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APPLICATION OF ULTRASPHERICAL POLYNOMIALS TO NON-LINEAR OSCILLATIONS II. FREE OSCILLATIONS*

By

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1. Introduction. In a recent paper [1], Denman and Howard introduced a procedure for the linearization of the non-linear ordinary differential equation governing the free oscillation of the simple pendulum, by approximating the non-linear torque with an ultraspherical polynomial of degree one. This linearization yields an approximation for the period of the pendulum as a function of the amplitude of the motion. The present paper applies this procedure to the following non-linear functions: $ax + bx^3$, $\sinh x$ and $\tanh x$. Certain asymptotic results are obtained. The paper also extends the procedure to the *cubic* ultraspherical polynomial approximation and applies this to the non-linear functions $\sin x$ and $\sinh x$. All results are either compared with the exact expressions for the period, if they are available, or to numerical results, if they are not. This extension yields a marked improvement over the *linear* approximation, with little increase in complexity.

2. The general free oscillation problem. A general free oscillation problem is characterized by the differential equation

$$d^2x/dt^2 + f(x) = 0. \tag{1}$$

In this paper, $f(x)$ is assumed to be an odd (non-linear) function (related to a force, torque, or voltage in a non-linear physical problem). The general initial conditions are: $x(0) = x_0$ and $v(0) = v_0$, where $v = dx/dt$. In the oscillatory case, these conditions may be replaced by $x_0 = A$, $v_0 = 0$, where A is the amplitude of motion. The first integral of (1) is

$$(dx/dt)^2/2 + V(x) = E,$$

where $V(x)$ is a potential function ($dV/dx = f(x)$), and E is proportional to the (constant) energy of the system. The turning points of the motion are $(-A, A)$.

The period of oscillation τ for (1) can be written

$$\tau/4 = \int_0^A [2(E - V)]^{-1/2}\, dx,$$

where τ is in general a function of A. Since $E = V(A)$, one may write

$$\tau/2^{3/2} = \int_0^A [V(A) - V(x)]^{-1/2}\, dx. \tag{2}$$

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3. The ultraspherical polynomials. The ultraspherical polynomials on the interval $[-1, 1]$ are the sets of polynomials orthogonal on this interval with respect to the weight factors $(1 - x^2)^{\lambda - 1/2}$, each set corresponding to a value of $\lambda > -\frac{1}{2}$. They may be obtained from Rodrigues' formula [2]

$$P_n^{(\lambda)}(x) = A_n^{(\lambda)}(1 - x^2)^{-\lambda+1/2}(d/dx)^n(1 - x^2)^{n+\lambda-1/2},$$

where $A_n^{(\lambda)}$ is a normalization factor.

Some subsets of the ultraspherical polynomials are: $\lambda = 0$, Tchebycheff polynomials of the first kind; $\lambda = \frac{1}{2}$, Legendre polynomials; $\lambda = 1$, Tchebycheff polynomials of the second kind; and $\lambda \to \infty$, the powers of x.

The ultraspherical polynomials on the interval $[-A, A]$ are defined as the sets of polynomials orthogonal on this interval with respect to the weight factors $[1 - (x/A)^2]^{\lambda-1/2}$, $\lambda > -\frac{1}{2}$. For a function $f(x)$ expandable in these polynomials one obtains

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\lambda)} P_n^{(\lambda)}(x/A), \tag{3}$$

where the coefficients $a_n^{(\lambda)}$ may be written

$$a_n^{(\lambda)} = \frac{\int_{-1}^{+1} f(Ax)P_n^{(\lambda)}(x)(1 - x^2)^{\lambda-1/2}\,dx}{\int_{-1}^{+1} [P_n^{(\lambda)}(x)]^2(1 - x^2)^{\lambda-1/2}\,dx}. \tag{4}$$

Since $a_n^{(\lambda)}P_n^{(\lambda)}(x/A)$ is unchanged if $P_n^{(\lambda)}(x/A)$ is multiplied by a constant, one may choose any convenient normalization factor $A_n^{(\lambda)}$.

For all odd $f(x)$, the approximate period τ^* (where * indicates a linear approximation) resulting from the *linear* ultraspherical polynomial approximation in $[-A, A]$, i.e., $f^*(x) = a_1^{(\lambda)}P_1^{(\lambda)}(x/A)$, can be expressed as

$$(\tau^*)_\lambda = 2\pi^{5/4}[\Gamma(\lambda + \tfrac{1}{2})A/4\Gamma(\lambda + 2)S]^{1/2}, \tag{5}$$

where

$$S = \int_0^1 x\, f(Ax)(1 - x^2)^{\lambda-1/2}\,dx, \tag{6}$$

for $\lambda > -\frac{1}{2}$. (This expression for τ^* actually converges for values of $\lambda > -2$ for many $f(x)$, but the accuracy of the approximate results for $-2 < \lambda < -\frac{1}{2}$ has been found to be rather poor; therefore this range is not considered in this paper).

4. Application of the linear ultraspherical approximation to some typical problems. The restoring forces or torques encountered in non-linear oscillation problems may be *roughly* classified as: hardening, softening, flattening, and bottoming. Examples from the first three classes will be examined here, and include: (I) (odd) cubic, $f(x) = ax + bx^3$; (II) sine, $f(x) = \sin x$, and hyperbolic sine, $f(x) = \sinh x$; and (III) hyperbolic tangent, $f(x) = \tanh x$.

I. Cubic non-linearity

The equation of free oscillation is taken as

$$d^2x/dt^2 + ax + bx^3 = 0. \tag{7}$$

Equation (7) can be classified according to whether a and b are positive or negative. The four combinations are shown in Table 1. Only the first three cases can yield bounded

TABLE 1

Classification	Name	Sign a	Sign b	Motion
Case 1	Hardening	+	+	Oscillatory
Case 2	Softening	+	−	Conditionally Oscillatory
Case 3	Softening-Hardening	−	+	Oscillatory
Case 4	Softening-Softening	−	−	Non-oscillatory

oscillatory motion; Case 4 is without any "restoring" force or torque regardless of the amplitude of motion. Case 3 is included both for completeness and because it will be used later (Section 6).

A. Exact solutions

Case 1. *Hardening cubic.* The expression for the period of (7) in terms of the complete elliptic integral of the first kind is well known [3]. The exact period is

$$\tau = 4K(k_1)/(a + bA^2)^{1/2}, \tag{8}$$

or, in dimensionless form,

$$\tau/\tau' = 2K(k_1)/\pi(1 + \nu)^{1/2}, \tag{9}$$

where $\tau' = 2\pi/a^{1/2}$ is the period of the system if $b = 0$, $K(k_1)$ is the complete elliptic integral of the first kind, $\nu = bA^2/a$ is a dimensionless quantity, and $k_1^2 = \nu/2(1 + \nu)$ is the modulus of the elliptic integral. The quantity ν is a measure of the non-linearity of the problem, since the maximum ratio of the non-linear to linear term in (7) is $bA^2/a = \nu$.

Case 2. *Softening cubic.* If b is negative, i.e., the non-linear spring is soft, a solution similar to (9) is obtained by making the transformation

$$k_2^2 = -\nu/(2 + \nu), \quad -1 < \nu \leq 0.$$

After some simplification, the period ratio becomes

$$\tau/\tau' = 2(1 + k_2^2)^{1/2}K(k_2)/\pi. \tag{10}$$

Note, however, that the above solution is no longer bounded, since $\nu \to -1$ implies $k_2^2 \to 1$, and $K \to \infty$. Such a ν (or amplitude) is called the critical ν (ν_c) (or amplitude A_c). Plots of τ/τ' against ν for Cases 1 and 2 are shown in Figs. 1 and 2 (in Fig. 2, a logarithmic scale is used to show the effects of large positive ν).

