SAINT-VENANT'S PROCEDURE AND SAINT-VENANT'S PRINCIPLE*

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1. Introduction. A procedure frequently employed in engineering applications of linear elasticity theory can be described as follows: The forces, $\mathbf{T}'$, acting on a small part, Γ_ϵ, of the surface, Γ, of a body are replaced by statically equivalent forces, $\mathbf{T}''$. These forces also act on Γ_ϵ but are distributed there in a manner which facilitates the solution of the appropriate equilibrium problem. Then the effects, $\mathcal{E}''$, (i.e. strains, displacements and stresses) for the modified problem are used in place of the undetermined effects, $\mathcal{E}'$, of the original problem at all points not too near Γ_ϵ. We call this, and closely related methods,[1] Saint-Venant's *procedure* since it was first employed in his classic memoir on torsion [4]. By means of the superposition principle of linear elasticity it follows that the validity of Saint-Venant's procedure is equivalent to the validity of the assertion that: *The effects, $\mathcal{E} \equiv \mathcal{E}' - \mathcal{E}''$, produced by the self-equilibrated forces, $\mathbf{T} = \mathbf{T}' - \mathbf{T}''$, acting on Γ_ϵ (with zero forces or displacements on appropriate parts of $\Gamma - \Gamma_\epsilon$) are negligible compared to the effects $\mathcal{E}''$ (at points not too near Γ_ϵ).*

If ϵ is a "typical" linear dimension of Γ_ϵ it is possible to obtain estimates of the effects $\mathcal{E}$ in the form

$$\mathcal{E} = \mathcal{O}(f(\epsilon)). \tag{1.0}$$

It trivially follows that

$$\mathcal{E}' = \mathcal{E}'' + \mathcal{O}(f(\epsilon)), \tag{1.1}$$

and hence if the term $\mathcal{O}(f(\epsilon))$ is "small" compared to $\mathcal{E}''$ then Saint-Venant's procedure is valid. Since, in applying the procedure, the effects $\mathcal{E}''$ are determined it is in principle possible to make the indicated comparison. Estimates of the form (1.0) were first obtained by Sternberg [5] and used for a slightly different purpose. We employ the same techniques to obtain similar estimates of the form:

$$\mathcal{E} = \mathcal{O}(T(\epsilon)A_\epsilon\epsilon^p). \tag{1.2}$$

Here $T(\epsilon)$ is the maximum magnitude of the load, A_ϵ is the area of Γ_ϵ and $p = 0$, 1 or 2 depending upon the conditions satisfied by $\mathbf{T}$. Under reasonable assumptions we determine the *necessary and sufficient* conditions which $\mathbf{T}$ must satisfy in order that $p = 1$ and 2. These results apply for surface portions Γ_ϵ which are neighborhoods of a point or of a straight "edge" on the body. Estimates of the effects produced by the application of several such loads on various small parts of the body surface are easily obtained from the given estimates.

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[1]Say a procedure in which $\mathbf{T}'$ and $\mathbf{T}''$ are not statically equivalent but have the same resultant force (see Section 5).

Since these considerations are related to what is called "Saint-Venant's principle" we conclude with a discussion of this principle.

2. Integral representations of effects. Let an elastic body with Lame constants λ and μ occupy a region D with piecewise smooth boundary Γ. In the absence of body forces we assume applied surface tractions, $\mathbf{T}(\mathbf{x})$, to be specified on some part, Γ_1, of Γ and the displacements, $\mathbf{U}(\mathbf{x})$ to be specified on the remainder of the surface, $\Gamma_2 = \Gamma - \Gamma_1$. The effects (i.e. displacement, strain, etc.) produced by the specified tractions and displacements can be represented at every point $\mathbf{y} \varepsilon D$ by means of appropriate fundamental solutions of the equilibrium equations. For example the dilatation $\theta(\mathbf{y})$ is given by the formula:

$$\theta(\mathbf{y}) = \frac{1}{c} \iint_{\Gamma_1} G_i(\mathbf{x}, \mathbf{y}) T_i(\mathbf{x})\, dS + \frac{1}{c} \iint_{\Gamma_2} S_{ij}(\mathbf{x}, \mathbf{y}) n_j(\mathbf{x}) U_i(\mathbf{x})\, dS. \tag{2.0}$$

Here $\mathbf{G}$ is the fundamental solution defined below, the S_{ij} are defined in (2.3) (as the components of the stress computed from the displacement field given by $\mathbf{G}$), $\mathbf{n}(\mathbf{x})$ is the outer unit normal to Γ, $c = \pi(\lambda + 2\mu)$, and the summation convention has been employed.

The vector $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is required to have a singularity specified by

$$\mathbf{G}(\mathbf{x}, \mathbf{y}) = -\frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{5/2}} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{2.1}$$

where $\mathbf{g}$ is twice continuously differentiable for $\mathbf{x}$ in any neighborhood of $\mathbf{y}$. The equilibrium equations to be satisfied by the components of $\mathbf{G}$ can be written as:

$$S_{ij,j}(\mathbf{x}, \mathbf{y}) = 0, \qquad \mathbf{x} \neq \mathbf{y}, \qquad i = 1, 2, 3, \qquad \mathbf{x}, \mathbf{y} \varepsilon D; \tag{2.2}$$

where we have introduced

$$S_{ij} \equiv \lambda G_{k,k} \delta_{ij} + \mu(G_{i,j} + G_{j,i}). \tag{2.3}$$

All differentiations, indicated by the usual indicial notation, are with respect to the components of $\mathbf{x}$. The appropriate boundary conditions on $\mathbf{G}$ are:

$$\left.\begin{array}{lll} \text{(a)} & S_{ij}(\mathbf{x}, \mathbf{y}) n_j(\mathbf{x}) = 0, & \mathbf{x} \varepsilon \Gamma_1 \\ \text{(b)} & G_i(\mathbf{x}, \mathbf{y}) = 0, & \mathbf{x} \varepsilon \Gamma_2 \end{array}\right\} i = 1, 2, 3. \tag{2.4}$$

It is easily verified, as in Love [2; pp. 233–236], that the formula (2.0) is valid provided $\mathbf{G}$ satisfies the above conditions.

