

$$A(x^+, t^+) = \int_0^{t^+} G_{i_j k_l}^{(1)}(x^+, t^+ - s) u_{k, l_j}(x^+, s) ds + \int_t^{t^+} G_{i_j k_l}^{(1)}(x^+, t^+ - s) u_{k, l_j}(x^+, s) ds. \tag{2.18}$$

Let Λ denote the compact set $\Omega \times [0, T]$, where Ω is a solid sphere centered at x and contained in \mathcal{R} while $T > t$. Then hypotheses (iv) and (vi) imply that $G_{i_j k_l}^{(1)}$ and u_{k, l_j} are bounded on Λ . Thus the second term in (2.18) tends to zero as $(x^+, t^+) \rightarrow (x, t)$. Moreover conditions (ii), (v), and (vii) yield

$$G_{i_j k_l}^{(1)}(x^+, t^+ - s) u_{k, l_j}(x^+, s) \rightarrow G_{i_j k_l}^{(1)}(x, t - s) u_{k, l_j}(x, s) \tag{2.19}$$

for almost every $s \leq t$, i.e., for $s \notin D_x$. Therefore the boundedness of $G_{i_j k_l}^{(1)}$ and u_{k, l_j} on Λ and Lebesgue's bounded convergence theorem imply

$$\lim_{(x^+, t^+) \rightarrow (x, t)} A(x^+, t^+) = \int_0^t G_{i_j k_l}^{(1)}(x, t - s) u_{k, l_j}(x, s) ds. \tag{2.20}$$

Clearly the same result follows for $\lim_{(x^-, t^-) \rightarrow (x, t)} A(x^-, t^-)$ and thus

$$[G_{i_j k_l}^{(1)} * u_{k, l_j}] = 0. \tag{2.21}$$

Equations (2.12), (2.17), and (2.22) imply $[F_i] = 0$. It therefore follows from (2.11), (2.16) that

$$(c_{i_j k_l} n_j n_l - \rho U^2 \delta_{ik}) a_k = 0, \tag{2.22}$$

and this completes the proof.

Acknowledgement. The authors are indebted to W. Edelstein who read the manuscript and offered many helpful criticisms. This research was supported in part (M. E. G.) by the Office of Naval Research under Contract Nonr 562(25) and in part (I. H) by the Institutes of Geophysics and Engineering at the University of Mexico.

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IRREVERSIBLE THERMODYNAMICS OF PERFECT PLASTICITY*

By N. MORGENSTERN, (*Imperial College of Science & Technology, London, England*)

1. Introduction. For the solution of boundary value problems in continuum mechanics, it is necessary to supplement the equation of continuity that expresses the conservation of mass and the equation of motion with the constitutive equation for the material under consideration. In the case of plasticity, the constitutive equation introduces the mechanical properties of the material by means of the relation between the

*Received October 11, 1963; revised manuscript received February 20, 1964.

stress and rate-of-strain tensors. For macro-plasticity, the constitutive equation can be regarded as wholly phenomenological in the sense that it completely describes the mechanical behaviour of the material under all possible loading arrangements without making any statement regarding the structure of the material or the relationship between the components of the material and its behaviour. Thoughts concerning the composition of the material may be useful in designing experiments in order to arrive at the correct constitutive equation, but they are not essential. Indeed, it is owing to the phenomenological nature of constitutive equations that we find the same description of mechanical behaviour applied to materials of widely differing compositions.

The study of the mechanical behaviour of materials usually exists on two levels. On the first, effort is directed toward obtaining a complete and consistent description of the response of the material to deformation, while on the second level the effort is directed toward reducing the macroscopic description to a theory based upon the structure of the material. It is usually necessary to invoke new phenomenological statements regarding the structural or micro-level in order to achieve this reduction. As an intermediate step in this process it may be useful to translate the behaviour described by the constitutive equation into thermodynamic terms which provides an alternative phenomenological theory. A thermodynamic interpretation of continuum mechanics is also of considerable interest in the development of a unified theory for the alternative rheological bodies.

