 - \gamma^2)^{1/2}}\right) H(\xi - \gamma) \\ &- \frac{1}{4}\frac{\xi\delta(\xi - \gamma)}{(\xi^2 - \gamma^2)^{1/2}}\Bigg], \end{aligned}$$

$$r_0 = [(x - x_0)^2 + (y - y_0)^2]^{1/2}, \qquad \xi = \frac{c_2 t}{r_0}, \qquad \gamma^2 = \frac{3 + 3^{1/2}}{4}, \qquad \beta^2 = \frac{3 - 3^{1/2}}{4},$$

$$k_1 = (3^{3/2} - 5)^{1/2}, \qquad k_2 = (3^{3/2} + 5)^{1/2},$$

and μ is one of the Lamé coefficients. Pekeris expresses $B_1(\xi)$ and $B_2(\xi)$ in terms of elliptic integrals as follows,

$$\begin{aligned} B_1 = -\frac{(3/2)^{1/2}}{8}\{6K(k) - 18\Pi(8k^2, k) &+ (6 - 4(3)^{1/2})\Pi[-(12(3)^{1/2} - 20)k^2, k] \\ &+ (6 + 4(3)^{1/2})\Pi[(12(3)^{1/2} + 20)k^2, k]\}, \end{aligned}$$

and

$$B_2(\xi) = -\frac{1}{4(\xi^2 - 1/3)^{1/2}}\{3K(k^{-1}) - 9\Pi(8, k^{-1}) - (2(3)^{1/2} - 3)\Pi[-(12(3)^{1/2} - 20), k^{-1}]$$
$$+ (2(3)^{1/2} + 3)\Pi[(12(3)^{1/2} + 20), k^{-1}]\},$$

where

$$k^2 = \tfrac{1}{2}(3\xi^2 - 1).$$

K and Π are defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{1/2}}, \qquad \Pi(n, k) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2 \theta)(1 - k^2 \sin^2 \theta)^{1/2}}.$$

To simplify the algebra resulting from a solution of the cubic equation for the Rayleigh wave velocity, the usual convention of assuming a value of one-fourth for Poisson's ratio has been followed. In this case, the dilatational wave velocity c_1 is $(3)^{1/2}$ times that of the shear wave velocity c_2, although for clarity both velocities have been carried along as separate constants when possible. The numerical factors $\frac{1}{4}$, β^2, and γ^2 are related to the algebraic equation for the Rayleigh wave velocity. In fact, the Rayleigh wave velocity is $c_R = c_2/\gamma$.

Displacement field produced by a moving pressure spot. A normal surface traction is suddenly applied to the elastic half-space $z < 0$ and thereafter maintained at a constant velocity ν along the positive x-axis. The displacement vector $\mathbf{u}$ satisfies

$$c_2^2\nabla^2\mathbf{u} + (c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{u}_{tt} \tag{11}$$

with the boundary conditions $\mathbf{T} = \mathbf{a}_z Q_0\, \delta(x - \nu t)\, \delta(y)$ and the initial conditions $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_t(\mathbf{x}, 0) = 0$. This problem differs from the one considered in Sect. 3 in that now there is a surface force and no (applied) body force, whereas in the above problem the reverse was true. Application of the dynamic Betti-Rayleigh theorem, Eq. (1), along with Eqs. (10) and (11), gives the following surface displacements resulting from the moving pressure spot,

$$u = \int_0^t Q_0 w^{(1)\prime}(\nu\tau, 0; x, y; t - \tau)\, d\tau, \qquad v = \int_0^t Q_0 w^{(2)\prime}(\nu\tau, 0; x, y; t - \tau)\, d\tau,$$
$$w = \int_0^t Q_0 w^{(3)\prime}(\nu\tau, 0; x, y; t - \tau)\, d\tau. \tag{12}$$

