2. W. Hahn, *Theorie und Anwendung der direkten Methode von Ljapunov*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 22, Springer-Verlag, 1959
3. R. E. Kalman, Y. C. Ho and K. S. Narendra, *Controllability of linear dynamical systems*, Contributions to Differential Equations, Vol. 1, Interscience, 1963, pp. 189–213
4. J. LaSalle and S. Lefschetz, *Stability by Liapunov's direct method with applications*, Academic Press, 1961
5. J. P. LaSalle, *The time optimal control problem*, Contributions to the Theory of non-linear Oscillations, vol. 5, Princeton University Press, 1960, pp. 1–24
6. W. T. Reid, *Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems*, Pacific J. Math. **13** (1963) 665–685
7. W. T. Reid, *Principal solutions of non-oscillatory linear differential systems*, to appear in J. Math. Anal. and Appl.

CONDITIONS FOR THE CAUSALITY OF NONLINEAR OPERATORS DEFINED ON A FUNCTION SPACE*

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Abstract. This note considers nonlinear operators T with both range and domain subsets of the space of complex n -vector-valued functions of the real variable t for $-\infty < t < \infty$. Conditions, in which energy-type quantities play a key role, are presented under which T is causal in the following sense:

Let $\mathfrak{D}(T)$ denote the domain of T and let $t_0 = \sup \{t' \mid \text{for all } f \in \mathfrak{D}(T), f(t) = 0 \text{ for almost all } t < t'\}$. Then T is causal if for an arbitrary $\delta > t_0$, $Tf = Tg$ a.e. on (t_0, δ) whenever f and g belong to $\mathfrak{D}(T)$ and $f = g$ a.e. on (t_0, δ) .

1. Notation. Let $\mathfrak{C}_n$ denote the space of complex measurable n -vector-valued functions of the real variable t for $-\infty < t < \infty$. The complex-conjugate transpose of an arbitrary $f \in \mathfrak{C}_n$ is written as f^* . With g and h arbitrary elements of $\mathfrak{C}_n$, and x either an arbitrary real number or ∞ , let $\langle g, h; x \rangle$ denote

$$\int_{-\infty}^x g^* h \, dt,$$

and let

$$||g; x|| = (\langle g, g; x \rangle)^{1/2}.$$

The symbol $\mathfrak{R}$ denotes the set of real-valued functions and

$$\mathfrak{L}_{2n} = \{f \mid f \in \mathfrak{C}_n, \langle f, f; \infty \rangle < \infty\}.$$

If $f \in \mathfrak{C}_n$, and $x < \infty$, then f_x is defined by

$$\begin{aligned} f_x &= f, & t &\leq x \\ f_x &= 0, & t &> x. \end{aligned}$$

2. Introduction. The external properties of a physical system can frequently be characterized by an operator relation of the form

$$g = Tf,$$

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in which g and f are n -vector-valued functions of t such that $\text{Re } (g^*f)$ represents the total instantaneous power entering the system. For example, for an electro-mechanical system, the j th component of f might be the voltage or force at the j th accessible point of the system, in which case g would be the corresponding n -vector of currents and velocities.

The notions of energy and causality play a central role in the study of the properties of physical systems. It is of interest therefore to consider the relation between energy type properties of a general operator T and the causality or lack of causality of T .

In order to be more specific, it is convenient to introduce the following definitions in which T denotes an arbitrary single-valued (not necessarily linear) operator with range and domain contained in $\mathcal{H}_n$. The domain of T is denoted by $\mathcal{D}(T)$.

Definition 1: We shall say that T is *passive* if for all real x and all $f \in \mathcal{D}(T)$, $\langle Tf, f; x \rangle$ is well-defined and finite, and $\text{Re } \langle Tf, f; x \rangle \geq 0$.

Definition 2. Let $t_0 = \sup \{t' \mid \text{for all } f \in \mathcal{D}(T), f(t) = 0 \text{ for almost all } t < t'\}$, with the understanding that $t_0 = -\infty$ if for every t' there exist a positive-measure subset S of $(-\infty, t')$, and an $f \in \mathcal{D}(T)$, such that $f(t) \neq 0$ for $t \in S$.

It is assumed throughout that $t_0 < \infty$.

Definition 3. We shall say that T is *causal* if for an arbitrary $\delta > t_0$, $Tf = Tg$ a.e. on (t_0, δ) whenever $f, g \in \mathcal{D}(T)$ and $f = g$ a.e. on (t_0, δ) .

In their interesting paper,¹ Youla, Castriota, and Carlin [1] take $t_0 = -\infty$ in the definition above and prove that a sufficient condition for a *linear passive* T to be causal is that there exist in $\mathcal{D}(T)$ n elements v_j ($j = 1, 2, \dots, n$) such that the $v_j(t)$ are linearly independent for almost every t in $(-\infty, \infty)$.

