# OPTIMUM IMPULSES FOR GRAVITY-FREE ROCKET MOTION*

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The equation of motion for a rocket maneuvering in a one-dimensional field-free space (coordinated by $x$) may be taken as

$$x^{\cdot\cdot} = -\theta M^{\cdot} c/M, \tag{1}$$

where $M(t)$ is the mass of the rocket at time $t$, $c(t)$ is the exhaust velocity, $\theta(t) = \pm 1$, and the point indicates differentiation with respect to time $t$. The "navigation" variable $\theta$ indicates that the rocket has the "choice" of instantaneously orienting itself one way or the other along the line without consuming any fuel in the process.

The problem considered is that of maneuvering a rocket with given $c(t) > 0$ from a specified position and velocity to another specified position and velocity in a specified time $T$. In certain special cases, e.g., $c(t)$ = const,. the answer to this problem is already known. In the more general case treated here, other methods may be used to establish the fact that the optimal "control" is impulsive and to derive the required number of impulses. This class of problems was treated first in Refs. 1-3 by modified classical variational calculus, and in Refs. 4, 5 by Pontryagin's maximum principle. Another investigation can be found in Ref. 6. In these references the exhaust speed $c$ is considered constant; however, all results are unaffected if $c$ is a prescribed $c(t)$. In Refs. 1-5 the mass flow rate $M^{\cdot}$ is taken to be bounded; hence, impulses correspond to the limiting case $M^{\cdot} \to -\infty$.† The following method, in addition to proving that the optimal "control" is impulsive for the stated problem, yields the number of impulses required, gives a procedure for finding where they should be applied, and a geometric interpretation of the result.

More precisely, we assume $c(t) > 0$, $T > 0$ given, and say that $x(t)$, $M(t)$, $\theta(t)$ satisfy condition $A$ if $M(t) > 0$ is monotone decreasing, $|\theta(t)| = 1$, and $x(0)$, $x^{\cdot}(0)$, $x(T)$, $x^{\cdot}(T)$, $M(0)$ assume given values. Among all $x(t)$, $M(t)$, $\theta(t)$ satisfying condition $A$, we are seeking an $x^*(t)$, $M^*(t)$, $\theta^*(t)$ such that (1) is satisfied and $M(T)$ is maximum. Now observe that the satisfaction of (1) implies:

$$x^{\cdot}(t) = x^{\cdot}(0) + \int_0^t \theta c \, d \ln M^{-1}, \tag{2}$$

$$x(t) = x(0) + x^{\cdot}(0)t + \int_0^t \left[ \int_0^\nu \theta c \, d \ln M^{-1} \right] d\nu. \tag{3}$$

In order to render the above problem accessible to the techniques utilized below, it will be necessary to reformulate the problem in measure-theoretic terminology.

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For given $\theta(t)$, $c(t)$, $M(t)$, if $x(t)$ is defined by (3), then (1) is automatically satisfied. Also, the requirement that $M(t)$ is monotone decreasing is equivalent to $d \ln M^{-1} \equiv d\mu$ is a non-negative measure. Also observe

$$\int_0^T \theta c \, d\mu = x^{\cdot}(T) - x^{\cdot}(0) \equiv \alpha, \tag{4}$$

$$\int_0^T \theta t \, c \, d\mu = T \, x^{\cdot}(T) - [x(T) - x(0)] \equiv \beta. \tag{5}$$

The relation (5) may be established using (4) and integrating the integral occurring in (3) by parts.

Using these observations, we may restate the problem as follows: Given $c(t) > 0$, $\alpha$, $\beta$, find $\theta(t)$, $\mu(t)$ such that $\mu \geq 0$, $|\theta(t)| = 1$, $\int c\theta \, d\mu = \alpha$, $\int tc\theta \, d\mu = \beta$, and $\int d\mu$ is minimized. There is clearly no loss of generality if we set $T = 1$ and interpret $\int$ without limits specified as the integral over the "whole space" [0, 1].