If $\Gamma_2 \equiv 0$ then $\mathbf{G}$ is the Betti vector for the second boundary value problem in elasticity. It is nonunique, in this case, to within an arbitrary rigid body motion.

Formulas exactly analogous to (2.0) can be derived for the components of displacement, $u_i(\mathbf{y})$, and of strain, $e_{ij}(\mathbf{y})$. The fundamental solutions appropriate to each of these quantities have singularities different from (2.1) but the boundary conditions (2.4) are unaltered (see Love [2; pp. 245–247]).

3. Loads in the neighborhood of a point. We consider the case in which $\mathbf{U}(\mathbf{x}) \equiv 0$ for all $\mathbf{x} \varepsilon \Gamma_2$ and $\mathbf{T}(\mathbf{x}) \equiv 0$ for all $\mathbf{x} \varepsilon \Gamma_1 - \Gamma_\epsilon$, where Γ_ϵ is a small portion of Γ_1 in the neighborhood of some particular point on Γ_1. The dilitation formula (2.0) then reduces to

$$\theta(\mathbf{y}) = \frac{1}{c} \iint_{\Gamma_\epsilon} G_i(\mathbf{x}, \mathbf{y}) T_i(\mathbf{x}) \, dS. \tag{3.0}$$

Without loss of generality we take the origin to be the specified point on Γ_1 (assumed a regular point of the surface), the positive x_3-axis in the direction of the outward normal to Γ and hence the (x_1, x_2)-plane is the tangent plane to Γ at $\mathbf{x} = \mathbf{0}$. In a neighborhood of the origin the surface can be represented by

$$\Gamma : x_3 = f(x_1, x_2) = ax_1^2 + bx_2^2 + \cdots, \tag{3.1}$$

provided the x_1- and x_2-axis are taken parallel to the directions of principle curvature.[2] Here $2a$ and $2b$ are the principle curvatures of Γ at $\mathbf{x} = \mathbf{0}$. For any ϵ in $0 < \epsilon < \epsilon_0$ we now define the small portion, Γ_ϵ, of Γ_1 by

$$\Gamma_\epsilon \equiv \{\mathbf{x} \mid x_3 = f(x_1, x_2), x_1^2 + x_2^2 \leq \epsilon^2\}. \tag{3.2}$$

The area of Γ_ϵ is

$$A_\epsilon = \iint_{\Gamma_\epsilon} dS = \iint_{x_1^2 + x_2^2 \leq \epsilon^2} [1 + f_{,1}^2(x_1, x_2) + f_{,2}^2(x_1, x_2)]^{1/2} \, dx_1 \, dx_2 = \pi\epsilon^2 + O(\epsilon^4). \tag{3.3}$$

The outer unit normal to Γ at the origin is

$$\mathbf{n}(\mathbf{0}) = (0, 0, 1). \tag{3.4}$$

The surface tractions applied on Γ_ϵ can be specified for each ϵ in $0 < \epsilon < \epsilon_0$ as

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}(x_1, x_2 : \epsilon); \qquad \mathbf{x} \in \Gamma_\epsilon \tag{3.5}$$

We introduce, for each such load, the quantities:

$$\begin{aligned} \mathbf{F}(\epsilon) &\equiv \iint_{\Gamma_\epsilon} \mathbf{T}(\mathbf{x}) \, dS, \qquad \mathbf{M}(\epsilon) \equiv \iint_{\Gamma_\epsilon} \mathbf{x} \times \mathbf{T}(\mathbf{x}) \, dS, \\ W_{ij}(\epsilon) &\equiv \iint_{\Gamma_\epsilon} x_i T_j(\mathbf{k}) \, dS, \qquad T(\epsilon) \equiv \sup_{\mathbf{x} \in \Gamma_\epsilon} |\mathbf{T}(\mathbf{x})|. \end{aligned} \tag{3.6}$$

Let $\delta > \epsilon_0$ be a fixed number such that $|\mathbf{x}| < \delta$ if $\mathbf{x} \in \Gamma_{\epsilon_0}$: Then we define D_δ, the interior of the body with the exclusion of a small spherical part at the origin, by:

$$D_\delta \equiv \{\mathbf{y} \mid \mathbf{y} \in D, |\mathbf{y}| > \delta\}. \tag{3.7}$$

Assuming $\mathbf{G}(\mathbf{x}, \mathbf{y})$ to have continuous second derivatives for $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x}$ on Γ_{ϵ_0} it can be shown by means of Taylor's Theorem that

$$G_i(\mathbf{x}, \mathbf{y}) = G_i^0(\mathbf{y}) + x_j G_{i,j}^0(\mathbf{y}) + R_i(\mathbf{x}, \mathbf{y}); \qquad \mathbf{x} \in \Gamma_\epsilon, \mathbf{y} \in D_\delta. \tag{3.8}$$

Here the superscript zero indicates evaluation at $\mathbf{x} = \mathbf{0}$. The remainder terms in the expansions (3.8) can be bounded by

$$|R_i(\mathbf{x}, \mathbf{y})| \leq M\epsilon^2, \qquad \mathbf{x} \in \Gamma_\epsilon, \mathbf{y}_\epsilon \in D_\delta. \tag{3.9}$$

[2]This choice of axes is not required in the present section. An arbitrary rotation about the x_3-axis merely replaces the right hand side of (3.1) by $f(x_1, x_2) = a'x_1^2 + b'x_1x_2 + c'x_2^2 + \cdots$. However, the specific form in (3.1) is required in Section 4.