Note that r_0 is now replaced by $r(\tau) = [(x - \nu\tau)^2 + y^2]^{1/2}$ in the integrals in Eq. (12). These integrals are more complicated than those encountered in Eq. (7). Again omitting the details of integration, the u surface displacement is

$$u(x, y, t) = \frac{Q_0}{\pi\mu}\nu y^2[E(\tau_+^{(1)}) - E(\tau_+^{(2)})]H\left(t - \frac{r}{c_2}\right) + \frac{Q_0}{\pi\mu}\nu y^2[-E(\tau_+^{(2)}) + E(\tau_-^{(2)})]S(\Delta_2)$$
$$+ \left[\frac{Q_0}{\pi\mu}\frac{(3/2)^{1/2}}{8}\frac{x}{N}\frac{c_2 t}{r}\left\{6K\left(\frac{1}{\kappa}\right) - 18\Pi\left(\frac{8}{\kappa^2}, \frac{1}{\kappa}\right)\right.\right.$$
$$\left.+ (6 - 4(3)^{1/2})\Pi\left(-\frac{12(3)^{1/2} - 20}{\kappa^2}, \frac{1}{\kappa}\right) + (6 + 4(3)^{1/2})\Pi\left(\frac{12(3)^{1/2} + 20}{\kappa^2}, \frac{1}{\kappa}\right)\right\}$$

$$+ \frac{Q_0}{\pi\mu}\nu y^2\{E(\tau_+^{(1)}) - E(0)\}\Big]H\Big(\frac{r}{c_2} - t\Big)H\Big(t - \frac{r}{c_1}\Big)$$

$$+ \frac{Q_0}{\pi\mu}\nu y^2[E(\tau_+^{(1)}) - E(\tau_-^{(1)})]S(\Delta_1) + \Big[\Big[\frac{Q_0}{4\pi\nu\mu}\frac{1}{(t^2 - r^2/c_1^2)^{1/2}}\Big[-3K(\kappa)$$

$$+ \frac{N}{r_{c_1}^2}\Big\{3 - 3\frac{y^2}{r_{2c_2}^2}\frac{\nu^2}{(2c_2)^2} + \frac{1}{3}\frac{y^2}{r_{cR}^2}\frac{\nu^2}{c_R^2} + \frac{1}{3}\frac{y^2}{r_{c_2/\beta}^2}\frac{\nu^2}{(c_{2/\beta})^2}\Big\}\Pi(-b^2, \kappa)$$

$$+ 9\Big(1 - \frac{N}{r_{2c_2}^2}\Big)\Pi(8, \kappa) + (2(3)^{1/2} - 3)\Big(1 - \frac{N}{r_{cR}^2}\Big)\Pi(-12(3)^{1/2} + 20, \kappa)$$

$$-(2(3)^{1/2} + 3)\Big(1 - \frac{N}{r_{c_2/\beta}^2}\Big)\Pi(12(3)^{1/2} + 20, \kappa)\Big]$$

$$- \frac{Q_0}{\pi\mu}\frac{c_2}{\nu}\frac{1}{r_{c_2}}(2/3)^{1/2}\Big[\Big(-3\frac{r_{c_2}^2}{r_{2c_2}^2} + \frac{3 - 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{cR}^2} + \frac{3 + 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{c_2/\beta}^2}\Big)$$

$$\cdot\Big\{1 - \tfrac{1}{2}(3/2)^{1/2}\frac{c_2}{\nu}\frac{r_{c_1}}{|y|}\log\frac{((3/2)^{1/2}c_2r_{c_1}/\nu\,|y| + 1)}{((3/2)^{1/2}c_2r_{c_1}/\nu\,|y| - 1)}\Big\}$$

$$-3\Big(1 - \frac{r_{c_2}^2}{r_{2c_2}^2}\Big)\{1 - 2^{-3/2}\tan^{-1}2^{3/2}\} + \frac{3 - 3^{1/2}}{2}\Big(1 - \frac{r_{c_2}^2}{r_{cR}^2}\Big)(1 - 2^{-5/2}k_2$$

$$\cdot\log\{(2^{-3/2}k_2 + 1)/(2^{-3/2}k_2 - 1)\} + \frac{3 + 3^{1/2}}{2}\Big(1 - \frac{r_{c_2}^2}{r_{c_2/\beta}^2}\Big)$$

$$\cdot\{1 - 2^{-3/2}k_1\tan^{-1}(2k_2)\}\Big]\Big]\Big]H\Big(t - \frac{r}{c_2}\Big) - \frac{2Q_0}{\pi\mu}\frac{c_2}{\nu}\frac{1}{r_{c_2}}(2/3)^{1/2}\Big[\Big(-3\frac{r_{c_2}^2}{r_{2c_2}^2}$$