An examination of the relationship between the theory of plasticity and the thermodynamics of irreversible processes has been initiated in several papers by Ziegler [1, 2], Wehrli and Ziegler [3], Kluitenberg [4], and others. It has usually been assumed that the internal energy is a function of the plastic strains and that the rate of entropy production is a maximum. In what follows it will be argued that during isothermal perfectly plastic deformation the change in internal energy and the rate of internal entropy production are both zero. Then if the rate of change of entropy per unit volume be taken as the plastic work of distortion and it be assumed that the entropy is a maximum, the plastic potential due to von Mises [5] is obtained. For isothermal deformation of an isotropic medium it is not necessary to introduce Onsager's relations. The resulting analysis is wholly consistent with an extension of the entropy concept for plastic deformation suggested by Bridgman [6].

Following Bridgman, the stress-strain relation for an ideal elastic-plastic body is shown in Fig. 1. Owing to the requirements of the first law of thermodynamics, the energy at any given point *A* must be path-independent although this point could be reached by different paths composed of elastic and plastic deformations. It is also required that the mechanical work done on any loop be equal to the total heat absorbed. Since the heat and work are equal on the elastic paths of any loop, the work of plastic deformation must be completely converted into heat which flows out of the body, inducing an increase in the entropy of the environment. Experiments by Taylor and Quinney [7] showed that during plastic deformation of copper and iron, at least 85% of the work of deformation was dissipated in the form of heat. Bridgman's argument then led him to the conclusion that the internal entropy production is zero and he states ". . . there is no progressive disorganization within the body as it is carried through one plastic cycle after another. The behaviour is what would be produced by internal slippage on one or another well organized plane, that is, a purely mechanical affair with no associated change of entropy."

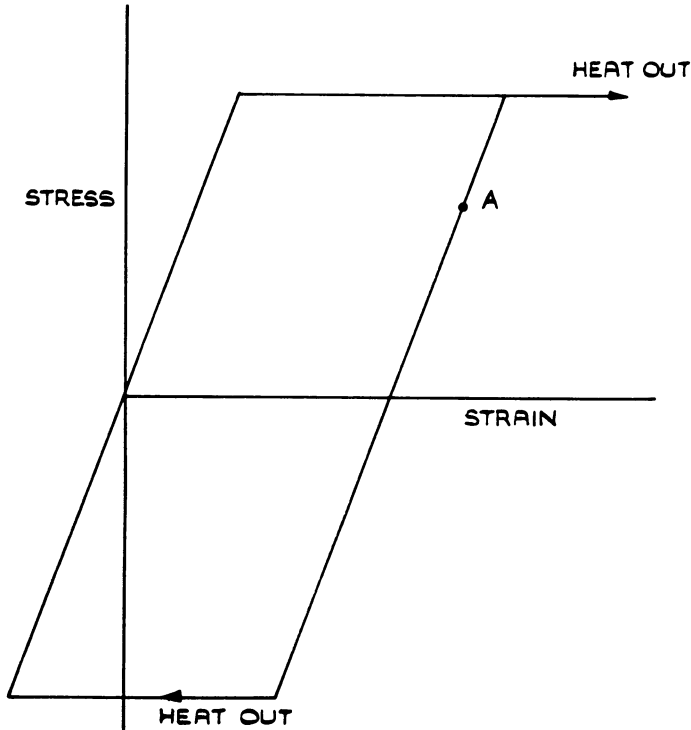


FIG. 1. Stress-Strain Behavior of an Ideal Elastic-Plastic Material.

2. The extremum principle of perfect plasticity. Only isothermal perfectly plastic deformation will be considered in the following. Assuming small deformations the plastic strain rate tensor ϵ'_{ij} is defined in terms of the velocities u_i by

$$\epsilon'_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (1)$$

where the subscript i following a comma denotes partial differentiation with respect to the rectangular Cartesian coordinate x_i . The equations of equilibrium are

$$\sigma_{ij,i} + X_i = 0, \quad (2)$$

where σ_{ij} is the stress tensor and X_i the vector of the body force per unit volume.

It is well known [8], that the first law of thermodynamics for a unit volume of a continuous medium subject to infinitesimal deformation may be expressed by

$$\frac{du}{dt} = \sigma_{ij}\epsilon'_{ij} - q_{i,i}, \quad (3)$$

where du/dt represents the rate of change of the specific internal energy per unit volume, the product $\sigma_{ij}\epsilon'_{ij}$ is the rate of plastic work, and q_i denotes the heat flux vector.