The constant M depends only upon bounds on the first[3] and second derivatives of the $G_i(\mathbf{x}, \mathbf{y})$ and the first and second derivatives of $f(x_1, x_2)$ for $\mathbf{x} \varepsilon \Gamma_{\epsilon_0}$ and $\mathbf{y} \varepsilon D_\delta$. Thus M depends implicitly on ϵ_0 and δ but is independent of ϵ for all $0 < \epsilon < \epsilon_0$.

Using (3.6) and (3.8) in (3.0) the dilatation becomes

$$c\theta(\mathbf{y}) = G_i^0(\mathbf{y})F_i(\epsilon) + G_{i,j}^0(\mathbf{y})W_{ji}(\epsilon) + \iint_{\Gamma_\epsilon} R_i(\mathbf{x}, \mathbf{y})T_i(\mathbf{x})\, dS; \qquad \mathbf{y} \varepsilon D_\delta . \tag{3.10}$$

Recalling the boundary conditions (2.4a) and the choice of coordinates such that (3.4) applies we find that

$$\begin{aligned} G_{\nu,3}^0(\mathbf{y}) + G_{3,\nu}^0(\mathbf{y}) &= 0; \qquad \nu = 1, 2; \\ (\lambda + 2\mu)G_{3,3}^0(\mathbf{y}) + \lambda[G_{1,1}^0(\mathbf{y}) + G_{2,2}^0(\mathbf{y})] &= 0. \end{aligned} \tag{3.11}$$

Writing the components of $\mathbf{M}(\epsilon)$ in terms of the $W_{ij}(\epsilon)$ it follows from (3.11) in (3.10) that

$$\begin{aligned} c\theta(\mathbf{y}) = {} & G_i^0(\mathbf{y})F_i(\epsilon) + G_{2,3}^0(\mathbf{y})M_1(\epsilon) + G_{1,3}^0(\mathbf{y})M_2(\epsilon) \\ & + G_{\nu,\mu}^0(\mathbf{y})W_{\mu\nu}(\epsilon) - \frac{\lambda}{\lambda + 2\mu} G_{\nu,\nu}^0(\mathbf{y})W_{33}(\epsilon) + \iint_{\Gamma_\epsilon} R_i(\mathbf{x}, \mathbf{y})T_i(\mathbf{x})\, dS; \quad \mathbf{y} \varepsilon D_\delta . \end{aligned} \tag{3.12}$$

Here Greek subscripts ν and μ take only the values 1 and 2.

From (3.1) and (3.2) it follows that on $\Gamma_\epsilon : x_3 = ax_1^2 + bx_2^2 + \mathcal{O}(\epsilon^3)$.
Recalling (3.3) and (3.6) we obtain the estimates, valid in general,

$$F_i(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^2), \qquad W_{\nu j}(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^3), \qquad W_{3j}(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^4). \tag{3.13}$$

Using these estimates and (3.9) in (3.12) we conclude that, for all $\mathbf{y} \varepsilon D_\delta$:

$$\begin{aligned} &\text{(a)} \quad c\theta(\mathbf{y}) = G_i^0(\mathbf{y})F_i(\epsilon) + G_{2,3}^0(\mathbf{y})M_1(\epsilon) + G_{1,3}^0(\mathbf{y})M_2(\epsilon) + G_{\nu,\mu}^0(\mathbf{y})W_{\mu\nu}(\epsilon) + \mathcal{O}(T(\epsilon)\epsilon^4); \\ &\text{(b)} \quad c\theta(\mathbf{y}) = G_i^0(\mathbf{y})F_i(\epsilon) + \mathcal{O}(T(\epsilon)\epsilon^3); \\ &\text{(c)} \quad \theta(\mathbf{y}) = \mathcal{O}(T(\epsilon)\epsilon^2). \end{aligned} \tag{3.14}$$

With no additional restrictions on the applied surface tractions (3.14c) is the best (i.e. highest order in ϵ) general estimate that can be obtained. This may be concluded by considering the special load $\mathbf{T}(x_1, x_2; \epsilon) \equiv (1, 1, 0)$.

If the *resultant of the applied forces vanishes* i.e. $\mathbf{F}(\epsilon)$ 0, then from (3.14b) it follows that

$$\theta(\mathbf{y}) = \mathcal{O}(T(\epsilon)\epsilon^3), \qquad \mathbf{y} \varepsilon D_\delta . \tag{3.15}$$

It is in fact sufficient for the validity of this estimate that the resultant force satisfies

$$F_i(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^3); \qquad i = 1, 2, 3. \tag{3.16}$$

These conditions are also necessary if the three functions $G_i^0(\mathbf{y})$ are linearly independent. Note that if the $G_i^0(\mathbf{y})$ are dependent then the vectors $\mathbf{G}^0(\mathbf{y})$ must lie in the same plane for all $\mathbf{y} \varepsilon D_\delta$. This is impossible for any body, as follows from (2.1), if $|\mathbf{y}|$ is sufficiently