$$+ \frac{3 - 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{cR}^2} + \frac{3 + 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{c_2/\beta}^2}\Big)\Big\{1 - \frac{(3/2)^{1/2}}{2}\frac{c_2}{\nu}\frac{r_{c_1}}{|y|}$$

$$\cdot\log\frac{((3/2)^{1/2}c_2r_{c_1}/\nu\,|y| + 1)}{((3/2)^{1/2}c_2r_{c_1}/\nu\,|y| - 1)}\Big\}$$

$$- 3\Big(1 - \frac{r_{c_2}^2}{r_{2c_2}^2}\Big)\{1 - 2^{-3/2}\tan^{-1}(2^{3/2})\}$$

$$+ \frac{3 - 3^{1/2}}{2}\Big(1 - \frac{r_{c_2}^2}{r_{cR}^2}\Big)(1 - 2^{-5/2}k_2\log\{(2^{-3/2}k_2 + 1)/(2^{-3/2}k_2 - 1)\})$$

$$+ \frac{3 + 3^{1/2}}{2}\Big(1 - \frac{r_{c_2}^2}{r_{c_2/\beta}^2}\Big)\{1 - 2^{-3/2}k_1\tan^{-1}(2k_2)\}\Big]S(\Delta_2)$$

$$+ \frac{Q_0}{4\mu}\frac{(x - \nu t)t + y^2\nu/c_R^2}{r_{cR}^2(t^2 - r^2/c_R^2)^{1/2}}H(t - r/c_R). \tag{13}$$

The symbols used in the above displacement are defined by

$$\tau_\pm^{(i)} = \frac{(x\nu - c_i^2t) \pm c_ir_{c_i}}{\nu^2 - c_i^2}, \qquad N = x(x - \nu t) + y^2, \qquad \kappa = \frac{2^{1/2}r}{c_1(t^2 - r^2/c_1^2)^{1/2}},$$

$$r_\alpha = [(x - \nu t)^2 + (1 - \nu^2/\alpha^2)y^2]^{1/2}, \qquad b^2 = \frac{2}{3}\frac{y^2}{r_{c_1}^2}\frac{\nu^2}{c_2^2}.$$

The term $S(\Delta_i)$ is a unit step function with value 1 inside the "triangular region" $r > c_i t$, $t - x/\nu - ((\nu/c_i)^2 - 1)^{1/2} \mid y \mid/\nu > 0$ and $x > \mid y \mid ((\nu/c_i)^2 - 1)^{-1/2}$ and value 0 outside this region. This triangular region is thus the surface intersection of the conical regions discussed in Sect. 3. $E(\tau)$ is the indefinite integral,

$$E(\tau) = \int \frac{F_1(\eta)}{[-r^2(\tau) + \nu(t - \tau)(x - \nu\tau)]^2}\, d\tau,$$

where

$$F_1(\eta) = \eta \int_{3^{-1/2}}^{\eta} \frac{v(v^2 - 1/3)^{1/2}(1 - v^2)^{1/2}(12 - 24v^2)}{(\eta^2 - v^2)^{1/2}(3 - 24v^2 + 56v^4 - 32v^6)}\, dv, \qquad \eta = \frac{c_2(t - \tau)}{r(\tau)}.$$

The double integral $E(\tau)$ is cumbersome but one of the integrations can be performed. A somewhat lengthy calculations leads to

$$[E(\tau_1) - E(\tau_2)] = \frac{(3/2)^{1/2}}{8} \frac{c_2^2}{\nu^3 \mid y \mid^3} \frac{\partial}{\partial\lambda^2} [6J(\tau_1, \tau_2, 0, \pi/2) - 18J(\tau_1, \tau_2, 8, \pi/2)$$

$$+ (6 - 4(3)^{1/2})\{H(r_{cR}^2)(J(\tau_1, \tau_2, - 12(3)^{1/2} + 20, \theta_0) + M(\tau_1, \tau_2, - 12(3)^{1/2} + 20))$$

$$+ H(-r_{cR}^2)(J(\tau_1, \tau_2, - 12(3)^{1/2} + 20, \pi/2))\}$$

$$+ (6 + 4(3)^{1/2})J(\tau_1, \tau_2, 12(3)^{1/2} + 20, \pi/2)],$$

where $J(\tau_1, \tau_2, \alpha, \phi)$ and $M(\tau_1, \tau_2, \alpha)$ (τ_1, τ_2, α and ϕ are generic symbols) are given by