We require one further restatement of the problem before proceeding. This restatement accentuates the role played by the function $f(t) \equiv c(t)^{-1}$ rather than $c(t)$ itself. If the measure $\nu$ is defined by $\nu(s) = \int_s c \, d\mu$ for any subset $S$ of [0, 1] then $c = d\nu/d\mu$, $f = d\mu/d\nu$, and $\int g \, d\mu = \int g(d\mu/d\nu) \, d\nu = \int gf \, d\nu$ for any $g$.

We may thus reformulate the preceding problem as follows. Given $f(t) > 0$, $\alpha$, $\beta$, find $\theta(t)$, $\nu(t)$ such that $\nu \geq 0$, $|\theta(t)| = 1$, $\int \theta \, d\nu = \alpha$, $\int t\theta \, d\nu = \beta$, and $\int f \, d\nu$ is minimized.

The answer to this problem is simply described, and has an interesting geometric interpretation. It will be proved that the measure $\nu$ is either a one or two-point measure; i.e., $\nu$ is concentrated either all at one point, or all at two points. The meaning of this statement in terms of the original problem is that one or, at the most, two impulses always yield the optimal mode of transfer.

In order to describe at what point or point pair the measure is concentrated, we shall utilize a simple geometric figure. In Figs. 1-4, the upper (wavy) contour $AB$ is

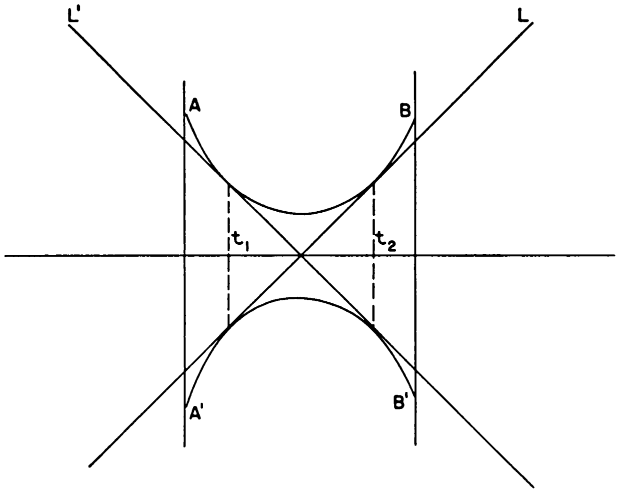


FIG. 1.

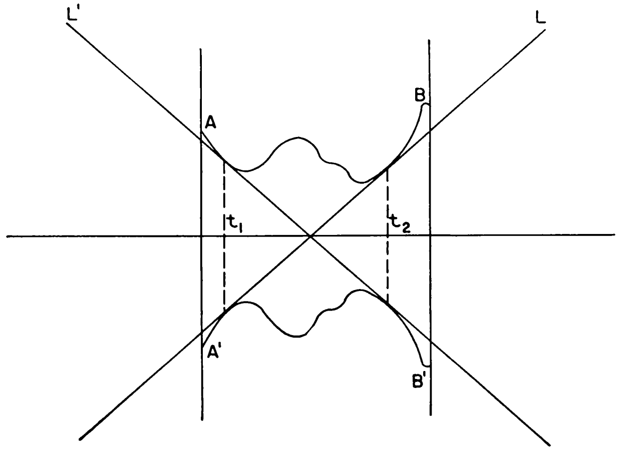


FIG. 2.

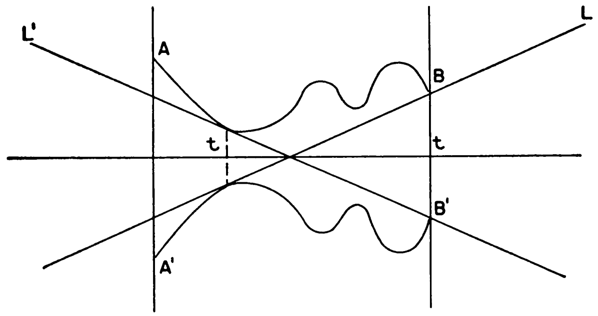


FIG. 3.