[3]In deriving (3.8) we cannot expand $G_i(t\mathbf{x}, \mathbf{y})$ about $t = 0$ for $\mathbf{x} \varepsilon \Gamma_\epsilon$ since all points on the line segment joining $\mathbf{0}$ and $\mathbf{x}$ need not lie in D or on Γ_ϵ. The expansion is taken about $x_1 = x_2 = 0$ with $x_3 = f(x_1, x_2)$ and the terms $\pm x_3 G_{i,3}^0(\mathbf{y})$ are added to the result.

small. However, since $\mathbf{y} \in D_\delta$ implies $|\mathbf{y}| > \delta$ the above argument is inconclusive in general. We conjecture that (3.16) *are the necessary and sufficient conditions for the validity of* (3.15) if D_δ has positive volume.[4] This result has been verified for many specific bodies for which $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is known explicitly (for the second boundary value problem, $\Gamma_2 = 0$).

A consideration of the special load $\mathbf{T}(x_1, x_2; \epsilon) = (0, 0, x_1)$ proves that the estimate (3.15) is the best that can be obtained for all loads satisfying (3.16).

If the forces applied on Γ_ϵ are *self-equilibrated*, i.e. $\mathbf{F}(\epsilon) = \mathbf{M}(\epsilon) = \mathbf{0}$, then (3.14a) reduces to

$$c\theta(\mathbf{y}) = G^0_{\nu,\mu}(\mathbf{y}) W_{\mu\nu}(\epsilon) + \mathcal{O}(T(\epsilon)\epsilon^4), \qquad y \in D_\delta . \tag{3.17}$$

The same expression results if we only require that

$$\begin{aligned} F_i(\epsilon) &= \mathcal{O}(T(\epsilon)\epsilon^4), \\ M_i(\epsilon) &= \mathcal{O}(T(\epsilon)\epsilon^4); \qquad i = 1, 2, 3. \end{aligned} \tag{3.18a}$$

Of course the estimate (3.15) still applies and is the best that can be concluded under the above conditions. The special load $\mathbf{T}(x_1, x_2; \epsilon) \equiv (x_1, x_2, 0)$ can be used to verify this fact. However, if in addition to (3.18a) we require the load to satisfy

$$W_{\nu\mu}(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^4), \qquad \nu, \mu = 1, 2 \tag{3.18b}$$

then it follows from (3.17) that

$$\theta(\mathbf{y}) = \mathcal{O}(T(\epsilon)\epsilon^4), \qquad \mathbf{y} \in D_\delta . \tag{3.19}$$

Furthermore if the nine functions $G^0_i(\mathbf{y})$, $G^0_{\nu,i}(\mathbf{y})$ are linearly independent functions (for $\mathbf{y} \in D_\delta$) then *conditions* (3.18a, b) *are the necessary and sufficient conditions for the validity of the estimate* (3.19). It should be noted that since $M_3 = W_{12} - W_{21}$ there are only nine independent conditions in (3.18).

Recalling (3.2) and (3.3) it follows that (3.18b) is satisfied if

$$\iint_{x_1^2+x_2^2 \le \epsilon^2} x_\nu T_\mu(x_1, x_2; \epsilon)\, dx_1\, dx_2 = 0; \qquad \nu, \mu = 1, 2. \tag{3.20}$$

These conditions are fullfilled, for instance, if $T_\mu(x_1, x_2; \epsilon)$, $\mu = 1, 2$, are even functions of x_1 and x_2. However, since all of our results remain valid for any rotation of the (x_1, x_2)-plane about the x_3-axis we may conclude that: *the estimate* (3.19) *is valid for any self-equilibrated load* (3.5) *whose tangential components are symmetric with respect to at least one pair of orthogonal tangent lines at the origin.*

A limited number of sufficient conditions for the validity of the estimate (3.19) when $\mathbf{T}(\mathbf{x})$ consists of point loads have been stated by von Mises [3]. All of these are special cases of the statement italicized above. Sternberg [5] has derived a number of conditions, which include and generalize those of von Mises, for which (3.19) is valid at a fixed point $\mathbf{y} \in D_\delta$. All of his conditions are included in (3.18) (as they must be if these are really necessary) and some are special cases of the italicized statement.

[4]The interpretation of $\mathbf{G}^0(\mathbf{y})$ as the displacement at $\mathbf{x} = \mathbf{0}$ caused by a point source of dilitation located at $\mathbf{y}$ makes this conjecture seem plausible. As the point source moves throughout some volume it seems unlikely that an *unconstrained* point on the surface would remain in a fixed plane.

Both von Mises and Sternberg devote special attention to self-equilibrated loads which are parallel, that is the special case of (3.5):

$$\mathbf{T}(\mathbf{x}) = \mathbf{k}t(x_1, x_2; \epsilon), \qquad |\mathbf{k}| = 1. \tag{3.21}$$

We examine such loads in the Appendix and show that no new results are obtained.