$$J(\tau_1, \tau_2, \alpha, \phi) \equiv \int_0^{\phi} k^{-1/2} \log \left[\frac{1 + \alpha/2\,(3p_2 - 1)\sin^2\theta}{1 + \alpha/2\,(3p_1 - 1)\sin^2\theta} \frac{D_1}{D_2}\right] d\theta,$$

$$M(\tau_1, \tau_2, \alpha) \equiv \int_{\theta_0}^{\pi/2} (-k)^{-1/2} \tan^{-1}\left(\frac{L_1 - L_2}{1 + L_1 L_2}\right) d\theta,$$

with

$$D_i = \frac{\alpha}{2}\left\{1 - \frac{1}{2}(3p_i - 1)\sin^2\theta\right\}\left\{(\lambda^2 - 1/3)\sin^2\theta + \frac{2}{3\alpha}\right\}$$

$$+ \frac{1}{4}\left\{1 + \frac{\alpha}{2}(3p_i - 1)\sin^2\theta\right\}\{(\lambda^2 - 1/3)\sin^2\theta - 2/3\}$$

$$- \frac{1}{2}\sin\theta[\alpha(\alpha + 1)]^{1/2}\left\{(\lambda^2 - 1/3)\sin^2\theta + \frac{2}{3\alpha}\right\}^{1/2}$$

$$\cdot\left\{1 - \frac{1}{2}(3p_i - 1)\sin^2\theta\right\}^{1/2}(\lambda^2 - p_i)^{1/2},$$

$$L_i = M_i/N_i,$$

$$M_i = 2(\alpha + 1)\left[(\lambda^2 - 1/3)\sin^2\theta + \frac{2}{3\alpha}\right] - \left[(\lambda^2 - 1/3)\sin^2\theta + \frac{2(\alpha + 2)}{3\alpha}\right]$$

$$\cdot\left\{1 + \frac{\alpha}{2}(3p_i - 1)\sin^2\theta\right\},$$

$$N_i = 2\sin\theta[-\alpha(\alpha+1)]^{1/2}\left\{(\lambda^2 - 1/3)\sin^2\theta + \frac{2}{3\alpha}\right\}^{\frac{1}{2}} \cdot\left\{1 - \frac{1}{2}(3p_i - 1)\sin^2\theta\right\}^{1/2}(\lambda^2 - p_i)^{1/2},$$

$$k = \frac{9}{4}\alpha(\alpha+1)\sin^2\theta\left[(\lambda^2 - 1/3)\sin^2\theta + \frac{2}{3\alpha}\right], \qquad \lambda^2 = [(x-\nu t)^2 + y^2]\frac{c_2^2}{\nu^2 y^2},$$

$$\cos\theta_0 = \frac{r_{cR}}{r_{c_1}}, \qquad p_i = \frac{c_2^2(t-\tau_i)^2}{r^2(\tau_i)}.$$

In the above expressions, in order to avoid ambiquity, $(\lambda^2 - p)^{1/2}$ is to be replaced by $c_2[-r(\tau) + \nu(t-\tau)(x-\nu\tau)/r(\tau)]/\nu \mid y \mid$ after differentiation with respect to λ^2, but before insertion of the proper values for the generic symbols τ_1 and τ_2. Note also that $p(\tau_\pm^{(1)}) = \frac{1}{3}$, $p(\tau_\pm^{(2)}) = 1$, and $p(0) = (c_2 t/r)^2$.

The v surface displacement is given by

$$v(x, y, t) = -\frac{Q_0}{\pi\mu}\nu(x-\nu t)y[E(\tau_+^{(1)}) - E(\tau_+^{(2)})]H\left(t - \frac{r}{c_2}\right)$$

$$- \frac{Q_0}{\pi\mu}\nu(x-\nu t)y[-E(\tau_+^{(2)}) + E(\tau_-^{(2)})]S(\Delta_2)$$

$$+ \left[\frac{Q_0}{\pi\mu}\frac{y}{N}\frac{c_2 t}{r}\frac{(3/2)^{1/2}}{8}\left\{6K\left(\frac{1}{\kappa}\right) - 18\Pi\left(\frac{8}{\kappa^2}, \frac{1}{\kappa}\right)\right.\right.$$