Case 3. *Softening-hardening cubic.* This case possesses rather interesting properties: for x small, the linear term dominates the cubic term, and vice-versa for large x; thus the force is away from the center for small x, but restoring for large x. The potential curve is shown in Fig. 3. For initial conditions represented by the energy E_1, the motion is periodic, but not symmetric about the origin. The conditions represented by E_2, however,

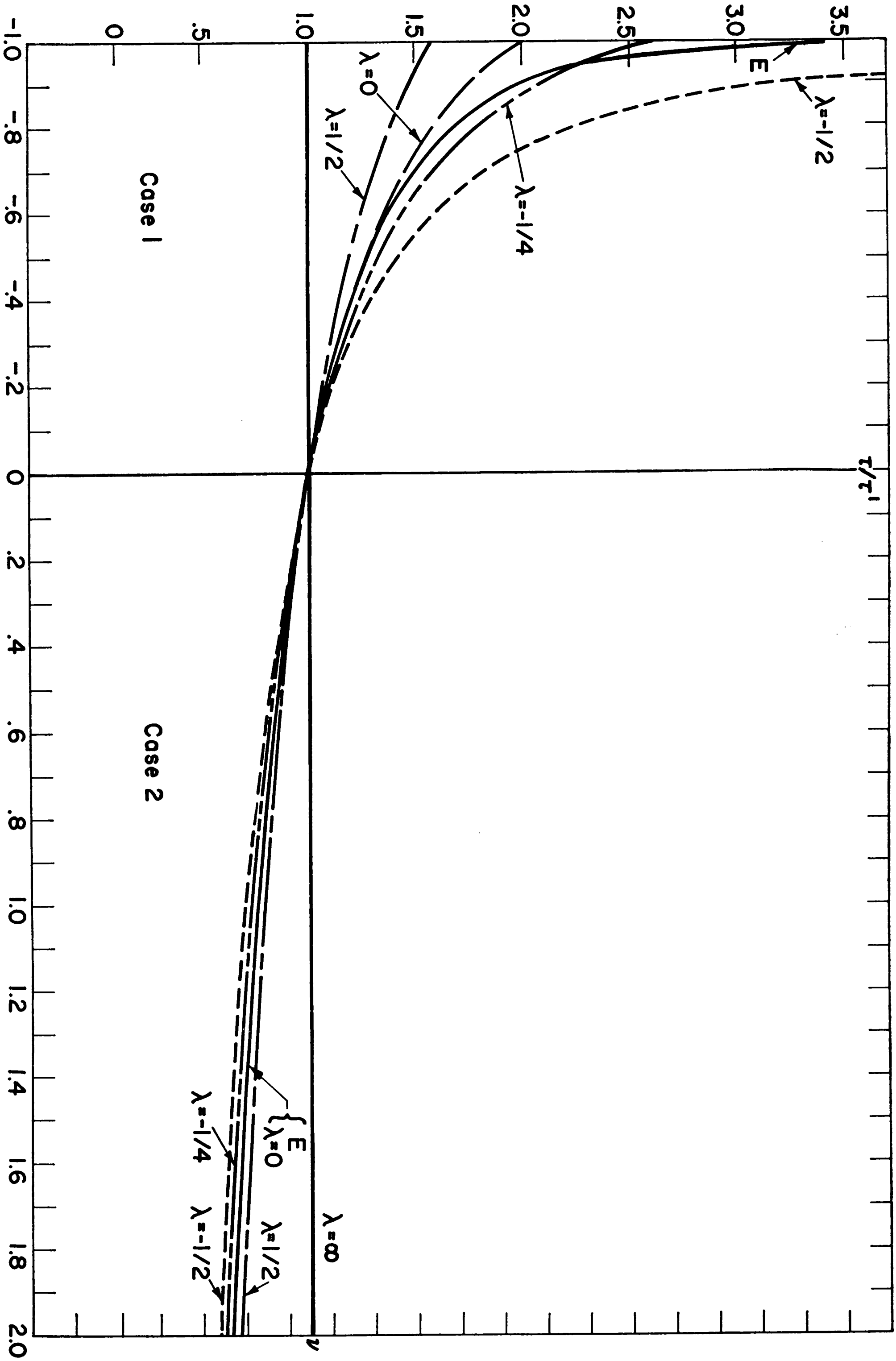

Fig. 1. Period Ratio τ/τ' vs. Non-linearity Factor ν. Hardening and softening cubic springs (Cases 1 and 2).

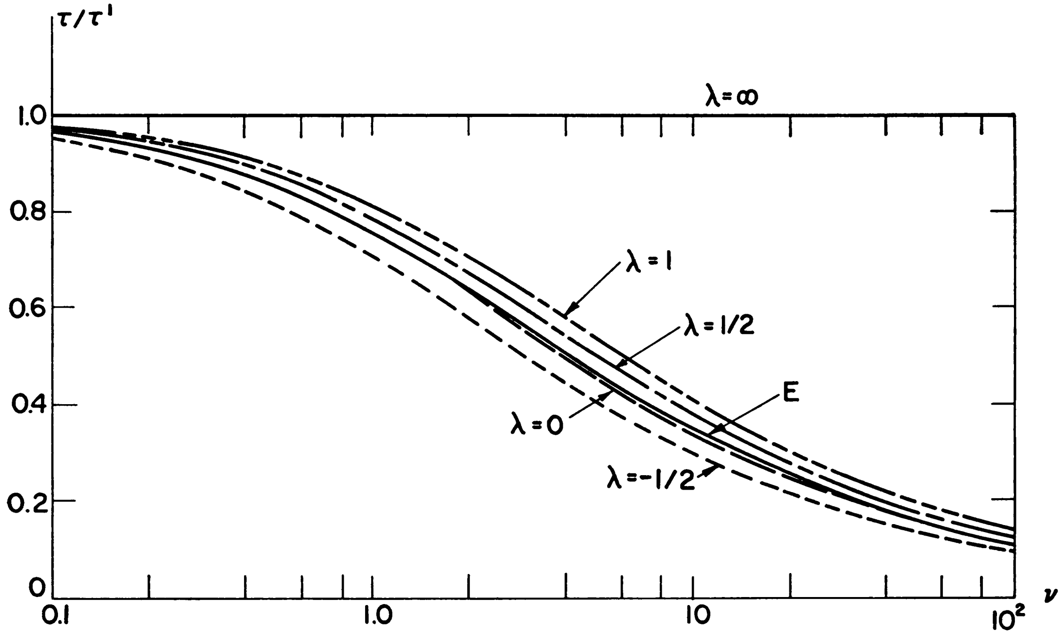

FIG. 2. Period Ratio τ/τ' vs. Large Non-linearity Factor ν. Hardening cubic spring.