4. Loads in the neighborhood of an "edge". We now consider bodies, D, whose surface Γ, has a tangent plane along a line segment, E, which we call an edge. Without loss of generality the coordinate system can be chosen such that, if $2L$ is the length of the edge,

$$E \equiv \{\mathbf{x} \mid x_1 = 0, |x_2| \le L, x_3 = 0\}, \tag{4.0}$$

and the x_3-axis is normal to Γ. The surface in the neighborhood of E is then represented by (3.1) with $b = 0$. For each h in $0 < h < h_0$ we define a neighborhood of the edge, Γ_h, by

$$\Gamma_h = \{\mathbf{x} \mid x_3 = f(x_1, x_2), |x_1| \le h, |x_2| \le L\}. \tag{4.1}$$

The area of Γ_h is

$$A_h = 4Lh + \mathcal{O}(h^3), \tag{4.2}$$

and the unit normal to Γ_h along the edge is

$$\mathbf{n}(\mathbf{x}) = (0, 0, 1), \qquad \mathbf{x} \in E. \tag{4.3}$$

If we again take $\mathbf{U}(\mathbf{x}) = 0$ on Γ_2 and $\mathbf{T}(\mathbf{x}) = 0$ on $\Gamma_1 - \Gamma_h$ where $\Gamma_h \subseteq \Gamma_1$, then the dilatation formula (3.0) applies with Γ_ϵ replaced by Γ_h. However for each Γ_h we shall require that the load be uniformly distributed along its length; that is

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}(x_1; h); \qquad \mathbf{x} \in \Gamma_h. \tag{4.4}$$

In analogy with D_δ we define D_Δ as the interior of D excluding a cylinder of radius Δ about E. Here $\Delta > h_0$ is such that if $\mathbf{y} \in \Gamma_{h_0}$ then $|\mathbf{y}| < \Delta$. Again, employing Taylor's Theorem it can be shown that:

$$G_i(\mathbf{x}, \mathbf{y}) = G_i^0(x_2, \mathbf{y}) + x_\xi G_{i,\xi}^0(x_2, \mathbf{y}) + \mathcal{O}(h^2); \qquad \mathbf{x} \in \Gamma_h, \mathbf{y} \in D_\Delta. \tag{4.5}$$

Here and throughout this section the subscript $\xi = 1, 3$, and the superscript zero indicates evaluation on E.

We now introduce the quantities:

$$f_i(h) \equiv \int_{-h}^{h} T_i(x_1; h)\, dx_1, \qquad w_{\xi i}(h) \equiv \int_{-h}^{h} x_\xi T_i(x_1; h)\, dx_1,$$

$$g_i(\mathbf{y}) \equiv \int_{-L}^{L} G_i^0(x_2; y)\, dx_2, \qquad g_{i\xi}(\mathbf{y}) \equiv \int_{-L}^{L} G_{i,\xi}^0(x_2, \mathbf{y})\, dx_2, \tag{4.6}$$

$$T(h) = \sup_{|x_1| \le h} |\mathbf{T}(x_1; h)|.$$

Using this notation, (4.4) and (4.5) in the appropriate form of (3.0) we obtain

$$c\theta(\mathbf{y}) = g_i(\mathbf{y}) f_i(h) + g_{i\xi}(\mathbf{y}) w_{\xi i}(h) + \mathcal{O}(T(h)h^3), \qquad \mathbf{y} \in D_\Delta. \tag{4.7}$$

The boundary conditions (2.4a) evaluated on E, with normal given by (4.3), may be integrated along E to yield:

$$g_{i3}(\mathbf{y}) + g_{3i}(\mathbf{y}) = 0. \tag{4.8}$$

It also follows from (4.6) that in general:

$$f_i(h) = \mathcal{O}(T(h)h), \qquad w_{ij}(h) = \mathcal{O}(T(h)h^2), \qquad w_{3j}(h) = \mathcal{O}(T(h)h^3). \tag{4.9}$$

Using these results and (4.8) in (4.7), with the definition

$$m_2(h) \equiv w_{31}(h) - w_{13}(h), \tag{4.10}$$

we obtain for all $\mathbf{y} \in D_\Delta$:

$$\begin{aligned} &\text{(a)} \quad c\theta(\mathbf{y}) = g_i(\mathbf{y})f_i(h) + g_{13}(\mathbf{y})m_2(h) + g_{\nu 1}(\mathbf{y})w_{1\nu}(h) + \mathcal{O}(T(h)h^3); \\ &\text{(b)} \quad c\theta(\mathbf{y}) = g_i(\mathbf{y})f_i(h) + \mathcal{O}(T(h)h^2); \\ &\text{(c)} \quad \theta(\mathbf{y}) = \mathcal{O}(T(h)h). \end{aligned} \tag{4.11}$$

Thus for all "edge" loads of the form (4.4) the estimate (4.11c) is valid. If in addition these loads satisfy

$$f_i(h) = \mathcal{O}(T(h)h^2), \qquad i = 1, 2, 3; \tag{4.12}$$

then it follows from (4.11b) that

$$\theta(\mathbf{y}) = \mathcal{O}(T(h)h^2), \qquad \mathbf{y} \in D_\Delta . \tag{4.13}$$

Conditions (4.12) *are necessary and sufficient conditions for the validity of* (4.13) if the functions $g_i(\mathbf{y})$ for $\mathbf{y} \in D_\Delta$ are linearly independent.