$$\left. + (6 - 4(3)^{1/2})\Pi\left(-\frac{12(3)^{1/2} - 20}{\kappa^2}, \frac{1}{\kappa}\right) + (6 + 4(3)^{1/2})\Pi\left(\frac{12(3)^{1/2} + 20}{\kappa^2}, \frac{1}{\kappa}\right)\right\}$$

$$\left. - \frac{Q_0}{\pi\mu}\nu(x-\nu t)y\{E(\tau_+^{(1)} - E(0)\}\right]H\left(\frac{r}{c_2} - t\right)H\left(t - \frac{r}{c_1}\right)$$

$$- \frac{Q_0}{\pi\mu}\nu(x-\nu t)y[E(\tau_+^{(1)}) - E(\tau_-^{(1)})]S(\Delta_1) + \left[\left[\frac{Q_0}{\pi\mu}\frac{c_2}{\nu}\frac{1}{r_{c_2}}\frac{x-\nu t}{y}(2/3)^{1/2}\right.\right.$$

$$\cdot\left[\left(-3\frac{r_{c_2}^2}{r_{2c_2}^2} + \frac{3 - 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{cR}^2} + \frac{3+3^{1/2}}{2}\frac{r_{c_2}}{r_{c_2/\beta}^2}\right)\right.$$

$$\cdot\left\{1 - \frac{(3/2)^{1/2}}{2}\frac{c_2}{\nu}\frac{r_{c_1}}{|y|}\log\frac{((3/2)^{1/2}c_2 r_{c_1}/\nu\mid y\mid + 1)}{((3/2)^{1/2}c_2 r_{c_1}/\nu\mid y\mid - 1)}\right\}$$

$$- 3\left(1 - \frac{r_{c_2}^2}{r_{2c_2}^2}\right)\{1 - 2^{-3/2}\tan^{-1}2^{3/2}\}.$$

$$+ \frac{3 - 3^{1/2}}{2}\left(1 - \frac{r_{c_2}^2}{r_{cR}^2}\right)(1 - 2^{-5/2}k_2\log\{2^{-3/2}k_2 + 1)/(2^{-3/2}k_2 - 1)\})$$

$$\left. + \frac{3+3^{1/2}}{2}\left(1 - \frac{r_{c_2}^2}{r_{c_2/\beta}^2}\right)\{1 - 2^{-3/2}k_1\tan^{-1}(2k_2)\}\right]$$

$$+ \frac{Q_0}{4\pi\mu\nu}\frac{1}{(t^2 - r^2/c_1^2)^{1/2}}\left[+3\frac{x}{y}K(\kappa) - \frac{(x-\nu t)}{y}\frac{N}{r_{c_1}^2}\left\{+3 - 3\frac{y^2}{r_{2c_2}^2}\frac{\nu^2}{(2c_2)^2}\right.\right.$$

$$\left. + \frac{1}{3}\frac{y^2}{r_{cR}^2}\frac{\nu^2}{c_R^2} + \frac{1}{3}\frac{y^2}{r_{c_2/\beta}^2}\frac{\nu^2}{(c_{2/\beta})^2}\right\}\Pi(-b^2, \kappa)$$

$$+ 9\left(\frac{x - vt}{y}\frac{N}{r_{2c_2}^2} - \frac{x}{y}\right)\Pi(8, \kappa) + (2(3)^{1/2} - 3)\left(\frac{x - vt}{y}\frac{N}{r_{cR}^2} - \frac{x}{y}\right)\Pi(-12(3)^{1/2} + 20, \kappa)$$

$$- (2(3)^{1/2} + 3)\left(\frac{x - vt}{y}\frac{N}{r_{c_2/\beta}^2} - \frac{x}{y}\right)\Pi(12(3)^{1/2} + 20, \kappa)\Big]\Big]\Big]H\left(t - \frac{r}{c_2}\right)$$

$$+ 2\frac{Q_0}{\pi\mu}\frac{c_2}{v}\frac{1}{r_{c_2}}\frac{x - vt}{y}(2/3)^{1/2}\left[\left(-3\frac{r_{c_2}^2}{r_{2c_2}^2} + \frac{3 - 3^{1/2}}{2}\frac{r_{c_2}^2}{r_{cR}^2} + \frac{3 + 3^{1/2}}{2}\left(\frac{r_{c_2}^2}{r_{c_2/\beta}^2}\right)\right.\right.$$

