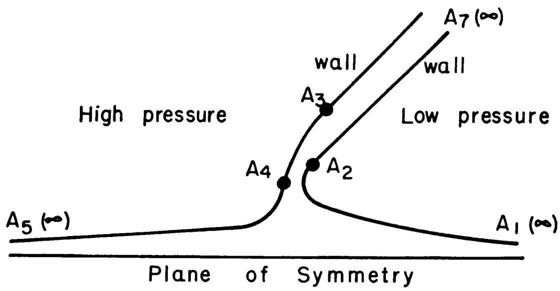

FIG. 6. Partially channelled incoming jet.

poles of $df/d\zeta$ at $\pm a_3$ by simple poles at $\pm a_7$, where $1 < a_7 < a_3$. Now (4.2) becomes

$$\frac{df}{d\zeta} = -\frac{2\beta}{w} \cdot \frac{1}{\zeta(\zeta^2 - a_7^2)(\zeta^2 - a_5^2)}, \tag{5.1}$$

while (3.2) remains unchanged. The presence of the additional parameter a_7 will make it possible to vary the location of A_3 in Fig. 6, for example, while the locations of A_2 and A_4 and the directions of the walls are held constant. The calculation of the various jet widths and rates of mass flow are perfectly straightforward exercises which we shall not repeat. It should be remarked that if there is an inflection point A_4 the wall A_7A_3 can be extended into the high pressure region again, just as in the discussion of Fig. 5.

SOME PROPERTIES OF INFINITE LINES*

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The transmission line equations in the lossless case may be written in the form

$$\frac{du(x)}{dx} + p(x)u(x) - i\omega v(x) = 0, \tag{1a}$$

$$\frac{dv(x)}{dx} - p(x)v(x) - i\omega u(x) = 0, \tag{1b}$$

where

$$p(x) = \frac{1}{2Z_0(x)} \frac{dZ_0(x)}{dx}, \tag{2}$$

and the characteristic impedance $Z_0(x)$ is real. In Ref. [1], solutions of Eqs. (1a) and (1b) were considered which have the asymptotic behavior,

$$\lim_{x \to \infty} u(x, \omega) \exp(-i\omega x) = 1 \tag{3a}$$

$$\lim_{x \to \infty} v(x, \omega) \exp(-i\omega x) = 1, \tag{3b}$$

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and for which the input admittance,

$$Y(\omega) = \frac{v(0, \omega)}{u(0, \omega)}, \tag{4}$$

is a rational function of the real frequency variable ω. It was shown that if $Y(\omega)$ satisfies certain realizability conditions, the solutions to Eqs. (1a) and (1b) can be expressed by

$$u(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0 \tag{5a}$$

$$v(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu^*(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0, \tag{5b}$$

where $f_\nu(x)$ and $f_\nu^*(x)$ satisfy, for $x \geq 0$,

$$-\rho_\nu \sum_{\mu=1}^{n} f_\mu(x) \frac{\exp[i(\kappa_\mu + \kappa_\nu)x]}{\kappa_\mu + \kappa_\nu} + f_\nu(x) + i\rho_\nu \exp(i\kappa_\nu x) = 0, \quad \nu = 1, 2, \cdots, n \tag{6a}$$

$$\rho_\nu \sum_{\mu=1}^{n} f_\mu^*(x) \frac{\exp[i(\kappa_\mu + \kappa_\nu)x]}{\kappa_\mu + \kappa_\nu} + f_\nu^*(x) - i\rho_\nu \exp(i\kappa_\nu x) = 0, \quad \nu = 1, 2, \cdots, n, \tag{6b}$$

and ρ_ν and κ_ν are the residues and poles, respectively, of $1 - u(0, -\lambda)/u(0, \lambda)$ in the upper half λ-plane. For infinite lines in this class an explicit solution to the synthesis problem is available [1]. The purpose of this note is to present some properties of these lines.

