THE GAIN OF A DEFOCUSED CIRCULAR APERTURE*

BY

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1. Introduction. Beam shaping and broadening in phased array applications are achievable through deliberate defocusing. The following analysis provides the generalized defocused pattern for a continuous circular aperture in order to arrive at the general expression for the on-axis gain. The pattern is derived as an infinite series of confluent hypergeometric functions, while the gain is obtained in terms of the incomplete Gamma function of complex argument. The type of defocusing considered is relevant to a constant amplitude illumination with an aperture phase distribution of the form $\psi = \beta_K r^K$, where β_K indicates the phase at the aperture boundary (maximum phase), and r is the normalized radius. The results, although pertinent to continuous apertures, are equally applicable to circular arrays.[1]

2. The defocused continuous circular aperture.

A. *The generalized pattern for defocusing of the form* $\beta_K r^K$.

The Fraunhofer Diffraction Pattern, as expressed in integral form, of a continuous circular aperture is[2]

$$g(\mu, \phi) = b^2 \int_0^{2\pi} \int_0^1 F(r, \phi') \exp(j\mu r \cos(\phi - \phi'))r \, dr \, d\phi'. \tag{1}$$

The coordinate system used is shown in Figure 1. The aperture radius is b, $F(r, \phi')$ is the illumination function, and

$$\mu = \frac{2\pi b}{\lambda} \sin \theta. \tag{2}$$

The aperture illumination function which will account for the particular defocusing philosophy under consideration is

$$F(r) = \exp(-j\beta_K r^K). \tag{3}$$

After integration with respect to ϕ' Eq. (1) becomes,

$$g(\mu) = 2\pi b^2 \int_0^1 J_0(\mu r) \exp(-j\beta_K r^K) r \, dr. \tag{4}$$

By representation as a series, such that

$$J_0(\mu r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\mu r}{2}\right)^{2n} \tag{5}$$

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[1]The circular array will be defined as being planar in which the aperture boundary describes a circle in contrast to an array of elements positioned on the periphery of a circle.

[2]S. Silver, *Microwave antenna theory and design*, vol. 12, McGraw-Hill, New York, 1949, p. 192.