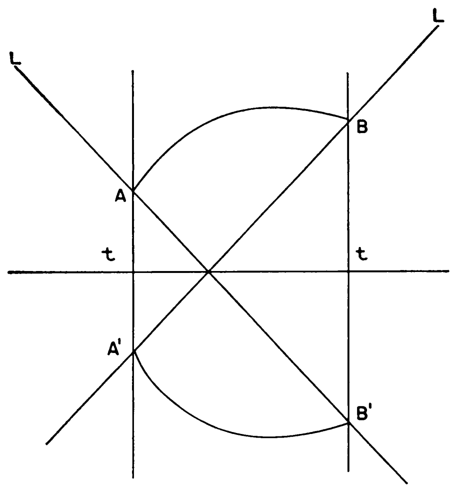


FIG. 4.

the graph of a function $f(t)$. The lower (wavy) contour $A'B'$ is the reflection of $AB$ in the $t$ axis, or alternately described as the graph of $-f(t)$. We shall refer to the "inside" of the closed contour $ABB'A'A$ as the "vase." The vase includes the boundaries $AB$, $BB'$, $B'A'$, $A'A$. Let $\mathfrak{L}$ be the class of straight lines not perpendicular to the $t$ axis. Any straight line $L$ in $\mathfrak{L}$ intersects the (extended) lines $AA'$ and $BB'$ at, say, the points $P$ and $Q$, respectively. We shall call $PQ$ the line segment generated by $L$, and say that $L$ is inside the vase if the line segment generated by $L$ is inside the vase. In general, we shall use the terminology $L$ and "line segment generated by $L$" interchangeably when there is no danger of confusion. Also, we shall call the (logical) union of the two contours $AB$, and $A'B'$ the "extended" $f$.

Suppose we consider a line $L$ in $\mathfrak{L}$ which is contained in the vase and intersects extended $f$ in at least one point. Such a line $L$, together with a selected point $P$ in which $L$ intersects extended $f$, will be called a one-point line with anchor point $P$. Similarly, consider a line $L$ in $\mathfrak{L}$ which is contained in the vase and intersects extended $f$ in at least two distinct points. Such a line $L$, together with two distinct selected points $P_1$, $P_2$, in which $L$ intersects extended $f$, will be called a two-point line with anchor points, $P_1$, and $P_2$. Note that every two-point line is also a one-point line.

A two-point line with anchor points $(t_1, \epsilon_1 f(t_1))$ and $(t_2, \epsilon_2 f(t_2))$, $0 \leq t_1 < t_2 \leq 1$, $\epsilon_i = \pm 1$, will be called a $(+, +)$ line if $(\epsilon_1, \epsilon_2) = (1, 1)$; $(+, -)$ line if $(\epsilon_1, \epsilon_2) = (1, -1)$; etc. In Figs. 1-4, line $L$ is a $(-, +)$ line; line $L'$ is a $(+, -)$ line which, in fact, is the reflection of line $L$. It is geometrically evident (see Figs. 1-4) that the vase generated by $f(t)$ contains a unique $(-, +)$ line $L$ and $(+, -)$ line $L'$ which are reflections of one another in the $t$ axis. The lines $L$ and $L'$ considered together form an inscribed cone of the vase and have common anchor points with abscissas $t_1$, $t_2$. The line pair $L$ and $L'$ will be referred to as the inscribed cone; the interval $(t_1, t_2)$ will be called the cone interval. Note that the cone interval may be the "whole" interval (0, 1) as in Fig. 4.

We shall now prove the aforementioned result. Recall that

$$\int \theta \, d\nu = \alpha, \tag{6}$$

$$\int t\theta \, d\nu = \beta, \tag{7}$$

and we wish to choose $\theta(|\theta| = 1)$ and $\nu(\nu \geq 0)$ such that (6), (7) are satisfied and $\int f \, d\nu$ is a minimum.