In general, $Z_0(x)$ will not be monotonic but will oscillate in the fashion of a damped sinusoid as illustrated in the example given in Ref. [1]. However, there exists a class of input admittances that yield a characteristic impedance which exhibits a monotone behavior. We shall denote these admittances as *RC* (*RL*) functions since they are also realizable by lumped networks consisting of resistors and capacitors (inductors). Necessary and sufficient conditions for an admittance function to be realizable as an infinite line of the class considered here were given in Theorem 2 of Ref. [1]. In order that an *RC* (*RL*) function $Y(is)$ satisfy these conditions it is necessary that it have the expansion (see Ref. [2])

$$Y(is) = 1 + \sum_{\nu=1}^{n} \frac{K_\nu}{s - \sigma_\nu}, \tag{7}$$

where the σ_ν are distinct negative real numbers and the residues are real and satisfy, for $\nu = 1, 2, \cdots, n$,

$$K_\nu \begin{cases} < 0 (RC) \\ > 0 (RL). \end{cases} \tag{8}$$

The complex variable in (7) is defined by $s = -i\lambda = \sigma - i\omega$. We shall now show that if in addition, $Y(0) \neq 0$, then the admittance defined by (7) satisfies all of the conditions of Theorem 2 and therefore is realizable as an infinite line having a solution of the form given in (5). It will be sufficient to prove the following.

Lemma 1. If $Y(is)$ is an *RC* or *RL* admittance function as defined by (7), the zeros of $G(is) = Ev[Y(is)]$ lie on the real axis of the s-plane. If, in addition, $Y(0) \neq 0$, all of the zeros are simple.

Proof. The ν-th term in (7) can be written

$$\frac{K_\nu}{s - \sigma_\nu} = F_\nu(\sigma, \omega) + \frac{T_\nu(\sigma, \omega)}{s}, \qquad \nu = 1, 2, \cdots, n, \tag{9}$$

where the F_ν and T_ν are given by

$$F_\nu(\sigma, \omega) = \frac{-K_\nu \sigma_\nu}{(\sigma - \sigma_\nu)^2 + \omega^2}, \qquad \nu = 1, 2, \cdots, n \tag{10a}$$

$$T_\nu(\sigma, \omega) = \frac{K_\nu(\sigma^2 + \omega^2)}{(\sigma - \sigma_\nu)^2 + \omega^2}, \qquad \nu = 1, 2, \cdots, n. \tag{10b}$$

Therefore, we can express (7) by

$$Y(is) = 1 + F(\sigma, \omega) + \frac{T(\sigma, \omega)}{s}, \tag{11}$$

where the functions

$$F(\sigma, \omega) = \sum_{\nu=1}^{n} F_\nu(\sigma, \omega); \qquad T(\sigma, \omega) = \sum_{\nu=1}^{n} T_\nu(\sigma, \omega) \tag{12}$$

are real for all σ and ω. The conclusion that the zeros of $G(is)$ are real is evident from

$$G(is) = \tfrac{1}{2}[Y(is) + Y(-is)]. \tag{13}$$

It follows from (7) that

$$\frac{dY(i\sigma)}{d\sigma}\begin{cases} > 0(RC) \\ < 0(RL) \end{cases}, \qquad \frac{dY(-i\sigma)}{d\sigma}\begin{cases} < 0(RC) \\ > 0(RL) \end{cases}, \tag{14}$$

and

$$\left| \frac{dY(-i\sigma)}{d\sigma} \right| \gtrless \left| \frac{dY(i\sigma)}{d\sigma} \right|, \qquad \sigma \gtrless 0. \tag{15}$$

Therefore,

$$\frac{dG(i\sigma)}{d\sigma} \neq 0, \qquad \sigma \neq 0, \tag{16}$$

and the zeros of $G(is)$ must be simple. The possibility of a zero with even multiplicity at the origin is ruled out by the condition $Y(0) \neq 0$.

In order to show the monotonicity of the characteristic impedance, we will need the following result.

Lemma 2. If $f_\nu(x)$ and $f_\nu^*(x)$ satisfy (6a) and (6b), respectively, then for all x in the interval, $0 \leq x < \infty$,

$$\text{Re}[\bar{f}_\nu(x) f_\nu^*(x)] \leq 0, \qquad \nu = 1, 2, \cdots, n. \tag{17}$$

The equality sign holds only if both $f_\nu(x)$ and $f_\nu^*(x)$ are identically zero for all $x \geq 0$.

Proof. If (5a) and (5b) satisfy (1a) and (1b), then it follows by direct substitution that

$$\frac{df_\nu(x)}{dx} + p(x) f_\nu(x) + i\kappa_\nu f_\nu^*(x) = 0, \qquad \nu = 1, 2, \cdots, n \tag{18a}$$

$$\frac{df_\nu^*(x)}{dx} - p(x)f_\nu^*(x) + i\kappa_\nu f_\nu(x) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{18b}$$

and

$$p(x) = \sum_{\nu=1}^{n} [f_\nu(x) - f_\nu^*(x)] \exp (i\kappa_\nu x). \tag{19}$$