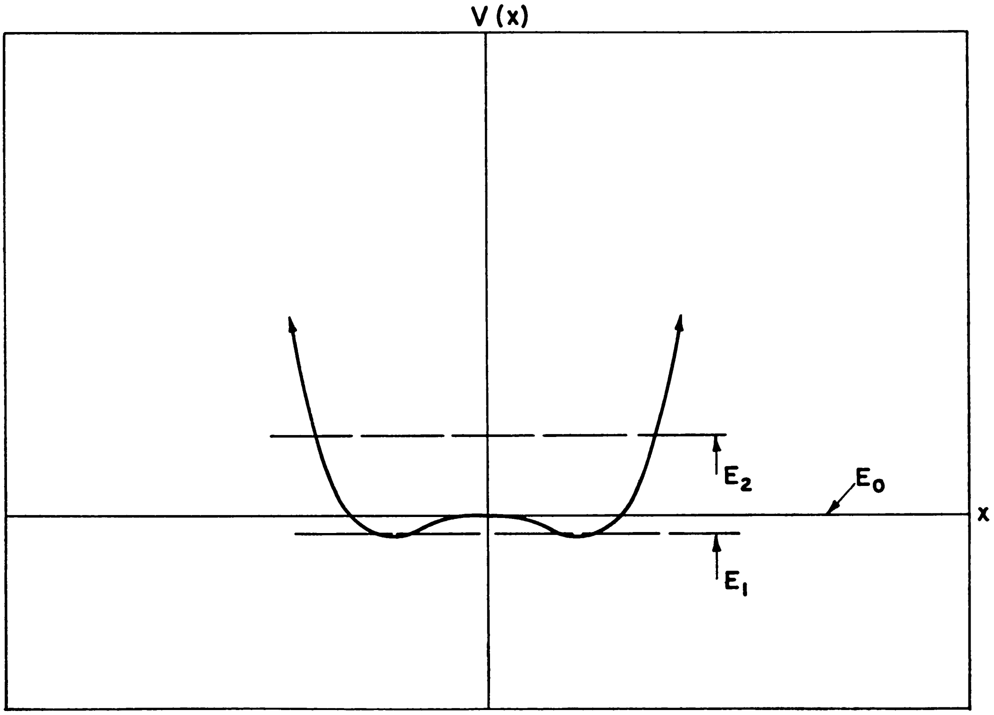

FIG. 3. Potential Curve for Softening-Hardening Cubic Spring (Case 3).

do give symmetric oscillations about the origin. The demarcation line is E_0 . Only the case represented by E_2 is considered here. Then one obtains

$$\tau(-a)^{1/2} = 4(-\nu - 1)^{-1/2}K(k_3), \tag{11}$$

where $\nu = bA^2/a < 0$, and $k_3^2 = \nu/2(1 + \nu)$. For $E > E_0$, ν is < -2, so that $\frac{1}{2} \leq k_3^2 < 1$. For $\tau' = 2\pi/|a|^{1/2}$,

$$\tau/\tau' = 2K(k_3)/\pi(-\nu - 1)^{1/2}. \tag{12}$$

Equation (12) is plotted as curve E in Fig. 4.

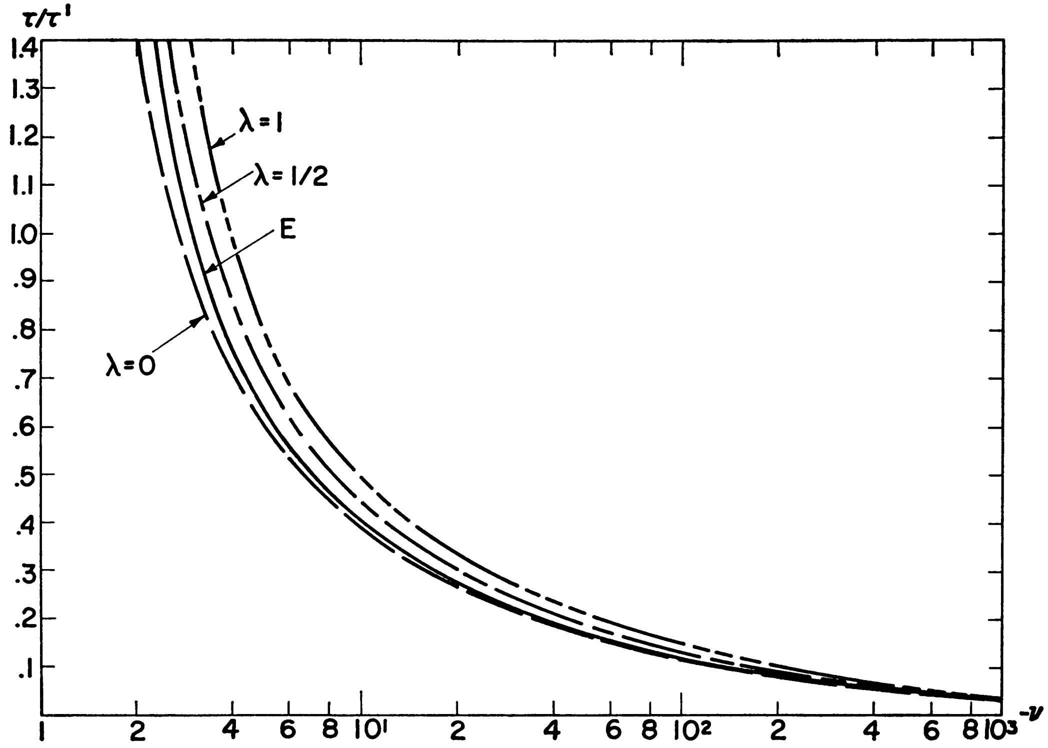

Fig. 4. Period Ratio τ/τ' vs. Non-linearity Factor ν. Softening-Hardening Cubic Spring (Case 3).

B. Linear ultraspherical polynomial approximation

In (7), if one expands $f(x) = ax + bx^3$ in ultraspherical polynomials in $[-A, A]$, one obtains the linear approximation

$$(ax + bx^3)_\lambda^* = [a + 3bA^2/2(\lambda + 2)]x.$$

This linear approximation, when substituted into (7), results in the approximate linear differential equation

$$d^2x/dt^2 + [a + 3bA^2/2(\lambda + 2)]x = 0. \tag{13}$$

Therefore, for Cases 1 and 2,

$$\tau^*/\tau' = [1 + 3\nu/2(\lambda + 2)]^{-1/2}, \qquad \nu \geq -1, \tag{14a}$$

and for Case 3,

$$\tau^*/\tau' = [-1 - 3\nu/2(\lambda + 2)]^{-1/2}, \quad \nu < -2, \tag{14b}$$

where $\tau' = 2\pi/|a|^{1/2}$, and $\nu = bA^2/a$. For $a = 0$, τ' and ν are not defined (Cases 1 and 3) but the results (8) and (11) approach $\tau = 4K(2^{1/2})/b^{1/2}A$, which is the correct result.

On Figs. 1, 2, and 4 are shown curves given by the above approximations for certain values of λ, for these cubic springs. (The linear Maclaurin series approximation for Cases 1 and 2 gives amplitude-independent τ^*, while for Case 3 it yields an imaginary one.)

II. Sine and Hyperbolic Sine.

A. Exact solutions

The governing differential equations are written

$$d^2x/dt^2 + \omega_0^2 \sin x = 0, \tag{15}$$

$$d^2x/dt^2 + \omega_0^2 \sinh x = 0. \tag{16}$$

Solutions of (15), both exact and for the linear ultraspherical polynomial approximation, are found in [1]. Equation (16) can also be solved in terms of the complete elliptic integral of the first kind. While the solution for (15) is oscillatory only if $A < \pi$, the solution to (16) has no such limitation; the exact period is given by

$$\tau/\tau_0 = (2/\pi) \operatorname{sech}(A/2) K[\tanh(A/2)], \tag{17}$$

where $\tau_0 = 2\pi/\omega_0$.

B. Linear ultraspherical polynomial approximation

If one expands $\sinh x$ in ultraspherical polynomials in $[-A, A]$ and truncates after the linear term, one obtains

$$(\sinh x)_\lambda^* = [\Gamma(\lambda + 2) I_{\lambda+1}(A)/(A/2)^{\lambda+1}]x, \tag{18}$$

where $I_n(x) = (-i)^n J_n(ix)$ is a modified Bessel function. The approximate period ratio τ^*/τ_0 for (16) is then

$$\tau^*/\tau_0 = [(A/2)^{\lambda+1}/\Gamma(\lambda + 2) I_{\lambda+1}(A)]^{1/2}. \tag{19}$$

The comparison between (19) and the exact solution (17) for various λ values is shown in Fig. 5.