Assume $\alpha > 0$. For any $t_1$, $t_2$ such that $0 \leq t_1 < t_2 \leq 1$, $t_i \neq \beta/\alpha \equiv \gamma$, set

$$\nu_1' = \frac{|t_2\alpha - \beta|}{t_2 - t_1}, \tag{8}$$

$$\nu_2' = \frac{|\beta - t_1\alpha|}{t_2 - t_1}, \tag{9}$$

$$\theta_1' = \frac{t_2\alpha - \beta}{|t_2\alpha - \beta|}, \tag{10}$$

$$\theta_2' = \frac{\beta - t_1\alpha}{|\beta - t_1\alpha|}, \tag{11}$$

If $\nu'$ is the two-point measure defined by $\nu_1'$ and $\nu_2'$ (i.e., $\nu'\{t_1\} = \nu_1'$, $\nu'\{t_2\} = \nu_2'$, and $\nu'(S) = \delta_{t1}\nu_1' + \delta_{t2}\nu_2'$ where $\delta_{ti} = 1$ if $t_i \epsilon S$ and 0 otherwise) and $\theta'$ is such that $\theta'(t_1) = \theta_1'$, $\theta'(t_2) = \theta_2'$, $\theta'(t)$ arbitrary for $t \neq t_1, t_2$, then $\theta'$, $\nu'$ satisfy (6), (7). Let

$$f(t_1) \equiv f_1, \qquad f(t_2) \equiv f_2, \qquad \epsilon_1' \equiv \theta_1'^{-1}, \qquad \epsilon_2' \equiv \theta_2'^{-1}.$$

Then

$$\begin{aligned} \int f\,d\nu' &= f_1\nu_1' + f_2\nu_2' \\ &= f_1\epsilon_1'\left(\frac{t_2\alpha - \beta}{t_2 - t_1}\right) + f_2\epsilon_2'\left(\frac{\beta - t_1\alpha}{t_2 - t_1}\right) \\ &= \epsilon_1'\alpha f_1 + \left(\frac{\epsilon_2' f_2 - \epsilon_1' f_1}{t_2 - t_1}\right)(\beta - t_1\alpha). \end{aligned} \tag{12}$$

Now for arbitrary $\theta$, ($|\theta| = 1$) and $\nu \geq 0$ satisfying (6) and (7) we have

$$\int f\,d\nu' = \int \theta\left[\epsilon_1' f_1 + \left(\frac{\epsilon_2' f_2 - \epsilon_1' f_1}{t_2 - t_1}\right)(t - t_1)\right] d\nu = \int \theta L_t\,d\nu,$$

where $L_t$ is the ordinate of the line $L$ joining $(t_1, \epsilon_1' f_1)$ and $(t_2, \epsilon_2' f_2)$. We can now observe that $\int f\,d\nu \geq \int f\,d\nu' = \int \theta L_t\,d\nu$ if $f \geq \theta L_t$ for all $\theta$. But $f \geq \theta L_t$ for all $\theta$ means $L$ is contained in the "vase."

In terms of the previously discussed geometric terminology, $\nu'$ and $\theta'$ provide the answer if $L$ is a two-point line; note that if both $t_1$ and $t_2$ are $< \gamma$, then relations (10) and (11) imply that $L$ must be a $(-, +)$ line. Similarly, if $t_1 < \gamma < t_2$ then $(+, +)$; if $\gamma < t_1 < t_2$ then $(+, -)$. Thus, $t_1$ and $t_2$ yield a $\nu'$ and $\theta'$ which solve the problem if $L$ is a two-point line satisfying the above $(\pm, \pm)$ relations. In addition, the minimum value of $\int f\,d\nu$ is $\alpha L_t(\gamma)$. (See (12)).

Let $(t_1, t_2)$ be the cone interval for $f(t)$ (see previous geometrical discussion). Let $L, L'$ be the inscribed cone (Figs. 1-4). It is now clear that if $\gamma \geq t_2$ then $t_1$ and $t_2$ suffice* and $L$ is the associated $(-, +)$ two-point line; if $\gamma \leq t_1$, then $t_1$ and $t_2$ suffice and $L'$ is the associated $(+, -)$ line. If $t_1 < \gamma < t_2$, then $t_1$ and $t_2$ no longer suffice, since neither $L$ nor $L'$ is a $(+, +)$ line.