On the basis of the experiments of Taylor and Quinney [7] it is suggested that for *perfectly* plastic isothermal flow, the rate of change of internal energy is zero and that all the work performed in plastic deformation is converted into heat. Equation (3) then becomes

$$\sigma_{ij}\dot{\epsilon}_{ij} = q_{i,i} \quad (4)$$

In the terminology of thermodynamics, isothermal, perfectly plastic deformation is an example of a closed system in which energy, but not matter, can be exchanged with the environment. For such a system the second law of thermodynamics is expressed by asserting the existence of the state function, entropy. The rate of change of the entropy dS/dt consists of the sum of two parts. The first is the rate of flow of entropy due to the interaction of the system with its exterior and is denoted by dS_e/dt ; the second is the rate of internal entropy production arising from irreversible changes within the system and it is denoted by dS_i/dt . The second law then states that

$$\frac{dS}{dt} = \frac{dS_e}{dt} + \frac{dS_i}{dt} \quad (5)$$

A detailed discussion of these considerations has been given by Prigogine [9].

For an irreversible process†

$$\frac{dS_i}{dt} \geq 0. \quad (6)$$

When the equality in (6) is satisfied in an irreversible process, the rate of internal entropy production is zero, and the entropy is a maximum. Equation (5) may be written as

$$\frac{ds}{dt} = \frac{ds_e}{dt} + \frac{ds_i}{dt}, \quad (7)$$

where s , s_e , and s_i denote the entropy, the flow of entropy and the internal entropy per unit volume, respectively.

For a unit volume, the flow of entropy is defined by

$$T \frac{ds_e}{dt} = -q_{i,i} \quad (8)$$

where T is the absolute temperature.

By substituting (7) and (8) into (4), and considering that heat is being removed, one readily shows that

$$\sigma_{ij}\dot{\epsilon}_{ij} = T \frac{ds}{dt} - T \frac{ds_i}{dt} \quad (9)$$

or

$$s_{ij}\dot{e}_{ij} + 3p\dot{\epsilon}_v = T \frac{ds}{dt} - T \frac{ds_i}{dt}, \quad (10)$$

where s_{ij} and \dot{e}_{ij} are the deviatoric parts of the stress and strain-rate tensors, $p\dot{\delta}_{ij}$ is the hydrostatic component of the stress tensor, $\dot{\epsilon}_v\delta_{ij}$ is the volumetric component of the strain-rate tensor, and δ_{ij} is Kronecker's delta.

The state of a perfectly plastic medium is also taken to be defined by the stresses and this condition is expressed by stating that the stresses satisfy the yield criterion

$$\phi(J_1, J_2, J_3) = 0, \quad (11)$$

where J_1 , J_2 , and J_3 are the invariants of the stress tensor σ_{ij} .

†For reversible processes, $dS_i/dt = 0$.

The energy dissipated in an irreversible volume change is taken as the measure of the internal entropy production and for perfectly plastic flow this is zero. Such a viewpoint is consistent with Bridgman's statement [6] that there is no progressive disorganization within the material undergoing plastic deformation. Either the nature of the deformation is such that it does not permit disorganization as might be assumed for slip along a defined crystallographic plane or the medium has attained a condition of maximum possible disorder as may be the case for a granular material shearing under conditions of no further volumetric strain.

It is now asserted that the conditions for perfectly plastic deformation is that the entropy be a maximum subject to the conditions (11) and that

$$-T \frac{ds_i}{dt} = 3p\epsilon_{\dot{v}} = 0. \quad (12)$$

Varying the deviatoric components of the stress tensor and using a Lagrangian multiplier, this condition is satisfied when

$$e_{i,i} = \lambda \cdot \frac{\partial \phi}{\partial s_{i,i}} \quad (13)$$

and the results given by the von Mises extremum principle are recovered, with $e_{i,i}$ such that $\epsilon_{i,i} = \epsilon_{\dot{v}} = 0$.

3. Concluding remarks. It has been shown that the theory of the plastic potential for perfectly plastic deformation can be derived from irreversible thermodynamics by equating the rate of work dissipated in irrecoverable volume change with the rate of internal entropy production. By imposing the condition that during isothermal deformation, all of the plastic work is dissipated in heat and assuming that the entropy is a maximum, the constitutive equation is obtained. This relationship, given by Eq. (13), is valid for granular materials for which the yield criterion depends upon the first stress invariant, as well as for metals.

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