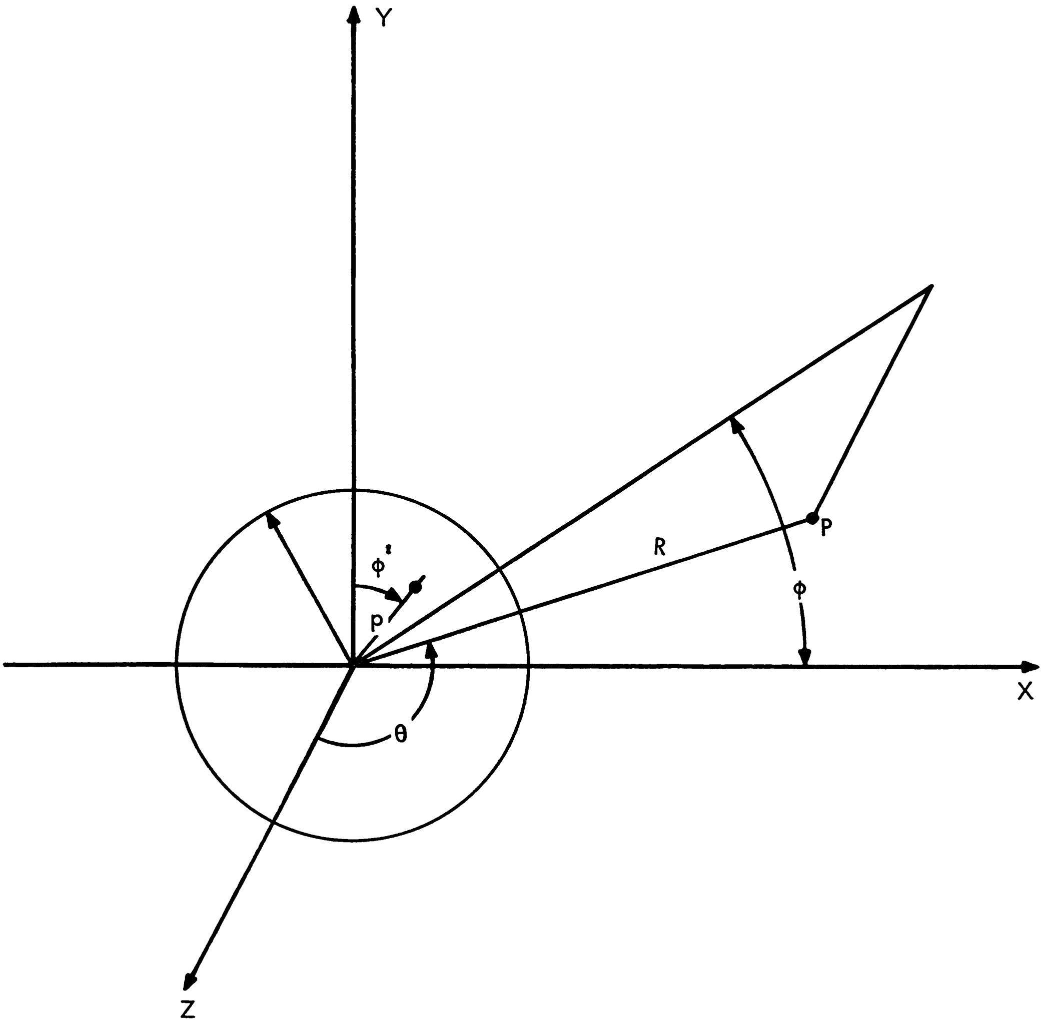

FIG. 1. (ϕ, θ, R) spatial spherical coordinate system centered on the z-axis; (ρ, ϕ') cylindrical aperture coordinates.

and

$$\exp(-j\beta_K r^K) = \sum_{m=0}^{\infty} \frac{(-j\beta_K)^m r^{mK}}{m!}. \tag{6}$$

The integrand in (4) may be expressed as the product of two infinite series:

$$g(\mu) = 2\pi b^2 \int_0^1 \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(-j\beta_K)^m}{m!(n!)^2} \left(\frac{\mu}{2}\right)^{2n} r^{2n+mK+1} \right\} dr. \tag{7}$$

After integrating (7), we find

$$g(\mu) = 2\pi b^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(-j\beta_K)^m}{(n!)^2 m!} \left(\frac{\mu}{2}\right)^{2n} \left[\frac{1}{mK + 2n + 2}\right] \tag{8}$$

or

$$g(\mu) = 2\pi b^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\mu}{2}\right)^{2n} \left\{ \frac{1}{(2n+2)} - \frac{j\beta_K}{(K+2n+2)} - \frac{\beta_K^2}{2!(2K+2n+2)} + j\frac{\beta_K^3}{3!(3K+2n+2)} + \cdots \right\}. \tag{9}$$

The series in brackets is

$$Y = \sum_{m=0}^{\infty} \frac{(-j\beta_K)^m}{m!(Km+2n+2)} \tag{10}$$

and may be re-written, letting $\alpha = 2n + 2/K$

$$Y = \frac{1}{(2n+2)} \left\{ 1 - j\frac{\beta_K(\alpha)}{(1+\alpha)} - \frac{\beta_K^2(\alpha)(1+\alpha)}{2!(1+\alpha)(2+\alpha)} + \cdots \right\}. \tag{11}$$

Here, $(2n+2)Y$ is Kummer's[3] confluent hypergeometric equation of the form

$${}_1F_1(a; c; X) = 1 + \frac{aX}{C} + \frac{a(a+1)X^2}{C(C+1)2!} + \cdots, \tag{12}$$

where

$$a_n = \frac{2}{K}(n+1), \qquad C_n = 1 + a = 1 + \frac{2}{K}(n+1) \tag{13}$$

and $X = -j\beta_K$.