$$\cdot\left\{1 - \frac{(3/2)^{1/2}}{2}\frac{c_2}{v}\frac{r_{c_1}}{|y|}\log\frac{((3/2)^{1/2}c_2r_{c_1}/v\,|y| + 1)}{((3/2)^{1/2}c_2r_{c_1}/v\,|y| - 1)}\right\}$$

$$- 3\left(1 - \frac{r_{c_2}^2}{r_{2c_2}^2}\right)\{1 - 2^{-3/2}\tan^{-1} 2^{3/2}\} + \frac{3 - 3^{1/2}}{2}\left(1 - \frac{r_{c_2}^2}{r_{cR}^2}\right)$$

$$\cdot(1 - 2^{-5/2}k_2 \log\{(2^{-3/2}k_2 + 1)/(2^{-3/2}k_2 - 1)\}) + \frac{3 + 3^{1/2}}{2}\left(1 - \frac{r_{c_2}^2}{r_{c_2/\beta}^2}\right)$$

$$\left.\cdot\{1 - 2^{-3/2}k_1\tan^{-1}(2k_2)\}\right]S(\Delta_2) + \frac{Q_0}{4\mu}\frac{y(t - xv/c_R^2)}{r_{cR}^2(t^2 - r^2/c_R^2)^{1/2}}H(t - r/c_R) \tag{14}$$

In general, the evaluation of the integrals involved in the expressions for $E(\tau)$ will almost certainly require numerical integration. However, in certain special cases, these integrals are not needed in the displacement expressions. From Eqs. (13) and (14), it is seen that on the line $y = 0$ the u and v displacements are expressible in terms of known functions (from symmetry in this case $v(x, 0, t) = 0$). Also on the moving line $x = vt$, the v (but not the u) displacement is free of the $E(\tau)$ function. Finally the normal surface displacement is found to be

$$w(x, y, t) = -\frac{Q_0}{16\mu}\left\{\left[-2(3)^{1/2}\frac{r_{c_1}}{r_{cR}^2} - 6\frac{r_{c_1}}{r_{2c_2}^2} + 2(3)^{1/2}\frac{r_{c_1}}{r_{c_2/\beta}^2}\right.\right.$$

$$+ \frac{(4 + 8(3)^{-1/2})^{1/2}}{r_{cR}^2}\frac{N}{(r^2 - c_R^2t^2)^{1/2}}H\left(\frac{r}{c_R} - t\right) + 2(3)^{1/2}\frac{N}{r_{2c_2}^2(4c_2^2t^2 - r^2)^{1/2}}$$

$$\left. - \frac{(-4 + 8(3)^{-1/2})^{1/2}}{r_{c_2/\beta}^2}\frac{N}{(c_2^2t^2/\beta^2 - r^2)^{1/2}}\right]H\left(t - \frac{r}{c_1}\right) + \left[-2(2 + 3^{1/2})\frac{r_{c_2}}{r_{cR}^2}\right.$$

$$+ 2\frac{r_{c_2}}{r_{2c_2}^2} - 2(2 - 3^{-1/2})\frac{r_{c_2}}{r_{c_2/\beta}^2} + \frac{(4 + 8(3)^{-1/2})}{r_{cR}^2}\frac{N}{(r^2 - c_R^2t^2)^{1/2}}H\left(\frac{r}{c_R} - t\right)$$

$$\left. - 2(3)^{1/2}\frac{N}{r_{2c_2}^2(4c_2^2t^2 - r^2)^{1/2}} + \frac{(-4 + 8(3)^{-1/2})^{1/2}}{r_{c_2/\beta}^2}\frac{N}{(c_2^2t^2/\beta^2 - r^2)^{1/2}}\right]H\left(t - \frac{r}{c_2}\right)$$

$$+ \left[-4(3)^{1/2}\frac{r_{c_1}}{r_{cR}^2} - 12\frac{r_{c_1}}{r_{2c_2}^2} + 4(3)^{1/2}\frac{r_{c_1}}{r_{c2/\beta}^2}\right]S(\Delta_1)$$

$$\left.+ \left[-4(2 + 3^{1/2})\frac{r_{c_2}}{r_{cR}^2} + 4\frac{r_{c_2}}{r_{2c_2}^2} - 4(2 - 3^{1/2})\frac{r_{c_2}}{r_{c_2/\beta}^2}\right]S(\Delta_2)\right\}. \tag{15}$$