If the edge load (4.4) is such that

$$f_i(h) = \mathcal{O}(T(h)h^3), \qquad m_2(h) = \mathcal{O}(T(h)h^3), \tag{4.14a}$$

Then (4.11a) reduces to

$$c\theta(\mathbf{y}) = g_{\nu 1}(\mathbf{y})w_{1\nu}(h) + \mathcal{O}(T(h)h^3), \qquad \mathbf{y} \in D_\Delta .$$

The estimate (4.13) is the best that can be generally concluded in this case. However, if we impose the two further conditions

$$w_{1\nu}(h) = \mathcal{O}(T(h)h^3), \qquad \nu = 1, 2, \tag{4.14b}$$

it then follows that

$$\theta(\mathbf{y}) = \mathcal{O}(T(h)h^3), \qquad \mathbf{y} \in D_\Delta . \tag{4.15}$$

If the six functions $g_i(\mathbf{y})$, $g_{i1}(\mathbf{y})$ for $\mathbf{y} \in D_\Delta$ are linearly independent then the *conditions* (4.14) *are necessary and sufficient for* (4.15) *to hold.*

In order to relate the conditions (4.12) or (4.14) to the resultant force or moment of the load (4.4) we compute:

$$\begin{aligned} \mathbf{F}(h) &= \iint_{\Gamma_h} \mathbf{T}(\mathbf{x})\, dS, \\ &= \int_{-L}^{L} \int_{-h}^{h} \mathbf{T}(x_1\,;h)\, dx_1\, dx_2 + \mathcal{O}(T(h)h^3), \\ &= 2L\mathbf{f}(h) + \mathcal{O}(T(h)h^3); \end{aligned} \tag{4.16a}$$

and similarly:

$$\mathbf{M}(h) \equiv \iint_{\Gamma_h} \mathbf{x} \times \mathbf{T}(\mathbf{x})\, dS, \\ = 2L(0, m_2(h), w_{12}(h)) + \mathcal{O}(T(h)h^3). \tag{4.16b}$$

Thus (4.12) is equivalent to $\mathbf{F}(h) = \mathcal{O}(T(h)h^2)$, the first of (4.14a) is equivalent to $\mathbf{F}(h) = \mathcal{O}(T(h)h^3)$, the second of (4.14a) is equivalent to $M_2(h) = \mathcal{O}(T(h)h^3)$ and (4.14b) is equivalent to $M_3(h) = \mathcal{O}(T(h)h^3)$ and $w_{11}(h) = \mathcal{O}(T(h)h^3)$. If the *edge load is self-equilibrated and* $T_1(x_1, h)$ *is an even function of* x_1 (so that $w_{11}(h) = 0$) all conditions (4.14) are satisfied and *the estimate* (4.15) *applies*. For example if $T_1 \equiv 0$, so that the self-equilibrated edge load is parallel to the normal plane to Γ containing E, then (4.15) is valid.

5. Theorems relating to Saint-Venant's procedure. The form of the estimates of the dilatation derived in Sections 3 and 4 are equally valid for the displacement components, the strain components and the stress components. All of these quantities we call the "effects" and denote them collectively at any point $\mathbf{y}$ by the symbol: $\mathcal{E}(\mathbf{y})$.

Let an elastic body have specified displacements on a part of its surface, Γ_2, (which may be null) and specified tractions on the remainder Γ_1. The tractions on a part Γ_ϵ of Γ_1 we call $\mathbf{T}'(\mathbf{x}; \epsilon)$ and the effects in the body we call $\mathcal{E}'(\mathbf{y})$. We also consider the related problem in which the conditions on Γ_2 and $\Gamma_1 - \Gamma_\epsilon$ are retained but on Γ_ϵ the surface tractions are replaced by $T''(\mathbf{x}; \epsilon)$. The effects in this problem are denoted by $\mathcal{E}''(\mathbf{y})$. However, by superposition $\mathcal{E}(\mathbf{y}) \equiv \mathcal{E}'(\mathbf{y}) - \mathcal{E}''(y)$ are the effects produced in the body by zero displacement on Γ_2, zero traction on $\Gamma_1 - \Gamma_\epsilon$ and the traction $\mathbf{T}(\mathbf{x}, \epsilon) \equiv \mathbf{T}'(\mathbf{x}; \epsilon) - T''(\mathbf{x}; \epsilon)$ on Γ_ϵ. This is precisely the type of problem considered in Sections 3 and 4, for special forms of Γ_ϵ, and the estimates obtained there must apply for $\mathcal{E}(\mathbf{y})$ and $\mathbf{T}(\mathbf{x}; \epsilon)$. Thus we conclude the following

Theorem I: *If Γ_ϵ is the neighborhood of a regular point, P, on the surface Γ of the body D which has area A_ϵ, is contained in a sphere of radius $\delta > \epsilon_0$ about P and has maximum linear dimension ϵ then for all $\mathbf{y} \in D_\delta$ and $\epsilon \leq \epsilon_0$:*

$$\mathcal{E}'(\mathbf{y}) = \mathcal{E}''(\mathbf{y}) + \mathcal{O}(T(\epsilon)A_\epsilon\epsilon^p), \tag{5.0}$$

where

(a) $p = 0$ *for all* $\mathbf{T}'$ *and* $\mathbf{T}''$;
(b) $p = 1$ *if* $\mathbf{T}'$ *and* $\mathbf{T}''$ *have the same resultant force*;
(c) $p = 2$ *if* $\mathbf{T}'$ *and* $\mathbf{T}''$ *have the same resultant force and moment and the components of* $(\mathbf{T}' - \mathbf{T}'')$ *tangential to* Γ *are symmetric with respect to at least one pair of orthogonal tangent lines to* Γ *at* P.

The proof of the above result is essentially contained in Section 3, only trivial modifications being required. If the linear independence of the appropriate three or nine functions for each of the fundamental solutions is postulated then the statement of the theorem can be strengthened to include necessary and sufficient conditions on $\mathbf{T}'$ and $\mathbf{T}''$ in order that (5.0) be valid for $p = 1$ or $p = 2$.