In general, the radiation pattern will be

$$g(\mu) = \pi b^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2(n+1)} \left(\frac{\mu}{2}\right)^{2n} \{{}_1F_1(a; C; -j\beta_K)\}. \tag{14}$$

B. The Variation in Gain with Maximum Phase (β_K).

1. *The general gain expression.* The field intensity on the boresight axis ($\mu = 0$) is the first term of Equation (14). Since for the initial term, $n = 0$,

$$g_0 = \pi b^2 \, {}_1F_1(a; C; -j\beta_K). \tag{15}$$

In each case $C = 1 + a$ so that[4]

$${}_1F_1(a; 1+a; -j\beta_K) = \frac{a}{(j\beta_K)^a} \gamma(a; j\beta_K), \tag{16}$$

where $\gamma(a; j\beta_K)$ is the incomplete Gamma function defined integrally as

$$\gamma(a; j\beta_K) = \int_0^X \epsilon^{-Y} Y^{(a-1)} \, dY \{\mathrm{Re}\, a > 0,\ X = j\beta_K\} \tag{17}$$

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[3]L. J. Slater, *Confluent hypergeometric functions,* Cambridge, 1960, p. 2.
[4]Slater, *loc. cit.,* p. 96.

and equivalently represented in series form as

$$\gamma(a; X) = \epsilon^{-X} X^{a} \sum_{s=0}^{\infty} \frac{X^{s}}{(a)s + 1}, \tag{18}$$

where

$$(a)_{s+1} = a(a+1)(a+2) \cdots (a+s). \tag{19}$$

The relative directive gain, as indicated by the square of the on-axis amplitude when normalized to the uniform case, may be described either as, the confluent hypergeometric function of complex argument[5]

$$G = \left| {}_1F_1\left(\frac{2}{K}; 1 + \frac{2}{K}; -j\beta_K\right)\right|^2, \tag{20}$$

or more specifically, as related to the incomplete Gamma function by

$$G = \left| \sum_{s=0}^{\infty} \frac{(j\beta_K)^s}{\left(\frac{2}{K}\right)_s} \right|^2. \tag{21}$$

2. *Gain relations for linear defocusing.* In the linear case $K = 1$, we have

$$G_1 = |{}_1F_1(2; 3; -j\beta_K)|^2, \tag{22}$$

which may be depicted in closed form as[6]

$$G_1 = \left| \frac{1 - \exp(-j\beta_1) - j\beta_1 \exp(-j\beta_1)}{-\beta_1^2} \right|^2. \tag{23}$$

Therefore,

$$G_1 = \beta_1^{-4}\left\{4 \sin^2 \frac{\beta_1}{2} + \beta_1(\beta_1 - 2 \sin \beta_1)\right\}. \tag{24}$$

3. *Gain relations for quadratic defocusing.* In this case $K = 2$, and $a = 1$. As shown by Slater,[7]

$$G_2 = \left| \frac{1 - \exp(-j\beta_2)}{j\beta_2} \right|^2 \tag{25}$$

since

$$\gamma(1; j\beta_2) = 1 - \exp(-j\beta_2) \tag{26}$$

and finally,[8]

$$G_2 = \left| \frac{\sin \beta_2/2}{\beta_2/2} \right|^2 \tag{27}$$

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[5]Note: ${}_1F_1(a; C; 0) = 1$, Slater, *op. cit.*, p. 116.

[6]D. Middleton, *Introduction to statistical communications*, McGraw-Hill, 1960, p. 1074, Eq. (A.1.1.9d).

[7]Slater, *op. cit.*, p. 96.

[8]Silver, *op. cit.*, p. 199.