Since some of the important features of the u and v displacements may still be hidden in the $E(\tau)$ integrals, the discussion will be confined to the normal displacement w. From the previous results for the displacements produced by a moving load in an infinite elastic space, it might be expected that w would be singular on the lines $r_{c_1} = 0$ and

$r_{c_2} = 0$. This is not the case as Eq. (15) indicates. The normal displacement does become unbounded on the circle $r = c_R t$ as could be anticipated from the Rayleigh wave character of Pekeris' [10] solution. The reciprocal square root singularity remains the same also. When the moving load velocity ν exceeds the Rayleigh wave velocity c_R, a much stronger singularity is introduced by the term r_{cR}^{-2} . A careful examination of Eq. (15) shows that the w displacement is not singular everywhere on the lines $r^2_{cR} = 0$, but only on those rays extending back from the moving pressure spot and tangent to the Rayleigh wave-front circle. These rays are indicated by the dashed lines in Figure 2. It should also be noted that r^2_{cR} can be negative, in contrast R^2_{c1} and $R^2_{c_2}$ of the infinite space solution which are limited to positive values by $S(C_1)$ and $S(C_2)$ respectively.

The form in which the normal displacement is written in Eq. (15) is somewhat suggestive and a word of speculation on the displacement for a material which gives rise to more than one surface-wave velocity seems in order. Corresponding to the Rayleigh wave velocity c_R , there are also velocities $(2c_2)$ and (c_2/β) which could give rise to singular expanding circles on the surface of the elastic half-space, but are prevented from doing this in the present case because of the limitation of the material constants. There would also be the possibility of other singular lines at $r^2_{2c_2} = 0$ and $r^2_{c_2/\beta} = 0$.* For the present case (Poisson's ratio = one-fourth), this behavior lies outside the bounds of the problem and must therefore be suppressed.

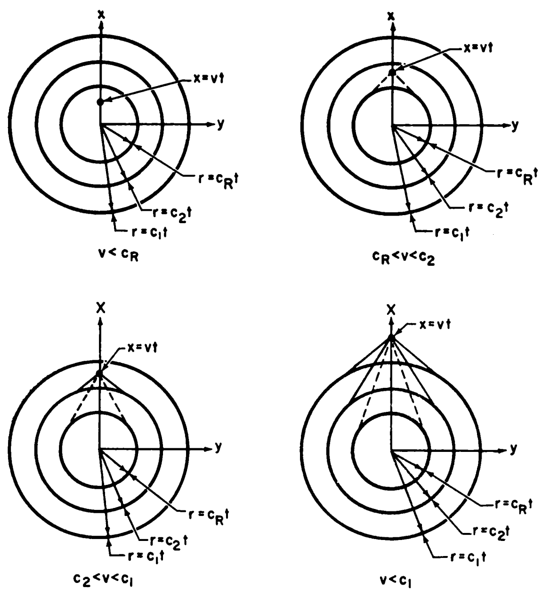

FIG. 2. Surface, wave-front patterns at time t for a moving pressure spot on the surface of an elastic half-space

(v = velocity of moving pressure spot, c_2 = velocity of shear waves,

c_l = velocity of dilatational waves, and c_R = velocity of Rayleigh waves)

*Of course, for different material properties, the roots of the algebraic equation determining the Rayleigh wave velocity would no longer be $2c_2$, $c_{2/\beta}$, and $c_{2/\gamma}$. These are used here merely for purposes of illustration.

References

1. I. S. Sokolnikoff, *Mathematical theory of elasticity.*, McGraw-Hill Book Company, Inc., New York (1956) p. 391
2. G. Eason, J. Fulton, and I. Sneddon, Phil. Trans. Roy. Soc. (London) (A) **248** (1956) 575
3. J. Mandel and A. Avramesco, C. R. Acad. Sci. (Paris) **252** (1961) 3730
4. D. D. Ang, Q. Appl. Math. **28** (1960) 251
5. F. L. DiMaggio and H. H. Bleich, J. Appl. Mech. **26** (1959) 678
6. L. Knopoff and A. F. Gangi, Geophysics **24** (1959) 681
7. P. M. Morse and H. Feshback, *Methods of Theoretical Physics*, McGraw-Hill Book Company, Inc., New York (1953), p. 1790
8. I. N. Sneddon and D. S. Berry, *Handbuch der Physik*, vol. 6, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959, p. 88
9. C. L. Pekeris, Proc. Nat. Acad. Sci. **41**, (1955) p. 629
10. C. L. Pekeris, Proc. Nat. Acad. Sci. **41**, (1955) 469