In analogy with the above we also have

Theorem II: *If Γ_h is the neighborhood of a regular edge, E, on the surface Γ of the body D which has area A_h, is contained in a circular cylinder of radius $\Delta > h_0$ about E and has maximum linear dimension h normal to E then for all $\mathbf{y} \in D_\Delta$ and $h \leq h_0$:*

$$\mathcal{E}'(\mathbf{y}) = \mathcal{E}''(\mathbf{y}) + \mathcal{O}(T(h)A_h h^p), \tag{5.1}$$

provided $\mathbf{T}'(\mathbf{x}; h)$ *and* $\mathbf{T}''(\mathbf{x}; h)$ *are independent of position along the edge and where*

(a) $p = 0$ *for all such* $\mathbf{T}'$ *and* $\mathbf{T}''$;
(b) $p = 1$ *for all such* $\mathbf{T}'$ *and* $\mathbf{T}''$ *which have the same resultant force;*
(c) $p = 2$ *for all such* $\mathbf{T}'$ *and* $\mathbf{T}''$ *which have the same resultant force and moment and whose component of* $(\mathbf{T}' - \mathbf{T}'')$ *tangential to* Γ *and normal to* E *is symmetric with respect to* E.

The proof of this theorem easily follows from the estimates in Section 4. Again stronger results are possible if appropriate assumptions of linear independence are made.

Obvious modifications of the above theorems can be obtained by considering neighborhoods of several distinct points P_k or distinct edges E_k or combinations of them on Γ. In these modifications the applied forces $\mathbf{T}'$ and $\mathbf{T}''$ are required to satisfy the indicated relations (i.e. same resultant forces, etc.) *over each neighborhood,* $\Gamma_\epsilon(P_k)$ or $\Gamma_h(E_k)$, *separately.*[5]

All of these results, stated or indicated, specify conditions under which the effects produced by one load might be approximated by the effects produced by another load. In fact they go beyond the version of Saint-Venant's procedure stated in the Introduction in that they admit the possibility of replacing one load by another not statically equivalent to the first (cases a and b in the theorems). The engineering usefulness of such applications must depend of course on the relative magnitudes of the terms in (5.0) or in (5.1).

Theorem I is an extension and generalization of the remarks made by von Mises in the concluding section of his well known paper [3] on Saint-Venant's principle. An extension to distributed loads was indicated by him. Such an extension and other generalizations can be obtained from the work of Sternberg [5] who first obtained estimates similar to those in Section 3. However, neither of these authors explicitly employed (in considering general bodies) the specific boundary conditions imposed on the fundamental solutions.

6. Saint-Venant's principle. The literature abounds with brief statements and discussions of what is referred to as "Saint-Venant's principle," although there seems to be no universally accepted statement of any such principle. However there is little doubt that what is *intended* is a somewhat general formulation of conditions under which the application of Saint-Venant's procedure is justified. One of the clearest such statements is given by Biezeno and Grammel [1; p. 102]. Using the present terminology their version is in brief: *The magnitudes of the effects* $\mathcal{E}(\mathbf{y})$ *decrease with distance from* Γ_ϵ . This account seems to be in close accord with the original discussion given by Saint-Venant [4]. It can in fact be proven that certain integrals of the effects over surfaces in the body decrease as the surfaces are moved further from Γ_ϵ (see Zanaboni [6]). However no estimates of rates of decay are given nor do such integral inequalities give pointwise information. Hence the above formulated "principle" cannot suffice to justify Saint-Venant's procedure in general. It should be observed that the coefficients implied

[5]Strengthened forms of the modified theorems, containing statements of necessary and sufficient conditions on the loads, could also be formulated. However, if there are p distinct points and q distinct edges, it would be required, for the validity of these forms, that $3(p + q)$ or $9p + 6q$ functions be linearly independent. It is not clear that such conditions of independence are generally reasonable.

by the $\mathcal{O}$-terms in the estimates of Sections 3 and 4 are nonincreasing functions of δ and Δ, respectively. Thus there is a suggestion of this decrease in effects with distance from Γ_ϵ contained in our results. For special bodies or even for special parts of general bodies (i.e. spherical regions) it may be possible to demonstrate an actual decay of the relevant coefficients.

Some versions of "Saint-Venant's principle" are essentially abbreviated forms of the italicized statement in the Introduction. Thus the form due to Love [2; p. 132] in effect just eliminates explicit reference to the comparison with the effects $\mathcal{E}''$. The resulting vague statement was interpreted by von Mises [3] to imply a different comparison, roughly as follows: If Γ_ϵ consists of two or more parts then the effects produced by loads which are self-equilibrated over each part are negligible compared to the effects produced by *any* other self-equilibrated load not self-equilibrated over each part. It was shown by von Mises [3] and Sternberg [5] that this is not true but that a modified form of this statement is valid. However the "modified Saint-Venant's principle" thus obtained is unnecessarily restrictive with regard to the applicability of Saint-Venant's procedure. Indeed the negative statements in [3] and [5] result from an attempt to compare the effects of one type of load with *all other loads* related to it in a very general way. Such comparisons are far from necessary in order to justify the procedure.

It should be pointed out, with regard to the italicized statement in the Introduction, that the boundary value problems which determine $\mathcal{E}'$ and $\mathcal{E}''$ differ in an essential manner from those which determine $\mathcal{E}$. The latter problems have homogeneous boundary conditions over all of the body surface except Γ_ϵ while this is not generally true of the data in the former problems. Thus it does not seem reasonable to be able to formulate non trivial conditions on $\mathbf{T}$ over Γ_ϵ which insure that $\mathcal{E}$ will be negligible compared to $\mathcal{E}'$ or $\mathcal{E}''$, independently of the conditions imposed on $\Gamma - \Gamma_\epsilon$. In fact the problems first treated by Saint-Venant using his procedure [4] had free surface conditions imposed everywhere but on Γ_ϵ. In such cases the problems for $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ have the same boundary conditions on $\Gamma - \Gamma_\epsilon$ and some sort of general comparison might seem more reasonable.

