ON THE FORMULATION AND ITERATIVE SOLUTION OF SMALL STRAIN PLASTICITY PROBLEMS*

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Abstract. This paper is concerned with a general method of formulation and iterative solution of small displacement plasticity problems, using the Hencky-Nadai hardening law as mathematical model for the material behavior. Beginning with a minimum energy principle for small thermal-mechanical strains under simple external loading, quasi-linear partial differential equations are formulated and a method of iteration by successive solutions is proposed. A finite-difference discretization of the equations (in two dimensions) is obtained through minimization of the total potential energy function, leading to positive definite symmetric matrices for general boundary configurations.

1. Introduction. The general stress-strain law of the linear incremental theory of small plastic deformations has been shown by Drucker to rest solely upon the fundamental postulate of material stability [1, 2], the extended stability postulate [3, 4], and the assumption of a smooth (regular) loading surface f in stress space. If the alternate assumption is made that a corner forms on the yield surface at the point of loading, a non-linear incremental theory results.

A significant body of experimental evidence has been reported since 1953 indicating the regular appearance of corners (see, for example, the discussions in [5] and [6]), and although there is also contrary evidence which tends to refute the concept of a corner carried with the point of loading, the matter does not appear to be sufficiently resolved to justify the acceptance of linearity and exclusion of non-linearity solely on experimental grounds. For the purpose of mathematical stress analysis it would seem particularly appropriate to utilize a non-linear incremental theory should that theory lead to a more feasible method of solution of complex problems. To this end a number of investigators have succeeded in rigorously justifying the relatively simple total stressstrain laws (deformation theory) within a theoretical framework of the fundamental postulate and singular loading surfaces [7-10]. In particular, Budiansky [8] has used the lack of uniqueness in direction of the incremental plastic strain vector at a corner (as given by Drucker's postulate [2]) to establish that the Hencky-Nadai deformation theory can be viewed as an integrable non-linear incremental theory that is mathematically and physically consistent for a range of loading paths including but not limited to proportional loading. In differential form the Hencky-Nadai law can be written -4

$$d\epsilon_{ij}^{P} = d\Lambda_1(J_2) s_{ij} + \Lambda_1(J_2) ds_{ij}$$
⁽¹⁾

where $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ is the deviatoric stress tensor and $\Lambda_1(J_2)$ is an experimentally specified function of the second deviatoric stress invariant. Budiansky's theory of the validity of this law has served as a theoretical basis for several papers [11–13] devoted to the solution of particular boundary value problems. In each of the problems considered

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it has been shown that the deviations of the stress paths from proportional loading are well within the permissable range established by Budiansky. It also has been shown [12, 13] that corresponding results based upon the simplest linear incremental theory of isotropic hardening (Laning hardening law) differ by only a small amount from the Hencky-Nadai deformation theory solutions. In light of these investigations it seems reasonable, from the viewpoint of mathematical stress analysis, to consider the use of total laws justified for many problems involving contained plastic deformations and simple external loading paths.

Accepting the Hencky-Nadai hardening law as an appropriate mathematical model of material behavior in small displacement plasticity problems, this paper is concerned with the formulation and numerical solution of these problems. Beginning with the variational principle for the total potential energy function, quasi-linear partial differential equations will be derived. An iterative method of successive solutions of these equations will be proposed and the question of convergence discussed. For the case of plane strain, two variable displacement problems, a finite-difference discretization based upon minimization of the total potential energy function will be presented. It will be proved that the resulting coefficients matrix is symmetric and positive definite for an arbitary boundary configuration and thermal-mechanical system of loading, thus assuring convergence of the "internal" iteration at each step of the successive solutions method.