III. Hyperbolic Tangent Non-linearity

A. Exact solution

The governing differential equation of motion is

$$d^2x/dt^2 + \omega_0^2 \tanh x = 0. \tag{20}$$

No exact solution of (20) in simple functions is known to exist. To provide comparison

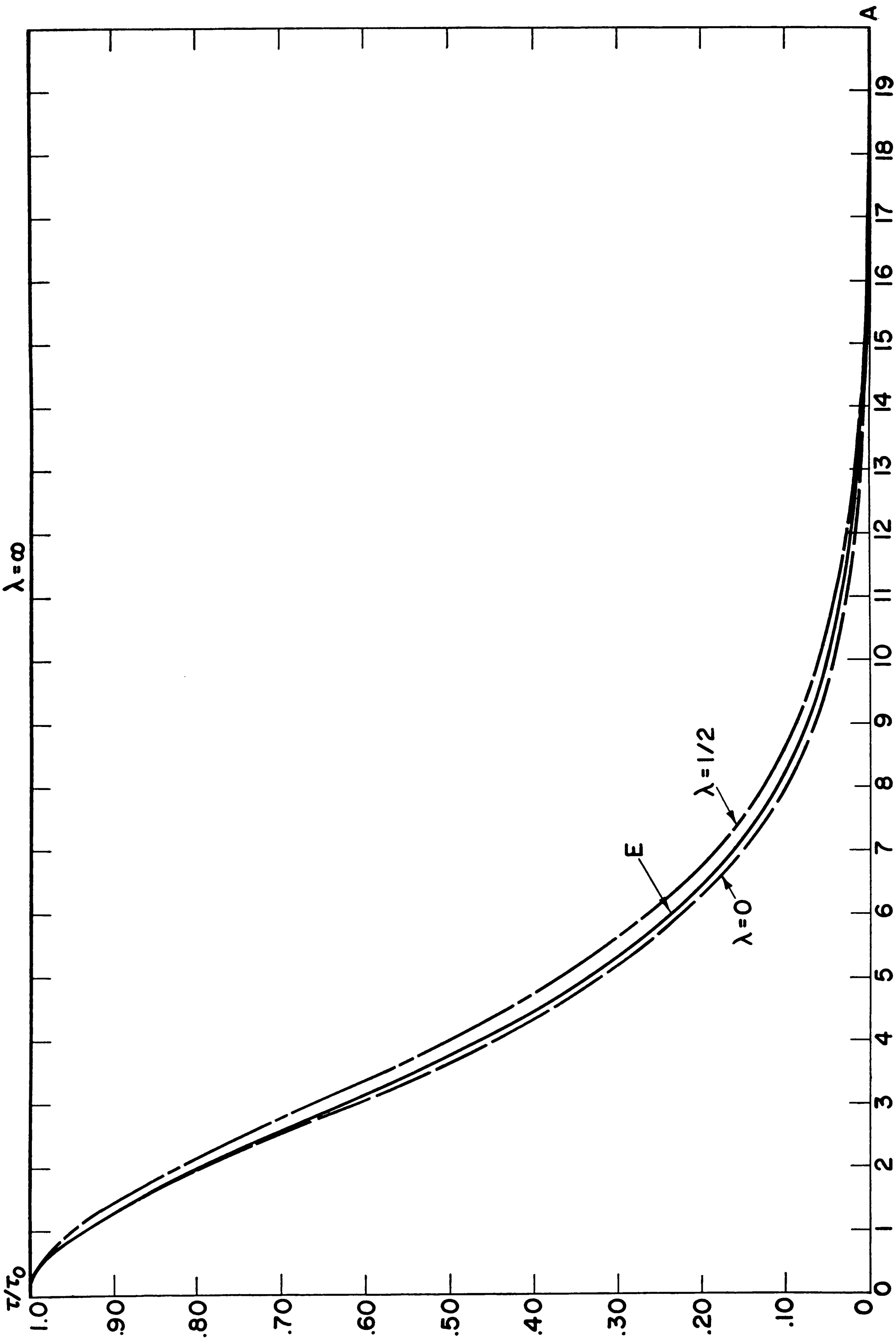

FIG. 5. Period Ratio τ/τ_0 vs. Amplitude A. Sinh spring.

for the approximate results, numerical quadrature is used to provide the "exact" results.

From (2) the exact period in this case can be written

$$\tau/\tau_0 = (2^{1/2}/\pi) \int_0^A [\ln (\cosh A/\cosh x)]^{-1/2} \, dx. \tag{21}$$

To remove the singularity at the upper limit for numerical quadrature, consider the expansion ($x^2 < \pi^2/4$)

$$\ln (\cosh x) = x^2/2 - x^4/12 + \cdots (-1)^{n-1}2^{2n-1}(2^{2n} - 1)B_{2n-1}x^{2n}/n(2n)!,$$

where B_{2n-1} are the odd Bernouilli numbers [4]. Equation (21) becomes

$$\tau/\tau_0 = (2^{1/2}/\pi) \int_0^A [(A^2 - x^2)/2 - (A^4 - x^4)/12 + \cdots]^{-1/2} \, dx. \tag{22}$$

Setting $x = A \cos \theta$, one obtains (for $A < \pi/2$)

$$\tau/\tau_0 = (2/\pi) \int_0^{\pi/2} [1 - A^2(1 + \cos^2 \theta)/6 + \cdots]^{-1/2} \, d\theta. \tag{23}$$

Equation (23) is now free of singularities and is susceptible to numerical integration. For small values A, one can truncate after the second term and express τ/τ_0 in terms of an elliptic integral

$$\tau/\tau_0 \doteq (2/\pi)[6/(6 - A^2)]^{1/2}K(k_4), \tag{24}$$

where $k_4^2 = A^2/(6 - A^2)$. For intermediate and large A, expand $\ln (\cosh x)$ in a Taylor series about A. Then

$$\ln \cosh x = \ln \cosh A + \tanh A \, (x - A) + \operatorname{sech}^2 A \, (x - A)^2/2$$
$$-\operatorname{sech}^2 A \tanh A \, (x - A)^3/3 + \cdots .$$

Setting

$$\tau/\tau_0 = (2/\pi)(6 \cosh^3 A/\sinh A)^{1/2} \int_0^{A^{1/2}} [3 \cosh^2 A - \tfrac{3}{2}y^2 \operatorname{ctnh} A - y^4 + \cdots]^{-1/2} \, dy.$$

If one truncates the series after the third term, the above integration can be carried out by means of an elliptic integral of the first kind [5]