In the discussions of this paper we have considered a fixed body and arbitrarily small portions, Γ_ϵ, of the surface of the body. However, for problems concerning "thin" bodies subject to end or edge loads the dimensions of Γ_ϵ are also typical dimensions of the body. Thus as $\epsilon \to 0$ the body approaches some degenerate form. In a variety of such bodies it can be shown [7] that the effects produced by special self-equilibrated end or edge tractions are estimated by

$$\mathcal{E}(\mathbf{y}) = \mathcal{O}\left(\exp\left[-\frac{ar}{\epsilon}\right]\right). \tag{6.0}$$

Here r is the distance of $\mathbf{y}$ from the end or edge and a is some positive constant. For several bodies which, as $\epsilon \to 0$, become shells, $a = \pi$, while for solid circular cylinders,[6] $a = j_{2,1} > \pi[j_{2,1}$ is the first zero of $J_2(x)]$. In these cases a decay with respect to distance from Γ_ϵ, as in the statement of Biezeno and Grammel, is clearly demonstrated. However, even more striking is the dependence on ϵ which for all $r \geq \delta > 0$ yields a uniform estimate vanishing faster than any power of ϵ. All problems originally treated by Saint-Venant [4] are such "thin" bodies and it is generally assumed that similar estimates apply to them.

[6]This result is easily obtained from the analysis given by Love [2; pp. 327–328].

The application of Saint-Venant's procedure is easier to justify when estimates of the form (6.0) apply than when only estimates of the form (1.2) hold. Thus for some class of thin bodies, with all surfaces free except those on the ends or edges, some statement of "Saint-Venant's principle" could possibly satisfy all reported requirements of such a principle. However in a more general context it would seem that the proper emphasis belongs on the procedure and not, due perhaps to historical accident, on statements of any principle independent of its applications.

Appendix

Self-equilibrated parallel loads. If the surface tractions on Γ_ϵ are of the form (3.21) we write, recalling (3.6):

$$W_{ij}(\epsilon) = m_i(\epsilon)k_j , \qquad m_i(\epsilon) \equiv \iint_{\Gamma_\epsilon} x_i t(x_1 , x_2 ; \epsilon) \, dS. \tag{A1}$$

Then the conditions of equilibrium can be expressed as:

$$\text{(a)} \quad \iint_{\Gamma_\epsilon} t(x_1 , x_2 ; \epsilon) \, dS = 0; \qquad \text{(b)} \quad \begin{bmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0. \tag{A2}$$

The coefficient matrix in the moment condition (A2, b) has rank two and thus the general solution must have the form

$$m_i(\epsilon) = \alpha(\epsilon)k_i , \qquad i = 1, 2, 3. \tag{A3}$$

Since $|\mathbf{k}| = 1$ the scalar $\alpha(\epsilon)$ is given by

$$\alpha(\epsilon) = k_i m_i(\epsilon) = [m_i(\epsilon)m_i(\epsilon)]^{1/2}. \tag{A4}$$

Using these results in (3.17) yields

$$c\theta(\mathbf{y}) = \alpha(\epsilon)[G^0_{\nu,\mu}(\mathbf{y})k_\nu k_\mu] + \mathcal{O}(T(\epsilon)\epsilon^4), \qquad \mathbf{y} \in D_\delta . \tag{A5}$$

Hence (3.19) follows if

$$\alpha(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^4).$$

However since $m_3(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^4)$ we need only require that

$$m_\nu(\epsilon) = \mathcal{O}(T(\epsilon)\epsilon^4), \qquad \nu = 1, 2. \tag{A6}$$

These conditions imply those in (3.18b).

Note that if $m_i(\epsilon) \equiv 0$, $i = 1, 2, 3$ then (A2, b) is satisfied for all directions $\mathbf{k}$. This is the case of astatic equilibrium which is the necessary and sufficient condition that $\alpha(\epsilon) = 0$. If $k_3 \neq 0$ then from (A3) with $i = 3$:

$$\alpha(\epsilon) = \frac{m_3(\epsilon)}{k_3} = \mathcal{O}(T(\epsilon)\epsilon^4),$$

or similarly $m_\nu(\epsilon) = k_\nu m_3(\epsilon)/k_3 = \mathcal{O}(t(\epsilon)\epsilon^4)$. This is the case in which $\mathbf{k}$ is not tangent to Γ_ϵ at $\mathbf{x} = 0$. Of course both of the above cases imply the validity of (3.19) since they obviously imply (A6). It should also be observed that if $\alpha(\epsilon) \neq 0$ then

$$k_i = \frac{m_i(\epsilon)}{[m_j(\epsilon)m_j(\epsilon)]^{1/2}}; \qquad i = 1, 2, 3. \tag{A7}$$

Thus *the direction* $\mathbf{k}$ *of the self-equilibrated parallel load* (3.21) *is determined by the scaler function* $t(x_1, x_2; \epsilon)$ whenever the load is not in astatic equilibrium. Finally we remark that $\mathbf{k}$ may very well be in the tangent plane and the parallel load still satisfy (A2) and (A6) but not be in astatic equilibrium (for example, take $t(x_1, x_2; \epsilon) = x_1 + x_2$ and let Γ_ϵ be plane).

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