Appendix

Behavior of $f(\tau)$ in the range $0 < \tau < t$. As noted in Sect. 3, integrals of the type $\int_0^t g(\tau)H[f(\tau)]\, d\tau$ depend on the intervals of τ, in the range $0 < \tau < t$ for which $f(\tau) > 0$. Integrals of the type $\int_0^t h(\tau)\, \delta[f(\tau)]\, d\tau$ depend on the zeros of $f(\tau)$ in the range $0 \leq \tau < t$. It is easy to see that both questions depend essentially on the roots of the algebraic equation,

$$f(\tau) = t - \tau - \frac{R(\tau)}{c} = 0, \tag{A-1}$$

or

$$(\nu^2 - c^2)\tau^2 + 2(c^2 t - z\nu)\tau + R^2 - c^2 t^2 = 0,$$

where $R(\tau) = (x^2 + y^2 + (z - \nu\tau)^2)^{1/2}$ and $R = (x^2 + y^2 + z^2)^{1/2}$.

Solving formally for these roots gives,

$$\tau_{\pm} = \frac{(z\nu - c^2 t) \pm cR_c}{\nu^2 - c^2}, \tag{A-2}$$

where $R_c = [(z - \nu t)^2 + (1 - \nu^2/c^2)r^2]^{1/2}$ as previously defined. $\tau_{\pm}$ is a function of the five parameters r, z, ν, c, t and it should be recalled that $r \geq 0$, $-\infty < z < +\infty$, $0 \leq \nu < +\infty$, $c > 0$, and $t > 0$. The location of the $\tau_{\pm}$ roots in the $(0, t)$ interval obviously depends in some way on the relative values of these five parameters. An examination of Eq. (A-1) supplies the necessary information. The results are shown in figure A-1. Although these results have been derived explicitly for the moving source in an infinite elastic body, they also hold for the half-space problem of section IV provided z is replaced by x, r by y, and hence R_c by r_c. Figure A-1 shows the important features of $f(\tau)$. In detail, these curves may look different for special values of the parameters. For example on the z axis ($r = 0$), the curves are replaced by straight lines since now

$$f(\tau) = t - \tau - \frac{1}{c}\,|\, z - \nu\tau \,|$$

which is a linear equation in τ. Eq. (A-2) still applies however.

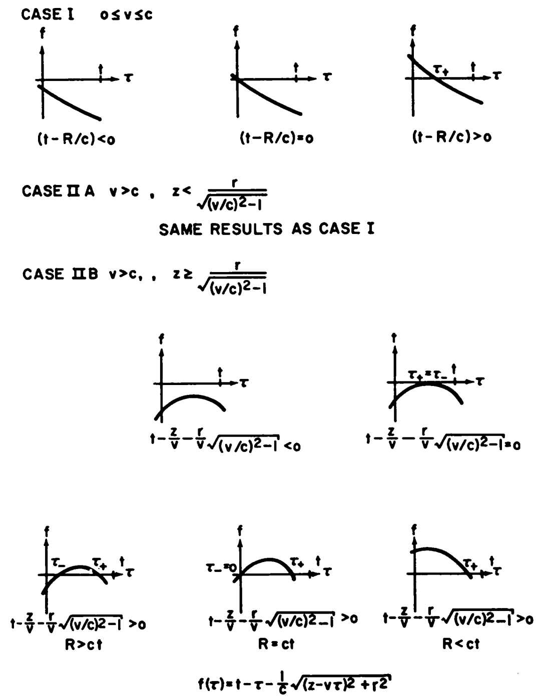

FIG. A-1. Behavior of $f(\tau)$ in the range $0 < \tau < t$

Finally, in evaluating the integrals in Eq. (12), integrals of the type

$$\int_0^t k(\tau)H[f_{c_i}(\tau)]H[-f_{c_R}(\tau)]\, d\tau,$$

are encountered where the velocity c in $f(\tau)$ is replaced by the appropriate subscript on $f(\tau)$. The addition of the function $H[-f_{cR}(\tau)]$ is an added complication but all the information which is needed to set the proper limits on the integral is still contained in Fig. A-1.