We may thus restrict the remaining portion of the proof to the case where $\gamma$ is properly inside the cone interval. In this case, $t_1$ and $t_2$ can suffice if the line joining $(t_1, f(t_1))$ and $(t_2, f(t_2))$ is a two-point $(+, +)$ line. This may occur (c.f. Fig. 4), and if it does, $t_1$ and $t_2$ suffice, and the problem is solved. We shall henceforth assume this line is not a two-point $(+, +)$ line. It is now convenient to discuss the two cases which may arise separately.

In the first case, there does exist a $(+, +)$ line with anchor points $t_3, t_4$, which is admissible relative to $\gamma$; i.e., $t_3 \leq \gamma \leq t_4$. In this case, $t_3$ and $t_4$ suffice. Fig. 5 illustrates this case, provided $t_3 \leq \gamma \leq t_4$; $L''$ is the associated $(+, +)$ line.

In the second case, there exists no $(+, +)$ line which is admissible relative to $\gamma$. For example, in Fig. 5, if $t_4 < \gamma < t_2$, then the only $(+, +)$ line $L''$ is not admissible, for both its anchor points are to the left of $\gamma$, and $L''$ would have to be $(-, +)$ to be

*We use the expression "$t_1$ and $t_2$ suffice" to indicate that the corresponding $\nu'$ and $\theta'$ defined above [See (8)–(11)] solve the problem. The case when one of the $t_i$ of the cone interval is equal to $\gamma$ follows in a similar manner; $\nu'$ then becomes a one-point measure concentrated at $t_i = \gamma$.

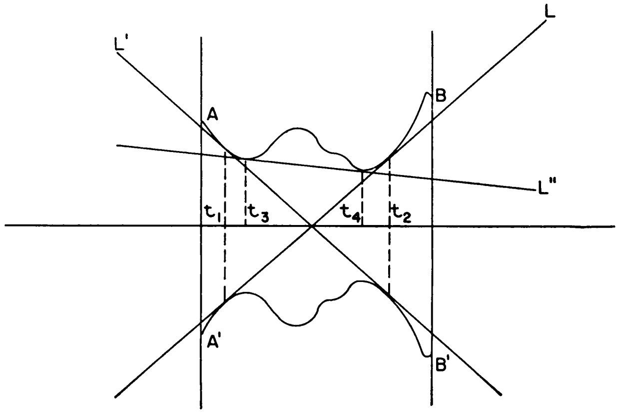


FIG. 5.

admissible. Also, in Fig. 1, if $t_1 < \gamma < t_2$, then there exists no admissible two-point line. It is geometrically evident that in this case the "tangent line" to $f$ at $\gamma$ is a one-point line. In fact, the one-point measure concentrated at $\gamma$ then provides the answer. Its value at $\gamma$ is $|\alpha|$ and the minimum value of $\int f\,d\nu$ is $|\alpha|\,f(\gamma)$, as a simple calculation shows.

The proof in the case $\alpha < 0$ is similar and will be omitted.

If $\alpha = 0$ and $\beta = 0$, then $\nu \equiv 0$ obviously solves the problem. If $\alpha = 0$, $\beta \neq 0$, and $0 \leq t_1 < t_2 \leq 1$, set

$$\nu_1' = \nu_2' = \frac{|\beta|}{t_2 - t_1}, \tag{13}$$

$$\theta_2' = -\theta_1' = \frac{\beta}{|\beta|}. \tag{14}$$

Then, as before, $\theta'$, $\nu'$ satisfy (6), (7). Also

$$\int f\,d\nu' = \left[\frac{f_2 - (-f_1)}{t_2 - t_1}\right]|\beta| = B\,|\beta|, \tag{15}$$

where $B$ is the slope of the line joining $(t_1, -f_1)$ and $(t_2, f_2)$. The minimum value $B^*$ of $B$ is the slope of the $(-, +)$ line $L$ of the inscribed cone. To show that $\int f\,d\nu \geq B^*\,|\beta|$, let $L$ be given by the function $L_t = A^* + B^*t$. Then $\int f\,d\nu \geq \int L_t\theta\,d\nu$ for all $\theta$, $\nu$ since $L$ is a two-point line. But

$$\int L_t\theta\,d\nu = \int (A^* + B^*t)\theta\,d\nu = A^*\int \theta\,d\nu + B^*\int t\theta\,d\nu = B^*\beta.$$