Multiply the conjugate of (18a) by $f_\nu^*(x)$ and (18b) by $\bar{f}_\nu(x)$. Adding and taking the real part, there results,

$$\frac{\partial}{\partial x} \operatorname{Re} [\bar{f}_\nu(x) f_\nu^*(x)] = \operatorname{Im} [\kappa_\nu][| f_\nu(x) |^2 + | f_\nu^*(x) |^2], \qquad \nu = 1, 2, \cdots, n. \tag{20}$$

Integrating and using the fact that

$$\lim_{x\to\infty} f_\nu(x) = 0, \qquad \nu = 1, 2, \cdots, n, \tag{21}$$

we obtain for arbitrary x_1,

$$\operatorname{Re} [\bar{f}_\nu(x_1) f_\nu^*(x_1)] = -\operatorname{Im} [\kappa_\nu] \int_{x_1}^{\infty} [| f_\nu(x) |^2 + |f_\nu^*(x) |^2]\, dx, \qquad \nu = 1, 2, \cdots, n. \tag{22}$$

Since $\operatorname{Im} [\kappa_\nu] > 0$, $\nu = 1, 2, \cdots, n$, the inequality in (17) is obtained, unless both $f_\nu(x)$ and $f_\nu^*(x)$ are zero for $x > x_1$. But if follows from Lemma 3 of Ref. [1] that the $f_\nu(x)$ and the $f_\nu^*(x)$ are bounded functions of x for all $x \geq 0$ and therefore are analytic in a domain including the positive x-axis. Consequently, if either function is zero for $x > x_1$, it must be zero everywhere, and the proof is complete.

We have seen that the admittance function given by (7) is realizable as an infinite line. A synthesis formula for the construction of the characteristic impedance is given in Theorem 4 of Ref. [1]. We now establish the monotonicity of the characteristic impedance for this special class of lines.

Theorem 1. If the input admittance of an infinite line is an *RC* (*RL*) function as defined by (7), then the synthesis formula given in Theorem 4 of Ref. [1] yields a characteristic impedance which is monotone increasing (decreasing) for all x in the interval $0 \leq x < \infty$.

Proof. For the class of lines considered here $u(0, \lambda)$ has the form (see Ref. [1])

$$u(0, \lambda) = \frac{\prod_{i=1}^{n} (\lambda - \mu_i)}{\prod_{i=1}^{n} (\lambda + \kappa_i)}, \tag{23}$$

where $\operatorname{Im} [\mu_i] < 0$, $i = 1, 2, \cdots, n$, and the κ_i have already been defined. It follows from

$$G(\lambda) = \frac{1}{u(0, \lambda)u(0, -\lambda)} \tag{24}$$

that $1 - G^{-1}(\lambda)$ has the expansion

$$1 - G^{-1}(\lambda) = \sum_{\nu=1}^{n} \left(\frac{k_\nu}{\lambda - \kappa_\nu} - \frac{k_\nu}{\lambda + \kappa_\nu}\right), \tag{25}$$

where

$$k_\nu = -u(0, \kappa_\nu) \lim_{\lambda \to \kappa_\nu} (\lambda - \kappa_\nu) u(0, -\lambda), \qquad \nu = 1, 2, \cdots, n. \tag{26}$$

In the case of RC or RL input admittances the κ_ν will be imaginary by Lemma 1; that is, $\kappa_\nu = i \mid \kappa_\nu \mid$, $\nu = 1, 2, \cdots, n$. Equation (25) becomes in the s-plane,

$$1 - G^{-1}(is) = -i \sum_{\nu=1}^{n} \left(\frac{k_\nu}{s - \mid \kappa_\nu \mid} - \frac{k_\nu}{s + \mid \kappa_\nu \mid} \right), \tag{27}$$

where we now evaluate the residues by

$$\begin{aligned} ik_\nu &= -\left\{ \frac{d}{ds} \left[\frac{G(is)}{G(is) - 1} \right] \right\}^{-1}_{s = \mid \kappa_0 \mid} \\ &= \left[\frac{dG(is)}{ds} \right]^{-1}_{s = \mid \kappa_\nu \mid}, \qquad \nu = 1, 2, \cdots, n. \end{aligned} \tag{28}$$

The right side of (28) is real and, from (13), (14), and (15), satisfies the inequality,

$$\left[\frac{dG(is)}{ds} \right]^{-1}_{s = \mid \kappa_\nu \mid} \begin{cases} < 0 (RC) \\ > 0 (RL). \end{cases} \tag{29}$$