$$\tau/\tau_0 \doteq (4/\pi)g \cosh A \, F(\psi, k_5), \tag{25}$$

where

$$k_5^2 = (1 - g^2)/2, \qquad g = [1 + (16/3) \sinh^2 A]^{-1/4},$$

and

$$\sin^2 \psi = 2/[1 + (2g^2 \sinh 2A)/A - g^2].$$

Retaining only two terms of the integrand yields the approximation

$$\tau/\tau_0 \doteq (4/\pi)(\cosh A) \sin^{-1} (A/\sinh 2A)^{1/2}. \tag{26}$$

For asymptotically large A, (26) approaches

$$\tau/\tau_0 \rightarrow (2/\pi)(2A)^{1/2} = 0.9003A^{1/2}. \tag{27}$$

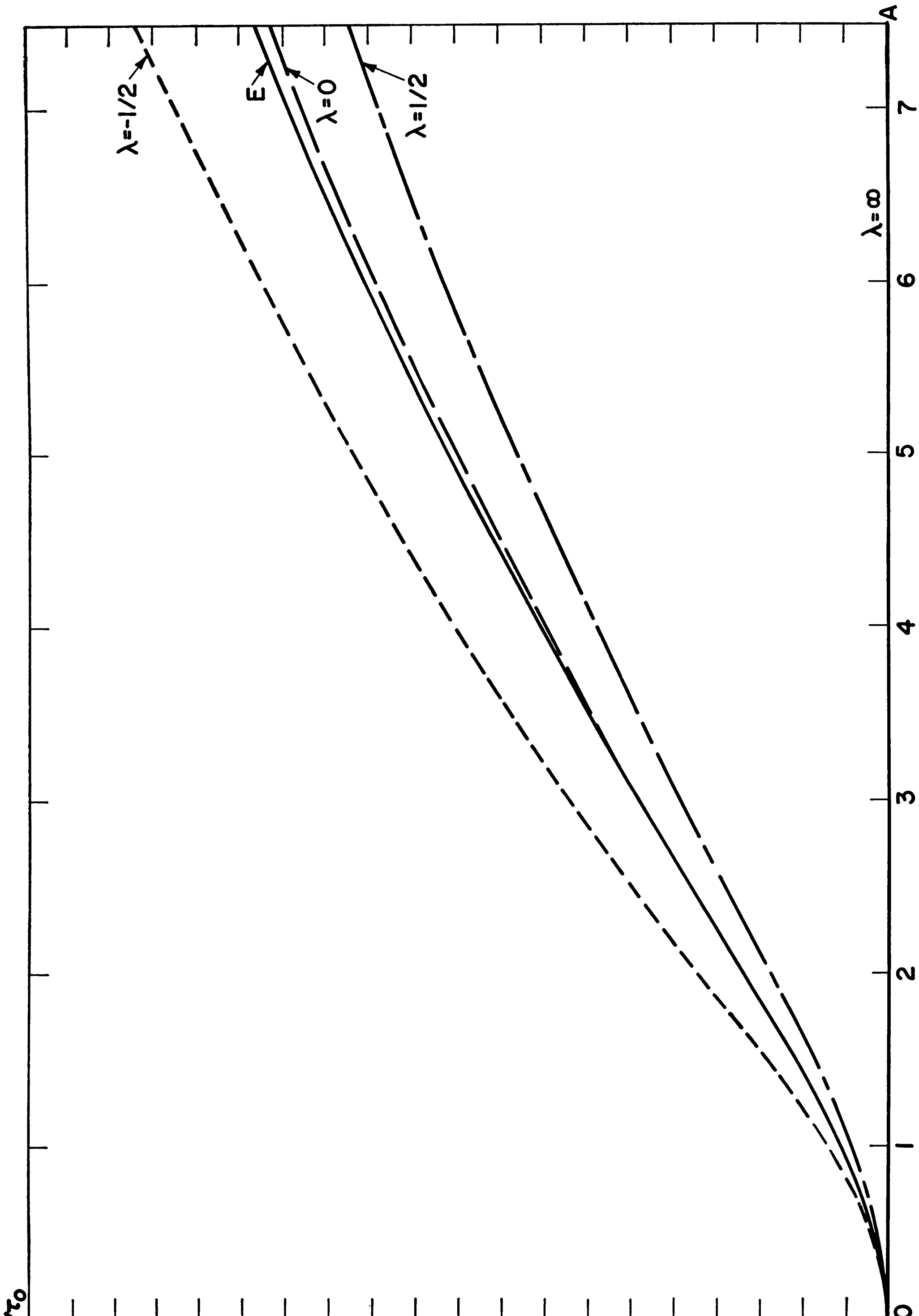

FIG. 6. Period Ratio τ/τ_0 vs. Amplitude A. Tanh spring.

Equations (24), (25), (26) and (27) span the entire range of A and overlap one another as far as graphical accuracy is concerned, and were used to construct the "exact" curve E in Fig. 6.

B. Linear ultraspherical polynomial approximation

For $f(x) = \omega_0^2 \tanh x$, S in (6) becomes

$$S = \omega_0^2 \int_0^1 x \tanh A\, x(1 - x^2)^{\lambda - 1/2}\, dx. \tag{28a}$$

This integral does not seem to exist in closed form in terms of simple functions. For $A < \pi/2$, $\tanh Ax$ can be expanded in a Maclaurin series, and

$$\tau^*/\tau_0 = \Big[\pi^{1/2} A/4\Gamma(\lambda + 2) \sum_{n=1}^{\infty} (-1)^{n-1}(2A)^{2n-1}(2^{2n} - 1)B_{2n-1}\Gamma(n + \tfrac{1}{2}) \cdot \Gamma(\lambda + n + 1)(2n!)^{-1}\Big]^{1/2}. \tag{28b}$$

Alternately, an integration of (28a) by parts yields

$$S = (2\lambda + 1)^{-1}\omega_0^2 A \int_0^1 (1 - x^2)^{\lambda + 1/2} \operatorname{sech}^2 Ax\, dx, \tag{28c}$$

which removes the singularity in the integrand of (28a) for $x = 1$, $\lambda < \frac{1}{2}$. Then τ^*/τ_0 can be expressed as

$$\tau^*/\tau_0 = \left\{\frac{8\Gamma(\lambda + 2)}{\pi^{1/2}\Gamma(\lambda + \frac{3}{2})} \int_0^1 \frac{(1 - x^2)^{\lambda + 1/2}}{e^{2Ax}(1 + e^{-2Ax})^2}\, dx\right\}^{-1/2}. \tag{29}$$

For small A, (28b) is convenient, while for larger A, numerical quadrature is used to evaluate (29).

However, certain values of λ yield somewhat simpler results. For example, if $\lambda = \frac{1}{2}$,

$$S = \omega_0^2 \int_0^1 x \tanh Ax\, dx. \tag{30}$$

Since, for $x \geq 0$, $A > 0$,

$$x \tanh Ax = x(1 - 2e^{-2Ax} + 2e^{-4Ax} - 2e^{-6Ax} + \cdots), \tag{31}$$

the period ratio for $\lambda = \frac{1}{2}$ can be expressed as

$$(\tau^*/\tau_0)_{1/2} = \left\{\frac{3}{2A}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2nA + 1)}{(nA)^2 e^{2nA}} - \frac{\pi^2}{12A^2}\right)\right\}^{-1/2}. \tag{32}$$

This expression is particularly convenient for large values of A, and for very large A, $(\tau^*/\tau_0)_{1/2}$ approaches $0.8165\, A^{1/2}$. For $\lambda \to -\frac{1}{2}$, (5) and (28c) yield

$$(\tau^*/\tau_0)_{-1/2} \to [A/\tanh A]^{1/2}. \tag{33}$$

Figure 6 compares the "exact" solution with the approximations corresponding to λ values of $-\frac{1}{2}$, 0, and $\frac{1}{2}$.

5. Limiting behavior of the linear ultraspherical polynomial approximation. Two special regions of interest are those for which A (or ν) approaches 0 and for A (or ν) large.

I. Lower Limit

For small oscillations in the neighborhood of the origin, one can expand $V(x)$ in (2) in a Maclaurin series

$$V(x) = V_0 + a_2x^2 + a_4x^4 + \cdots ,$$

or

$$V(A) - V(x) = a_2(A^2 - x^2)[1 + a_4(A^2 + x^2)/a_2 + \cdots],$$

where a_2 must be positive for *small* oscillations about the origin to exist. Setting $x/A = \cos \theta$, one finds

$$\tau/2^{3/2} = a_2^{-1/2} \int_0^{\pi/2} [1 + a_4A^2(1 + \cos^2 \theta)/a_2 + \cdots]^{-1/2} \, d\theta.$$

For sufficiently small values of A, one can expand the integrand by the Binomial Theorem, so that

$$\tau/2^{3/2} = a_2^{-1/2} \int_0^{\pi/2} [1 - a_4A^2(1 + \cos^2 \theta)/2a_2 + \cdots] \, d\theta.$$

The solution for small oscillation is, therefore,

$$\tau = 2\pi(2a_2)^{-1/2}[1 - 3a_4A^2/4a_2 + \cdots]. \tag{34}$$

In (6), setting $x = \cos \theta$, one obtains

$$S = \int_0^{\pi/2} f(A \cos \theta) \sin^{2\lambda}\theta \cos \theta \, d\theta. \tag{35}$$