In this note we present conditions under which nonlinear operators are causal. For example, a direct corollary of our main result asserts that if $\mathcal{D}(T) = \mathcal{L}_{2n}$ and T maps $\mathcal{L}_{2n}$ into itself such that there exists a functional $M(f)$ defined on $\mathcal{L}_{2n}$ with the properties that $M(f) = M(g)$ if $f = g$ a.e., $M(\beta f) \rightarrow 0$ as $\beta \rightarrow 0$, and $\|Th - Tk; \infty\| \leq M(h - k)$ for all $h, k \in \mathcal{L}_{2n}$, then T is causal if and only if

$$\text{Re } \langle Th - Tk, h - k; x \rangle \geq -M(h_x - k_x) \|h - k; x\|$$

for all real x and all $h, k \in \mathcal{L}_{2n}$.

A physical interpretation of some of the results is implied by Theorem 3, in which attention is focused on the "weak linearization" of T about an arbitrary element of its domain. For example, with T a mapping of $\mathcal{L}_{2n}$ into itself, if every weak linearization of T yields a passive operator, then Theorem 3 asserts that $\text{Re } \langle Th - Tk, h - k; x \rangle \geq 0$ for all $h, k \in \mathcal{L}_{2n}$. Thus, it follows from our results that if $\mathcal{D}(T) = \mathcal{L}_{2n}$ and T maps $\mathcal{L}_{2n}$ into itself such that there exists a functional $M(f)$ with the properties stated above, T is causal if the weak linearization of T about an arbitrary element of $\mathcal{L}_{2n}$ yields a passive operator.

3. Results. Our principal result is the following theorem.

Theorem 1. Let T be a (not necessarily linear) mapping into $\mathcal{H}_n$, with domain any subset $\mathcal{D}(T)$ of $\mathcal{H}_n$ such that:

- (1) $\langle Th - Tk, h - k; x \rangle$ is well-defined and finite for all real x and all $h, k \in \mathcal{D}(T)$.
- (2) $\text{Re } \langle Th - Tk, h - k; x \rangle \geq U(h - k; x)$ for all real x and all $h, k \in \mathcal{D}(T)$, where $U(f; x)$ is a functional defined on $\mathcal{E}(T) = \{f \mid f = h - k \text{ with } h, k \in \mathcal{D}(T)\}$ such that
 - (a) $U(f; x) = U(g; x)$ if $f, g \in \mathcal{E}(T)$ and $f = g$ for almost all $t < x$.
 - (b) $\beta^{-1}U(\beta f; x) \rightarrow 0$ as $\beta \rightarrow 0$, with β real or pure imaginary.

(3) for any $v \in \mathfrak{D}(T)$, there exists a positive number σ_0 and a subset $\{v_i, 1 \leq j \leq n\}$ of $\mathfrak{H}_n$ such that

- (a) the $v_i(t)$ are linearly independent for almost all $t \in (t_0, \infty)$
- (b) $\{v + \sigma v_i, 1 \leq j \leq n\} \subset \mathfrak{D}(T)$ for every real or pure imaginary number σ such that $|\sigma| \leq \sigma_0$
- (c) for all real x , and σ real or pure imaginary, $\langle T(v + \sigma v_i) - T(v), v_i; x \rangle \rightarrow 0$ as $\sigma \rightarrow 0$.

Then T is causal.

Proof. Assume that $f, g \in \mathfrak{D}(T)$ and that $f = g$ a.e. on (t_0, δ) , where δ is an arbitrary real number such that $\delta > t_0$. Let $\{v_i, 1 \leq j \leq n\}$ have the properties stated in the theorem, with $v = g$.

Consider the following inequality which, according to the second hypothesis, is valid for all real x

$$\operatorname{Re} \langle T(g + \sigma v_i) - Tg - [Tf - Tg], g - f + \sigma v_i; x \rangle \geq U(g - f + \sigma v_i; x) \quad (1)$$

Because $f = g$ a.e. on $(-\infty, \delta)$, it follows that for $x < \delta$ and every real or pure imaginary σ such that $|\sigma| \leq \sigma_0$

$$\operatorname{Re} \{ \sigma \langle T(g + \sigma v_i) - T(g), v_i; x \rangle \} - \operatorname{Re} \{ \sigma \langle Tf - Tg, v_i; x \rangle \} \geq U(\sigma v_i; x). \quad (2)$$

This inequality implies that

$$\langle Tf - Tg, v_i; x \rangle = 0 \quad \text{for } x \in (-\infty, \delta), \quad (3)$$

since otherwise, in view of the assumptions that $\langle T(g + \sigma v_i) - Tg, v_i; x \rangle$ and $\sigma^{-1}U(\sigma v_i; x)$ approach zero as $\sigma \rightarrow 0$, there would exist an $x \in (-\infty, \delta)$ and a real or pure imaginary σ with $|\sigma| \leq \sigma_0$ for which (2) was not satisfied.