The ρ_ν in (6) are clearly related to the k_ν given by (26). In fact,

$$\rho_\nu = \frac{k_\nu}{u^2(0, \kappa_\nu)}, \qquad \nu = 1, 2, \cdots, n. \tag{30}$$

Since $u(0, \kappa_\nu)$ is real for imaginary κ_ν we see that in the case of RC or RL admittance functions the ρ_ν are imaginary and satisfy, for $\nu = 1, 2, \cdots, n$,

$$i\rho_\nu \begin{cases} = \mid \rho_\nu \mid (RL) \\ = -\mid \rho_\nu \mid (RC). \end{cases} \tag{31}$$

We now direct our attention to $p(x)$. Multiply (6a) by $f_\nu^*(x)/\rho_\nu$ and (6b) by $f_\nu(x)/\rho_\nu$ and add. Summing over ν and employing (19) we obtain

$$p(x) = 2 \sum_{\nu=1}^{n} \frac{f_\nu(x) f_\nu^*(x)}{i\rho_\nu}, \qquad x \geq 0. \tag{32}$$

From (18) it can be seen that both $f_\nu(x)$ and f_ν^* are real. Therefore, from Lemma 2 and (31) we find that for all $x \geq 0$,

$$p(x) \begin{cases} > 0 (RC) \\ < 0 (RL). \end{cases} \tag{33}$$

This proves the theorem since, from Theorem 4 of Ref. [1], $Z_0(x)$ in Eq. (2) is positive for $x \geq 0$.

The undulatory nature of the characteristic impedance in the general case suggests that it may be possible at a point where $dZ_0(x)/dx = 0$ to connect a uniform line so as to provide a continuous transition for $u(x, \omega)$ and $v(x, \omega)$ without exciting a reflected wave. If the uniform line were connected on the input side it would then be possible to obtain reflectionless transmission between two real impedances for all frequencies. An equivalent result was found to exist by Kay and Moses [3] for a class of transparent dielectric media defined over the whole line, $-\infty < x < \infty$. On the other hand, if the uniform line were connected on the output side, it would be possible to reduce the infinite line to a finite line terminated in a matched real impedance. However, it will now be shown that for the class of lines considered here, the above conjecture has no basis in fact. It will be sufficient to prove the following.

Theorem 2. Given any finite point $x_1 \geq 0$ on an infinite line of the class having solutions of the form given in (5). The admittance at this point will have the value

$$y(x_1, \omega) = \frac{v(x_1, \omega)}{u(x_1, \omega)} = 1 \tag{34}$$

if and only if the line is uniform, that is, $Z_0(x) =$ constant, for all $x \geq 0$.

Proof. Consider the analytic continuation of (5a) and (5b) in the upper half-plane. Equation (34) implies

$$\sum_{\nu=1}^{n} [f_\nu(x_1) - f_\nu^*(x_1)] \frac{\exp [i(\kappa_\nu + \lambda)x_1]}{i(\kappa_\nu + \lambda)} = 0, \qquad \sigma \geq 0. \tag{35}$$

Put $\lambda = -\bar{\kappa}_\mu$, $\mu = 1, 2, \cdots, n$. Then (35) becomes

$$\sum_{\nu=1}^{n} [f_\nu(x_1) - f_\nu^*(x_1)] A_{\mu\nu} = 0, \qquad \mu = 1, 2, \cdots, n, \tag{36}$$

where $A_{\mu\nu}$ is defined by

$$A_{\mu\nu} = -\frac{\exp [i(\kappa_\nu - \bar{\kappa}_\mu)x_1]}{i(\kappa_\nu - \bar{\kappa}_\mu)}, \qquad \mu, \nu = 1, 2, \cdots, n. \tag{37}$$

The matrix $A = [A_{\mu\nu}]$ is positive definite (see Lemma 3, Ref. [1]). That is, for arbitrary constants, b_μ, $\mu = 1, 2, \cdots, n$,

$$\sum_{\mu,\nu=1}^{n} b_\nu \bar{b}_\mu A_{\mu\nu} > 0, \tag{38}$$

unless the b_μ are identically zero. Therefore, the only solution to the system (35) is $f_\nu(x_1) - f_\nu^*(x_1) = 0$, $\nu = 1, 2, \cdots, n$. Adding (6a) and (6b) we find that

$$f_\nu(x_1) = f_\nu^*(x_1) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{39}$$

But this implies by Lemma 2 that both $f_\nu(x)$ and $f_\nu^*(x)$ are identically zero for all x. We conclude from (2) and (19) that $Z_0(x) =$ constant, $x \geq 0$.

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