Expanding $f(A \cos \theta)$ in its Maclaurin series,

$$f(A \cos \theta) = \sum_{m=0}^{\infty} c_m A^m \cos^m \theta.$$

Then (5) becomes

$$\tau^* = 2\pi\left[\pi^{1/2}\Gamma(\lambda + \tfrac{1}{2})A/4\Gamma(\lambda + 2) \sum_{m=0}^{\infty} c_m A^m \int_0^{\pi/2} \cos^{m+1}\theta \sin^{2\lambda}\theta \, d\theta\right]^{1/2}. \tag{36}$$

For $f(x)$ odd, (36) yields

$$\tau^* = 2\pi[c_1 + 3c_3A^2/2(2 + \lambda) + \cdots]^{-1/2}. \tag{37}$$

Since the c_i's and the a_i's are the Maclaurin series coefficients for $f(x)$ and $V(x)$ respectively, their relationships are

$$a_2 = c_1/2, \qquad a_4 = c_3/4, \cdots$$

Equation (37) becomes

$$\tau^* = 2\pi(2a_2)^{1/2}[1 - 3a_4A^2/2a_2(\lambda + 2) + \cdots] \tag{37'}$$

When one compares (37′) with the expression for the exact period (34), one notes that the first two terms agree if and only if $\lambda = 0$. Hence, for *small* free oscillations, $\lambda = 0$, corresponding to Tchebycheff polynomials of the first kind, provides the most accurate period approximation of all linear ultraspherical polynomials, for *all* odd analytic $f(x)$. (This result cannot be applied to the softening-hardening cubic; only the case $E \geq E_0$ is considered here, for which arbitrarily small oscillations about 0 do not exist.)

II. Large Asymptotic Limit

For large non-linearities, each case was examined individually.

A. Cubic non-linearity

For the hardening and softening-hardening cases (Cases 1 and 3), the exact solutions are given by (9) and (12). For $|\nu| \gg 1$, (9) and (12) yield

$$\lim_{|\nu|\to\infty} (\tau/\tau') = 2(1.8541)/|\nu|^{1/2}\pi,$$

while from (14a) and (14b), one finds

$$\lim_{|\nu|\to\infty} (\tau^*/\tau') = [2(2 + \lambda)/3\ |\nu|]^{1/2}.$$

Hence, the optimum value of λ, for large non-linearity, is found from

$$[2(\lambda + 2)/3]^{1/2} = 2(1.8541)/\pi, \tag{38}$$

which yields $\lambda = 0.0899$.

For the softening case (Case 2), the exact solution (10) and the approximation (14a) behave as follows:

$$\tau/\tau' \to \infty \quad \text{as} \quad \nu \to -1 \tag{39a}$$

$$\tau^*/\tau' \to \infty \quad \text{as} \quad \tfrac{3}{2}(2 + \lambda) \to -1. \tag{39b}$$

Hence, the value of λ which yields the correct critical ν_c is $\lambda = -\frac{1}{2}$.

For the softening-hardening spring (Case 3), the exact solution (12) becomes infinite for $\nu = -2$; the approximation (14b) gives the same result for $\lambda = 1$.

B. Sine and hyperbolic sine non-linearities

The exact solution for the free oscillation of a simple pendulum is

$$\tau/\tau_0 = (2/\pi)K(k),$$

where $k = \sin(A/2)$. Thus, τ/τ_0 approaches infinity as $A \to \pi$. The linear ultraspherical polynomial approximation yields [1]

$$\tau^*/\tau_0 = [(A/2)^{\lambda+1}/\Gamma(\lambda + 2)J_{\lambda+1}(A)]^{1/2}.$$

From a table of the first zero of $J_p(x)$ as a function of p, such as [6], one notes that $J_{1/2}(A)$ has its first zero at $A = \pi$. Hence, the same critical amplitude occurs for the linear approximation when the index λ is $-\frac{1}{2}$.

The exact solution by elliptic integrals for the sinh spring is given by (17). For $A \gg 1$, $k^2 \to 1$, and

$$\tau/\tau_0 \to C_1 A \exp(-A/2), \tag{40}$$

where C_1 is a constant. The linear ultraspherical polynomial approximation result is given by (19). For very large A [7],

$$I_\nu(A) \to e^A/(2\pi A)^{1/2},$$

and therefore,

$$\tau^*/\tau_0 \to C_2 A \exp(-A/2) \tag{41}$$

only if $\lambda = \frac{1}{2}$.

C. The hyperbolic tangent non-linearity

The asymptotic τ/τ_0 is given by (27). For very large A, $\tanh Ax \to 1$, $x > 0$, and (28a) and (5) yield

$$\tau^*/\tau_0 \to \{A\pi^{1/2}\Gamma(\lambda + \tfrac{3}{2})/2\Gamma(\lambda + 2)\}^{1/2}. \tag{42a}$$

Equating the coefficients of $A^{1/2}$ in (27) and (42a), one obtains

$$\pi^{5/2}\Gamma(\lambda + \tfrac{3}{2}) = 16\Gamma(\lambda + 2),$$

from which

$$\lambda = -0.075. \tag{42b}$$

These asymptotic results are summarized in Table 2, and are verified by the computations leading to Figs. 1, 2, 4, 5, and 6.

Table 2

Type of Spring	Optimum Value of λ for Large Non-Linearity (or Correct Critical Value)
cubic	
(1) hardening	.089
(2) softening	−.5
(3) softening-hardening	.089 (large ν)
	1.0 (correct ν_c)
sine	−.5
sinh	.5
tanh	−.075

6. Cubic ultraspherical polynomial approximation. If, instead of truncating the ultraspherical polynomial expansion of $f(x)$ in $[-A, A]$ after the linear term, one does so after the cubic, much more accurate results are to be expected. Then

$$f^{**}(x) = a_1^{(\lambda)}P_1^{(\lambda)}(x/A) + a_3^{(\lambda)}P_3^{(\lambda)}(x/A),$$

where ** denotes a cubic approximation. When this result is substituted in (1), the resulting (approximate) cubic differential equation can be solved exactly either by elliptic functions or, preferably, by utilizing the previously plotted "exact" results for the cubic springs. The sin and sinh cases will be considered below (the hyperbolic tangent has no accurate cubic approximation for large amplitudes, as is true for *any* flattening spring).

The evaluation of the ultraspherical polynomial coefficient $a_3^{(\lambda)}$ will be shown for $f(x) = \sin x$, that for $\sinh x$ being very similar. From (4)

$$a_3^{(\lambda)} = \frac{\int_0^\pi \sin(A \cos\theta) \sin^{2\lambda}\theta\, P_3^{(\lambda)}(\cos\theta)\, d\theta}{\int_0^\pi [P_3^{(\lambda)}(\cos\theta)]^2 \sin^{2\lambda}\theta\, d\theta}.$$

The denominator is a standard integral, while the numerator is an integral found in [8]. Thus it can be shown

$$(\sin x)_\lambda^{**} = \frac{\Gamma(\lambda+2)}{(A/2)^{\lambda+1}} [J_{\lambda+1}(A) + (\lambda+3)J_{\lambda+3}(A)]x - \frac{\Gamma(\lambda+4)J_{\lambda+3}(A)}{6(A/2)^{\lambda+3}} x^3, \qquad (43a)$$

$$= \left[\Lambda_{\lambda+1}(A) - \frac{A^2}{4(\lambda+2)} \Lambda_{\lambda+3}(A)\right]x - \tfrac{1}{6}\Lambda_{\lambda+3}(A)x^3, \qquad (43b)$$

where $\Lambda_\lambda(A) = \Gamma(\lambda+1)\, J_\lambda(A)/(A/2)^\lambda$. The function Λ_λ is plotted in [6], while (43a) agrees with the result given in [9] for $\lambda = 0$.