Replacing $L$ by the other inscribed cone line (which, in fact, is defined by the function $-L_t$), we get $\int -L_t\,\theta d\nu = -B^*\beta$. Thus $\int f\,d\nu \geq B^*\,|\beta|$.

This same result could have been obtained by taking the limit of the previous solution as $\alpha \to 0$. Thus, if $\beta > 0$ then

$$\left(\int f\,d\nu\right)_{\min} = \lim_{\alpha\to 0}\,[\alpha(A^* + B^*\beta\alpha^{-1})] = \lim_{\alpha\to 0}\,(\alpha A^* + \beta B^*) = \beta B^*.$$

A more detailed examination of a few examples may help to clarify the previous remarks. We assume $\alpha > 0$ and $\gamma \equiv \beta\alpha^{-1}$. We also use the symbol $(t_1, t_2, L)$ in this section to mean "the measure is concentrated at the abscissa pair $t_1, t_2$ according to formula (8), (9), and the minimum value of $\int f\, d\nu$ is given by $\alpha L_t(\gamma)$, where $L_t(\gamma)$ is the ordinate of the line $L$ at abscissa $\gamma$." Also, the symbol $(\gamma, \gamma, \text{TAN})$ shall mean "the measure is concentrated wholly at $\gamma$ and the minimum value of $\int f\, d\nu$ is $\alpha f(\gamma)$ where TAN refers to the tangent to $f$ at $\gamma$."

In Figs. 1-5, if $\gamma < t_1$, then $(t_1, t_2, L')$; if $\gamma > t_2$, then $(t_1, t_2, L)$. In Fig. 1, if $t_1 \leq \gamma \leq t_2$, then $(\gamma, \gamma, \text{TAN})$. In Fig. 5 (which is a more detailed Figure 2), if $t_1 \leq \gamma \leq t_3$ or $t_4 \leq \gamma \leq t_2$, then $(\gamma, \gamma, \text{TAN})$; if $t_3 < \gamma < t_4$, then $(t_3, t_4, L'')$.

We shall now consider the effect upon the problem and its solution of requiring that $\theta(t) = 1$. In this case, the problem may be formulated as: Given $f(t) > 0$, $\alpha$, $\beta$, $0 \leq \beta \leq \alpha$, find $\nu(t)$ such that $\nu \geq 0$, $\int d\nu = \alpha$, $\int t\, d\nu = \beta$, and $\int f\, d\nu$ is minimized.

Since $\beta = \int_0^1 t\, d\nu$ and $\nu \geq 0$, it is clear that $\beta \geq 0$. Since $t \leq 1$, $\beta = \int t\, d\nu \leq \int d\nu = \alpha$ so that $\beta \leq \alpha$. This explains the initial constraint $0 \leq \beta \leq \alpha$. If $\alpha = 0$, then $\beta = 0$ and $\nu \equiv 0$ is clearly the solution. We therefore assume $\alpha > 0$ and let $\gamma \equiv \beta\alpha^{-1}$. Then $0 \leq \gamma \leq 1$. The solution in this case follows the same pattern as in the previous theorem, except that only $(+, +)$ two-point lines are admissible; the one-point lines play a similar role.

An example might help clarify the effect of this requirement ($\theta = 1$). For $f$ as in Fig. 5, and $t_1 \leq \gamma \leq t_2$ there is no difference in solution; but for $\gamma < t_1$ or $t_2 < \gamma$, the tangent line must be used in the present case, yielding a higher value of the minimum $\int f\, d\nu$ than the lines $L'$ and $L$, respectively, which were admissible in the old case, but not now.

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