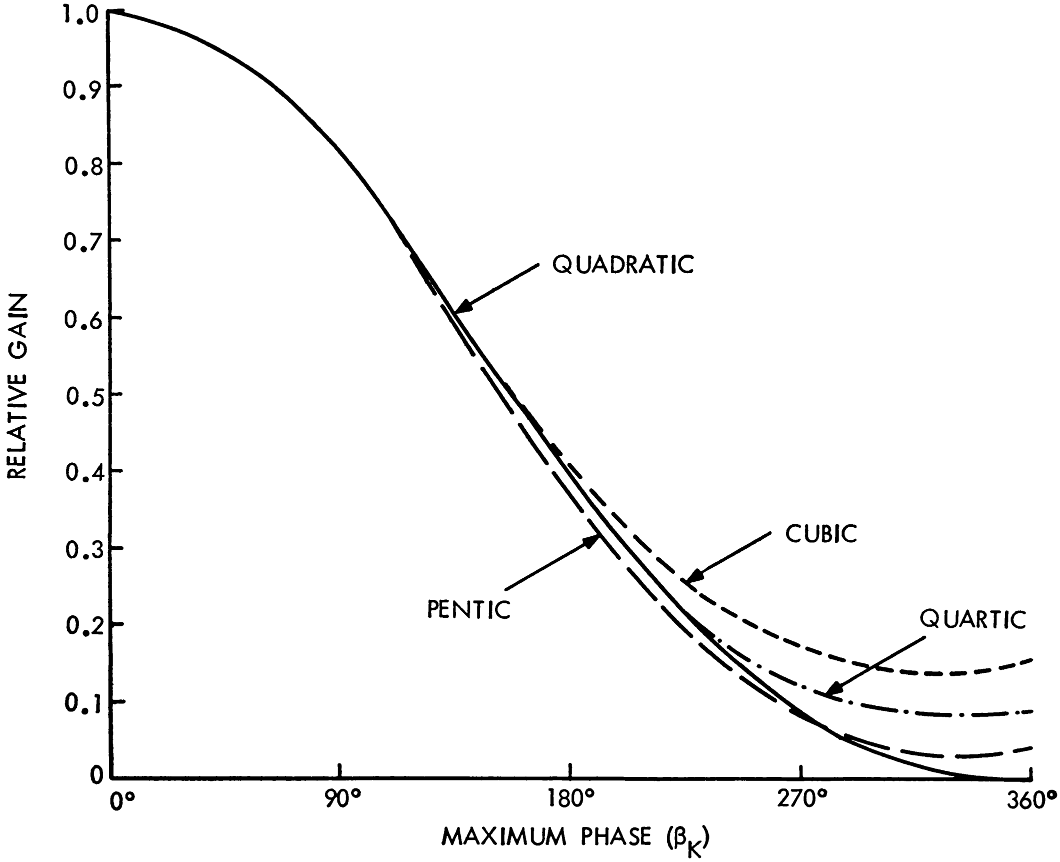

Fig. 2. The variation in relative gain with maximum phase.

4. *Gain relations for quartic defocusing.* With quartic defocusing ($K = 4$), we have

$$G_4 = |{}_1F_1(\tfrac{1}{2};\ \tfrac{3}{2};\ -j\beta_4)|^2. \tag{28}$$

The Fresnel integrals are, by definitions

$$C(\beta_4) = \int_0^{\beta_4} \frac{\cos T}{T^{1/2}}\, dt, \tag{29}$$

$$S(\beta_4) = \int_0^{\beta_4} \frac{\sin T}{T^{1/2}}\, dt, \tag{30}$$

and their identification with the confluent hypergeometric function is[9]

$$C(\beta_4) = \left[\frac{\beta_4}{2\pi}\right]^{1/2} \{{}_1F_1(\tfrac{1}{2};\ \tfrac{3}{2};\ -j\beta_4) + {}_1F_1(\tfrac{1}{2};\ \tfrac{3}{2};\ j\beta_4)\}, \tag{31}$$

[9] *Higher Transcendental Functions,* Bateman Manuscript Project, vol. 2, McGraw-Hill, 1953, p. 267, Eqs. 29 and 30.