The cubic ultraspherical polynomial approximation for $\sinh x$ is

$$(\sinh x)_\lambda^{**} = \frac{\Gamma(\lambda+2)}{(A/2)^{\lambda+1}} [I_{\lambda+1} - (\lambda+3)I_{\lambda+3}]x + \frac{\Gamma(\lambda+4)I_{\lambda+3}}{6(A/2)^{\lambda+3}} x^3. \qquad (44)$$

Consider the case $\lambda = 0$ and $f(x) = \omega_0^2 \sin x$. The differential equation (1) is replaced by

$$d^2x/dt^2 + [2\omega_0^2(J_1 + 3J_3)/A]x - (8\omega_0^2 J_3/A^3)x^3 = 0. \qquad (45)$$

Since $J_1 + 3J_3$ and J_3 are positive in $[0, \pi]$, (45) corresponds to the softening cubic of section 4, with

$$a = 2\omega_0^2(J_1 + 3J_3)/A, \qquad b = -8\omega_0^2 J_3/A^3.$$

Following solution (10), one finds, for $\lambda = 0$,

$$\tau^{**}[2\omega_0^2(J_1 + 3J_3)/A]^{1/2}/2\pi = (2/\pi)(1 + k_6^2)^{1/2}K(k_6), \qquad (46)$$

where τ^{**} is the approximation to the period resulting from the cubic ultraspherical approximation to $f(x)$,

$$k_6^2 = \frac{-\nu}{2+\nu} = \frac{2J_3}{J_1 + J_3}, \qquad 0 \le k_6^2 < 1,$$

and

$$\nu = -4J_3/(J_1 + 3J_3), \qquad -1 < \nu \le 0. \qquad (47)$$

The critical amplitude A_c for this cubic approximation is found from

$$k_6^2 = 2J_3/(J_1 + J_3) = 1, \quad \text{or} \quad J_1(A_c) = J_3(A_c).$$

The smallest solution of this equation is

$$(A_c^{**})_0 = 3.054, \tag{48}$$

compared with $A_c = \pi$ for the exact solution.

Corresponding to a given value of $A < A_c$ one can compute ν from (47). With this value of ν known, one finds the corresponding numerical value of τ/τ' on the exact solution curve (E) in Fig. 1. This numerical value is set equal to the left-hand side of (46), whereupon one can obtain the new period-ratio approximation τ^{**}/τ_0. A numerical example will show the manipulations.

Example: Letting the amplitude $A = 2$, one finds $\nu = -0.535$, which corresponds to $\tau/\tau' = 1.30$ in Fig. 1 (curve E). Hence, using (46), $\tau^{**}/\tau_0 = 1.32$.

Instead of using Fig. 1 to furnish the above results, one can, of course, evaluate (46) directly or equivalently, use

$$\tau^{**}/\tau_0 = (2/\pi)\Lambda_2^{-1/2}(A)\, K(k_6). \tag{49}$$

For comparison, using the value $A = 2$ of the above example, one obtains $\tau^{**}/\tau_0 = 1.3296$. The numerical difference between these results is due to graph-reading inaccuracy.

For $\lambda = \frac{1}{2}$, the cubic Legendre polynomial approximation for sin x yields

$$(\tau^{**}/2\pi)[3\omega_0^2(j_1 + 7j_3/2)/A]^{1/2} = 2(1 + k_7^2)^{1/2}K(k_7)/\pi, \tag{50}$$

where

$$\nu = -35j_3/(6j_1 + 21j_3), \qquad k_7^2 = 35j_3/(12j_1 + 7j_3),$$

and j_1 and j_3 are spherical Bessel functions, $j_n(x) = (\pi/2x)^{1/2}\ J_{n+1/2}(x)$. To establish the critical amplitude, let $k_7^2 = 1$, which results in the equation

$$3j_1(A_c) = 7j_3(A_c). \tag{51}$$

The smallest solution of (51) is

$$(A_c^{**})_{1/2} = 2.98. \tag{52}$$

Thus the critical amplitude A_c based on the cubic Legendre polynomial approximation is smaller than that from the Tchebycheff polynomials of the first kind, which in turn is smaller than the correct value. This means that the Legendre approximation becomes poorer at a smaller amplitude A. This is substantiated by the curves of Fig. 7, where these results and the exact solution are plotted. Included also in Fig. 7 is the *linear* approximation curve from [1] in order to demonstrate the improvement brought about by the cubic approximation.

For sinh x and $\lambda = 0$, one obtains from (44)

$$(\sinh x)_0^{**} = (2/A)(I_1 - 3I_3)x + [8I_3/A^3]x^3.$$

The cubic approximate differential equation of motion is

$$d^2x/dt^2 + [2\omega_0^2(I_1 - 3I_3)/A]x + [8\omega_0^2I_3/A^3]x^3 = 0. \tag{53}$$

When compared to (7) one notes

$$a = 2\omega_0^2(I_1 - 3I_3)/A, \qquad b = 8\omega_0^2I_3/A^3.$$

It is interesting to observe that while b will always be positive, a becomes *negative* for

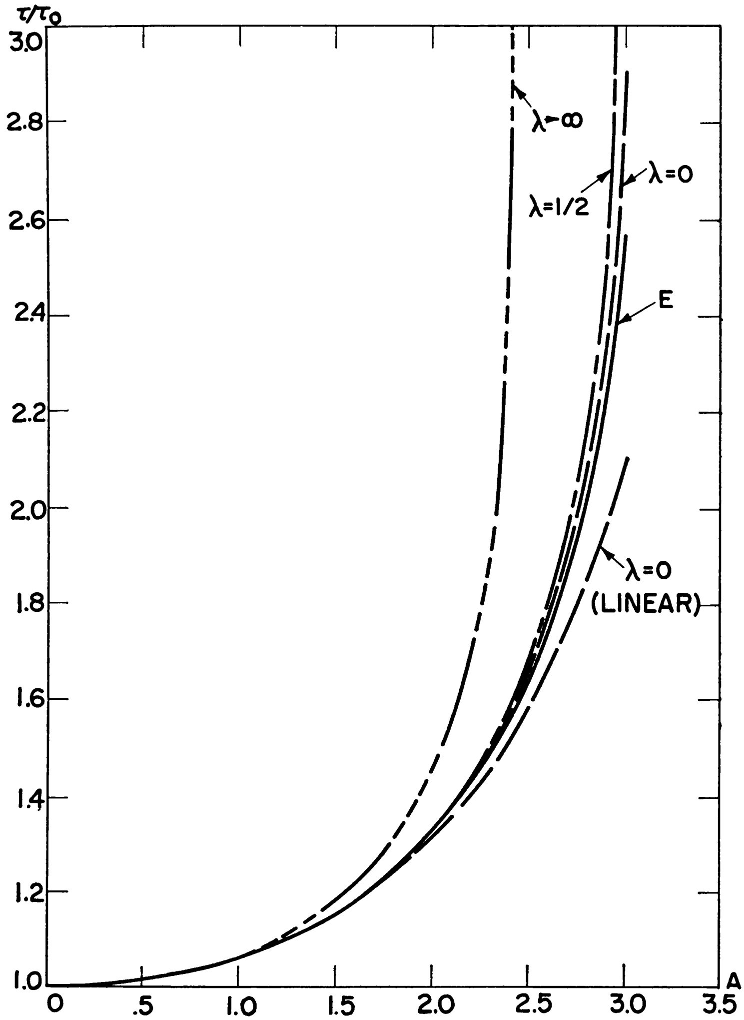

FIG. 7. Period Ratio τ/τ_0 vs. Amplitude A, Cubic Approximation. Sine spring.

certain values of the amplitude A, i.e., for $I_1 < 3I_3$ ($A > 3.9$). When a becomes negative, one must treat (53) as a "softening-hardening" (Case 3) cubic spring. In the present case,

$$\nu = bA^2/a = 4I_3/(I_1 - 3I_3). \tag{54}$$

For a given A, one can compute ν. If ν is positive, use Fig. 1 or 2; if ν is negative, use Fig. 4. Equate the period ratio found to $\tau^{**} |a|^{1/2}/2\pi$, and solve for the cubic-approximation period τ^{**}/τ_0 . An example is given to illustrate the procedure.