It is obvious that the derivative of the left-hand side of (3), with respect to x , vanishes for $x \in (-\infty, \delta)$. Thus

$$[Tf - Tg]^* v_i = 0 \quad \text{a.e. on } (-\infty, \delta)$$

for $j = 1, 2, \dots, n$, which, since the $v_i(t)$ are linearly independent for almost all $t \in (t_0, \infty)$, implies that

$$Tf = Tg \quad \text{a.e. on } (t_0, \delta).$$

This proves the theorem.[†]

Remarks. It is evident from the proof of Theorem 1 that the conclusion remains valid if $\mathfrak{D}(T) = \mathfrak{R} \cap \mathfrak{D}(T)$ and hypotheses 2(b), 3(b), and 3(c) are satisfied only for real β and real σ .

Observe that condition (3) of the theorem is satisfied except in rather pathological cases. A very simple example showing that a condition of the type 3(a) is actually needed is given in Ref. 1.

Theorem 2. Let T be a mapping of $\mathfrak{D}(T)$, a subset of $\mathfrak{H}_n$, into $\mathfrak{H}_n$ such that the first and third hypotheses of Theorem 1 are satisfied. For arbitrary real x , let $f_x \in \mathfrak{D}(T)$ whenever $f \in \mathfrak{D}(T)$. Suppose that there exists a functional $M(f; x)$ defined on $\mathfrak{E}(T) = \{f \mid f = h - k \text{ with } h, k \in \mathfrak{D}(T)\}$ such that $M(f; x) = M(g; x)$ if $f = g$ a.e., $M(\beta f; x) \rightarrow 0$

[†]After Theorem 1 had been proved, the author was informed by F. T. Boesch and D. C. Youla that they had independently obtained, in an unpublished study, a result similar to that to which Theorem 1 reduces for T linear.

as $\beta \rightarrow 0$, and $\|Th - Tk; x\| \leq M(h - k; x)$ for all x and all $h, k \in \mathfrak{D}(T)$. Suppose also that for all $f \in \mathfrak{E}(T)$, $\|f; x\| < \infty$ for $t_0 < x < \infty$. Then T is causal if and only if

$$\operatorname{Re} \langle Th - Tk, h - k; x \rangle \geq -M(h_x - k_x; x) \|h - k; x\|$$

for all real x and all $h, k \in \mathfrak{D}(T)$.

Proof. Suppose that T is causal. Then

$$\begin{aligned} |\operatorname{Re} \langle Th - Tk, h - k; x \rangle| &\leq \|Th - Tk; x\| \cdot \|h - k; x\| \\ &\leq \|Th_x - Tk_x; x\| \cdot \|h - k; x\| \\ &\leq M(h_x - k_x; x) \|h - k; x\|. \end{aligned}$$

Hence T causal implies that

$$\operatorname{Re} \langle Th - Tk, h - k; x \rangle \geq -M(h_x - k_x; x) \|h - k; x\|$$

for all real x and all $h, k \in \mathfrak{D}(T)$. As far as the sufficiency is concerned, observe that the functional $-M(h_x - k_x; x) \|h - k; x\|$ meets the conditions of the second hypothesis of Theorem 1.

Corollary 2(a). Let $\mathfrak{K}$ denote either $\mathfrak{L}_{2n}$ or $\mathfrak{R} \cap \mathfrak{L}_{2n}$. Suppose that T is a mapping of $\mathfrak{K}$ into itself with the property that there exists a functional $M(f)$ defined on $\mathfrak{K}$ such that $M(f) = M(g)$ if $f = g$ a.e., $M(\beta f) \rightarrow 0$ as $\beta \rightarrow 0$, and $\|Th - Tk; \infty\| \leq M(h - k)$ for all $h, k \in \mathfrak{K}$. Then T is causal if and only if

$$\operatorname{Re} \langle Th - Tk, h - k; x \rangle \geq -M(h_x - k_x) \|h - k; x\|$$

for all real x and all $h, k \in \mathfrak{K}$.

Proof: The first and third hypotheses of Theorem 1 are satisfied with $\mathfrak{D}(T) = \mathfrak{K}$ (for which $t_0 = -\infty$). In particular, for every $v \in \mathfrak{K}$ we may take v_i to be the n -vector with the j th component equal to e^{-it} and the remaining components equal to zero for all t .