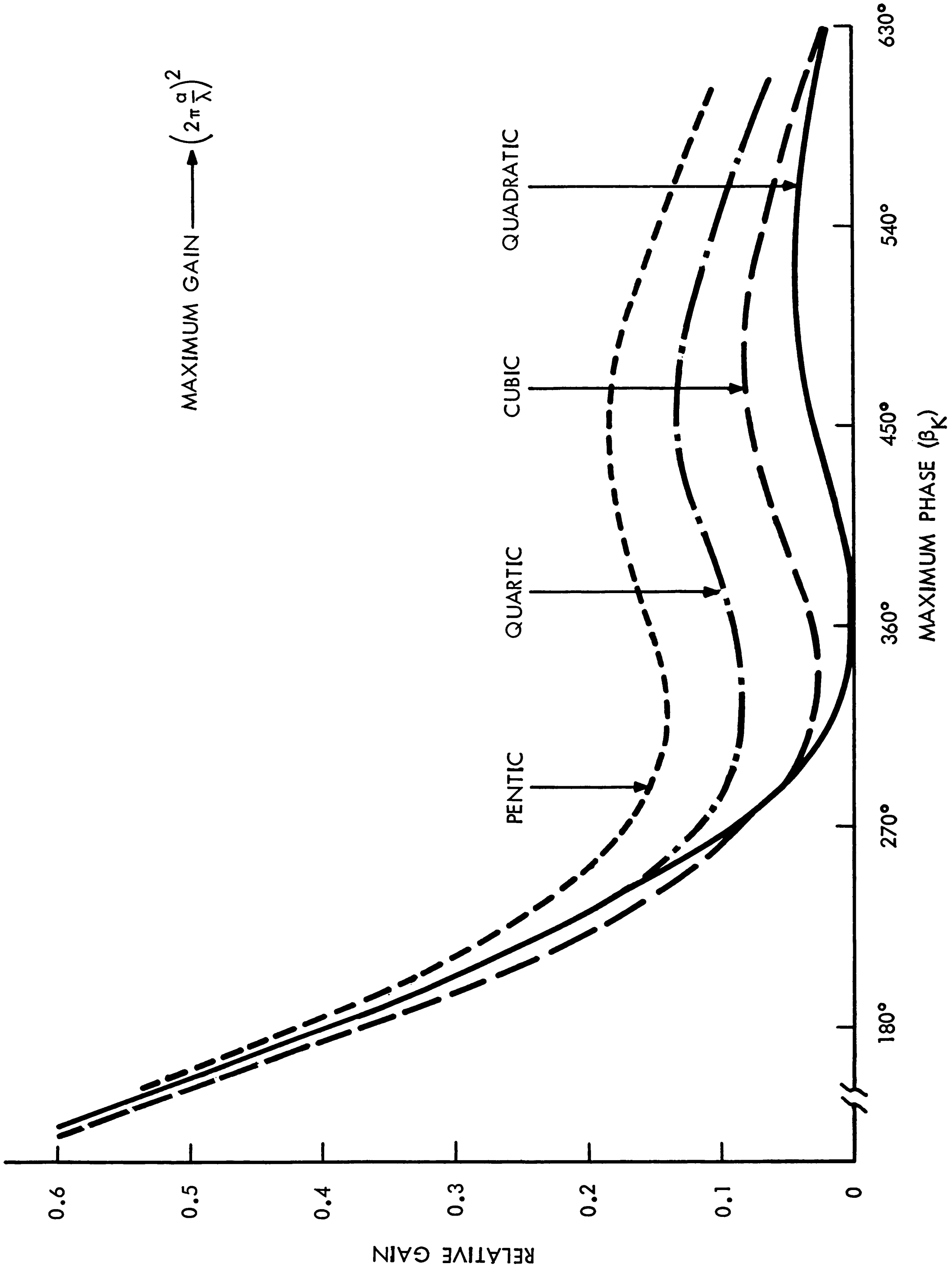

Fig. 3. The variation in gain with maximum phase as compared to the uniform case.

$$S(\beta_4) = j\left[\frac{\beta_4}{2\pi}\right]^{1/2}\{{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};-j\beta_4) - {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};j\beta_4)\}. \tag{32}$$

In manipulating the two previous equations, it is evident that,

$${}_1F_1(\tfrac{1}{2};\tfrac{3}{2};-j\beta_4) = \left[\frac{\pi}{2\beta_4}\right]^{1/2}\{C(\beta_4) - jS(\beta_4)\} \tag{33}$$

$$G_4 = \frac{\pi}{2\beta_4}\{C^2(\beta_4) + S^2(\beta_4)\}. \tag{34}$$

Figures 2 and 3 illustrate the variation in gain as a function of maximum phase for quadratic, cubic, quartic, and pentic radial phase variations. The gain shown is the on-axis gain normalized to that obtained with a uniformly illuminated circular aperture. For maximum defocusing phases greater than about 1.5π, the on-axis gain and the maximum gain are not necessarily concurrent.