Example: Given $A = 2$. Then $\nu = 0.8934$, and Fig. 1 gives $\tau/\tau' = 0.776$. Hence, $\tau^{**}/\tau_0 = 0.795$, while a solution of (53) directly in terms of elliptic integrals yields 0.7954.

Using the procedure shown for $\lambda = \frac{1}{2}$ also, one obtains results shown in Fig. 8. Note that the approximations for $\lambda = 0$ and $\lambda = \frac{1}{2}$ are so close to the exact solution that the

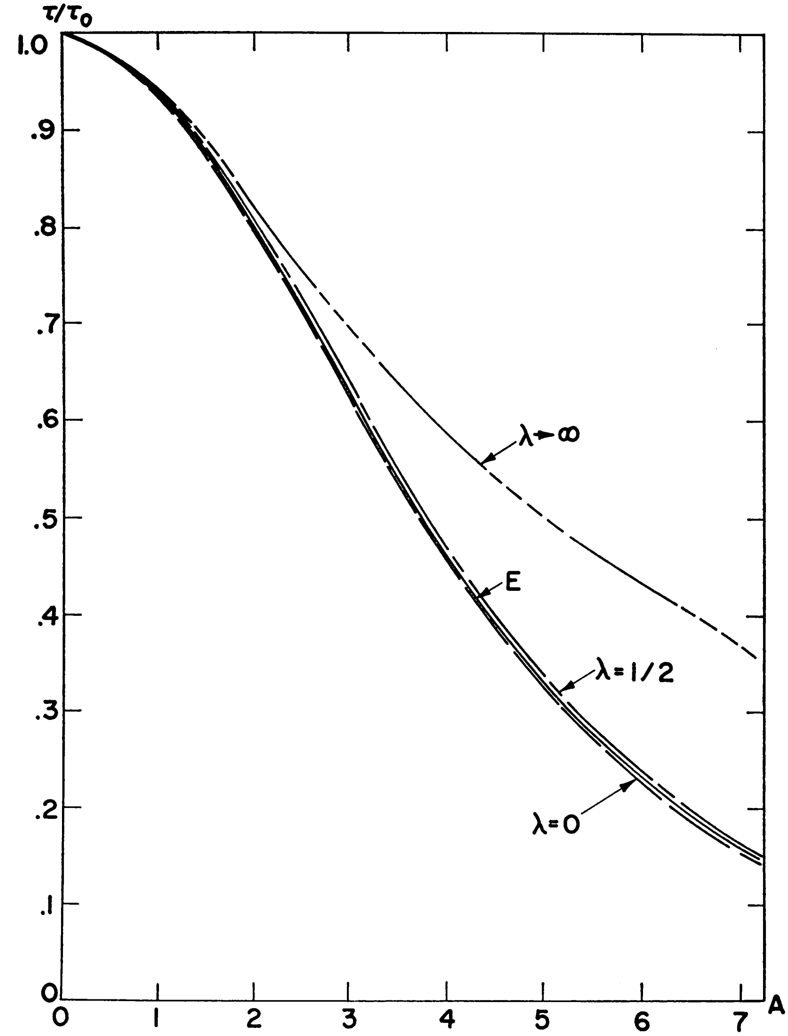

FIG. 8. Period Ratio τ/τ_0 vs. Amplitude A, Cubic Approximation. Sinh spring.

errors are hardly discernible. The result of the cubic Maclaurin series approximation has also been plotted for comparison.

7. Connection with method of Krylov and Bogoliuboff. The first approximation of Krylov and Bogoliuboff [10] yields the period approximation for (1)

$$(\tau)^*_{\text{K.B.}} = 2\pi\left[(1/A\pi)\int_0^{2\pi} f(A\cos\theta)\cos\theta\,d\theta\right]^{-1/2} \tag{55}$$

On the other hand, the linear ultraspherical polynomial approximation gives, from (5) and (6),

$$(\tau^*)_{\text{U.P.}} = 2\pi\left[\frac{4\Gamma(\lambda+2)}{\pi^{1/2}\Gamma(\lambda+\frac{1}{2})A}\int_0^{\pi/2} f(A\cos\theta)\sin^{2\lambda}\theta\cos\theta\,d\theta\right]^{-1/2}. \tag{56}$$

We note that (55) and (56) are equivalent for odd $f(x)$ when $\lambda = 0$ in (56). Hence a linear Tchebycheff polynomial approximation of $f(x)$ in (1) gives results identical to those of the first approximation of Krylov and Bogoliuboff.

Higher order approximations in the Krylov and Bogoliuboff method depend on an iteration procedure involving an approximate amplitude, whereas the cubic ultraspherical polynomial approximation yields a solution in elliptic integrals of the first kind, involving the correct amplitude. No general comparison was made.

8. Discussion. The characteristic feature of *any* ultraspherical polynomial approximation ($\lambda < \infty$) for arbitrary $f(x)$ in (1) is that it yields an approximate period which will in general depend on the amplitude of the motion. The exact solution of a non-linear oscillation problem also has this property, while the *linear* Maclaurin series approximation does not. It is this feature which stimulated these investigations, while the use of particular ultraspherical polynomials permitted comparison of results for various values of λ.

As was shown in Section 5 and indicated in all the graphical results, for small oscillations and $f(x)$ odd, the use of Tchebycheff polynomials ($\lambda = 0$) not only gives the best results of all the ultraspherical polynomials, but also τ^* agrees with the exact result to terms of order A^2. For large amplitudes or non-linearities, no such general result could be obtained. For the hardening cubics (Cases 1 and 3) and the hard sinh force, the λ value yielding the correct asymptotic form for τ^*, as A or ν becomes infinite, was small and positive, while $\lambda = 0$ gives smaller errors out to quite large amplitudes. For the only flattening spring considered here (tanh), the choice $\lambda = -0.075$ gives the correct asymptotic result for large amplitudes.

For the softening cubic (Case 2) and the soft sine force, oscillations at unlimited amplitudes cannot occur. Instead a critical amplitude exists at which the exact period becomes infinite. It can be shown that the linear ultraspherical polynomial approximation which produces the same critical amplitude corresponds to $\lambda = -\frac{1}{2}$. Although the integrals for the coefficients do not exist for this value, the polynomials and the result for $\tau^*_{-1/2}$ can be obtained by a limiting process. The numerical results were quite poor for the problems considered here, despite its yielding the correct ν_c or A_c.

The extension to cubic ultraspherical approximations yields the expected increase in the accuracy of the results τ^{**}. In comparing this extension with other higher order procedures, one notes that the cubic ultraspherical approximation method can be applied without a substantial increase in work, in contrast to higher order approximations in other methods, as the perturbation treatment. This is due to two factors. First, if the coefficient of the linear term $a_1^{(\lambda)}$ can be found, then $a_3^{(\lambda)}$ can usually be found in a similar manner. If $a_1^{(\lambda)}$ must be evaluated numerically, then $a_3^{(\lambda)}$ will probably have to be also, but with no greater difficulty. Secondly, the procedure uses the fact that not only are the exact results for any cubic known in terms of elliptic integrals, but also τ/τ' depends only on one parameter ν, so that one can obtain τ/τ' graphically or in tabular form *once* and use these results for the cubic ultraspherical approximations.

Finally, at no point in these approximation procedures was it necessary to restrict the non-linearity ν or the amplitude A to a small value. In fact, *all* ultraspherical polynomials ($\lambda < \infty$) gave reasonably good results as ν (or A) became infinite.

Given a (non-linear) $f(x)$ other than those studied here (and for which the amplitude does not become extremely large), then, if a single ultraspherical polynomial approximation is desired, the choice $\lambda = 0$ is indicated by the results given here.

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