With regard to part (c) of the third hypothesis,

$$\begin{aligned} |\langle T(v + \sigma v_i) - T(v), v_i; x \rangle| &\leq \|T(v + \sigma v_i) - T(v); x\| \cdot \|v_i; x\| \\ &\leq \|T(v + \sigma v_i) - T(v); \infty\| \cdot \|v_i; \infty\| \\ &\leq M(\sigma v_i) \|v_i; \infty\|. \end{aligned}$$

Thus for all real x , $\langle T(v + \sigma v_i) - T(v), v_i; x \rangle \rightarrow 0$ as $\sigma \rightarrow 0$.

Remark. For operators that represent physical systems, it is often the case that there exists a constant c such that $M(f)$ can be taken to be simply $c \|f; \infty\|$.

In the following theorem it is assumed that T is a mapping of $\mathfrak{L}_{2n}$ or $\mathfrak{R} \cap \mathfrak{L}_{2n}$ into itself and that T possesses a "weak linearization" (as opposed to the "strong linearization" involved, for example, in the definition of the Fréchet differential) about each element of its domain. A special case of the theorem asserts that the condition: $\langle Th - Tk, h - k; x \rangle \geq 0$ for all real x , is satisfied if and only if the linearization of T about an arbitrary element of its domain yields a passive operator.

Theorem 3. Let $\mathfrak{K}$ denote either $\mathfrak{L}_{2n}$ or $\mathfrak{R} \cap \mathfrak{L}_{2n}$. Suppose that T is a mapping of $\mathfrak{K}$ into itself such that for any $f, g \in \mathfrak{K}$ there exists a linear operator L_f , which maps $\mathfrak{K}$ into itself and depends continuously on f , with the property that

$$T(f + \lambda g) - T(f) = L_f(\lambda g) + r(f, \lambda g),$$

in which λ is a scalar and $\lambda^{-1} \|r(f, \lambda g); \infty\| \rightarrow 0$ as $\lambda \rightarrow 0$. Let $U(f; x)$ be a functional defined on $\mathcal{K}$ such that for every $f \in \mathcal{K}$ and every constant α , $U(\alpha f; x) = |\alpha|^2 U(f; x) + V(\alpha, f, x)$, in which $\alpha^{-2} V(\alpha, f, x) \rightarrow 0$ as $\alpha \rightarrow 0$. Then the following two statements are equivalent.

(i) $\operatorname{Re} \langle L_h(k), k; x \rangle \geq U(k; x)$ for all real x and all $h, k \in \mathcal{K}$

(ii) $\operatorname{Re} \langle Th - Tk, h - k; x \rangle \geq U(h - k; x)$ for all real x and all $h, k \in \mathcal{K}$.

Proof. The fact that (ii) implies (i) is an obvious consequence of the fact that (ii) implies that

$$\operatorname{Re} \langle L_f(\lambda g), \lambda g; x \rangle + \operatorname{Re} \langle r(f, \lambda g), \lambda g; x \rangle \geq U(\lambda g; x)$$

for all λ , all real x , and all $f, g \in \mathcal{K}$.

To show that (i) implies (ii), let $y(\alpha) = T[\alpha(h - k) + k]$ with $0 \leq \alpha \leq 1$ and $h, k \in \mathcal{K}$. Then

$$Th - Tk = y(1) - y(0) = \int_0^1 y'(\alpha) d\alpha,$$

in which, with $q = \alpha(h - k) + k$,

$$y'(\alpha) = \lim_{\Delta\alpha \rightarrow 0} [y(\alpha + \Delta\alpha) - y(\alpha)]/\Delta\alpha = L_q(h - k),$$

and hence

$$\begin{aligned} \operatorname{Re} \langle Th - Tk, h - k; x \rangle &= \operatorname{Re} \left\langle \int_0^1 y'(\alpha) d\alpha, h - k; x \right\rangle \\ &= \int_0^1 \operatorname{Re} \langle y'(\alpha), h - k; x \rangle d\alpha \\ &= \int_0^1 \operatorname{Re} \langle L_q(h - k); h - k, x \rangle d\alpha. \end{aligned}$$

If (i) holds, the last integral is at least equal to

$$\int_0^1 U(h - k; x) d\alpha = U(h - k; x).$$

This proves the theorem.

Remark: Observe that (i) implies (ii) even if $U(\alpha f; x) \neq |\alpha|^2 U(f; x) + V(\alpha, f, x)$.

REFERENCE

1. D. C. Youla, L. J. Castriota, and H. J. Carlin, *Bounded real scattering matrices and the foundations of linear passive network theory*, IRE PGCT, Vol. CT-6, March 1959.