2. Energy function and quasi-linear differential equations. Defining the strain energy density as

$$W(\epsilon_{ij}) = \int_0^{\epsilon_{ij}} \sigma_{ij} \, d(\epsilon_{ij} - \alpha T \delta_{ij}) \tag{2}$$

(wherein the usual summation convention of cartesian tensor calculus is adopted, α and T are the coefficients of thermal expansion and temperature rise, respectively, and δ_{ij} is the Kronecker delta), the corresponding extremum principle for the total potential energy function π is

$$\delta \left[\int_{R} W(\epsilon_{ij}^{t}) dR - \int_{R} F_{i} u_{i}^{t} dR - \int_{S_{T}} X_{i} u_{i}^{t} dS \right] \equiv \delta \pi^{t} = 0$$
(3)

where S_{τ} is that portion of the boundary surface S on which tractions are prescribed. The symbol δ is understood to represent a kinematically admissible variation in the true strain state and $\delta W(\epsilon_{ij}^t) = \sigma_{ij}^t \delta \epsilon_{ij}^t$, the superscript t denoting the true state. The displacements u_i are related to the strains ϵ_{ij} through the small strain-displacement relations*

 $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad R + S \tag{4}$

and the boundary tractions X_i are related to the stresses by the equations

$$\sigma_{ii}n_i = X_i \quad \text{on} \quad S. \tag{5}$$

The n_i are the direction numbers of the outward unit normal to S. (The extremum principle of Eq. (3), derived by Greenberg [14] for a general deformation theory of

^{*}A subscript preceded by a comma indicates differentiation with respect to the corresponding coordinate variable.

plasticity and by Drucker [15] for a non-linear elastic material, becomes a relative minimum principle for the Hencky-Nadai material, as shown by Kachanov [16] and discussed by Greenberg [14].)

Considering now the Hencky-Nadai total law and the stress-strain relations of uncoupled thermoelasticity,

$$\epsilon_{ij} - \alpha T \delta_{ij} = \frac{1}{2G} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \epsilon_{ij}^{P}$$
(6)

and

$$\epsilon_{ij}^{P} = \Lambda_{1}(J_{2})s_{ij} = \frac{3}{2}\frac{\tilde{\epsilon}^{P}}{\tilde{\sigma}}s_{ij} , \qquad (7)$$

the variational principle can be used to establish quasi-linear differential equations and natural boundary conditions for thermal-mechanical displacement problems. Introducing the parameter

$$\psi = \frac{\tilde{\epsilon}^P}{\tilde{\epsilon}^E} = E \frac{\tilde{\epsilon}^P}{\tilde{\sigma}} = \frac{E}{E_{***}} - 1$$
(8)

(where E_{sec} is the secant modulus of the stress-strain curve from the tensile test) and inverting Eqs. (6) and (7), a stress-strain relation in quasi-linear form is obtained:

$$\sigma_{ij} = 2G^* \epsilon_{ij} + \lambda^* e \delta_{ij} - (3\lambda + 2G) \alpha T \delta_{ij}$$

$$e \equiv \epsilon_{kk} .$$
(9)

where

The modified material properties denoted by (*) are defined in terms of the parameter ψ by the equations

$$E^* = \frac{E}{1+\psi} = E_{sec} , \qquad \nu^* = \frac{\nu + 0.5\psi}{1+\psi} ,$$

$$G^* = \frac{E^*}{2(1+\nu^*)} = G_{sec} , \qquad \lambda^* = \frac{\nu^* E^*}{(1+\nu^*)(1-2\nu^*)} .$$
(10)

The equivalent stress $\bar{\sigma}$ and equivalent plastic strain $\bar{\epsilon}^{P}$, related through the tensile test, are given as

$$\bar{\sigma} = \sqrt{3J_2} = \sqrt{\frac{3}{2}} \delta_{kl} \delta_{kl} , \qquad \tilde{\epsilon}^P = \tilde{\epsilon}^P(\bar{\sigma}) = \sqrt{\frac{2}{3}} \epsilon_{kl} \epsilon_{kl}^P \delta_{kl} . \tag{11}$$

 G_{see} is the secant modulus of the theoretical stress-strain curve of pure shear predicted by the tensile test, and

$$3\lambda^* + 2G^* \equiv \frac{E^*}{1 - 2\nu^*} = \frac{E}{1 - 2\nu} \equiv 3\lambda + 2G \tag{12}$$

where λ is the Lamé constant.

Substituting Eq. (9) into Eq. (2) and partially integrating, the strain energy density for the Hencky-Nadai material can be written

$$W(\epsilon_{ij}^{t}) = \int_{0}^{\epsilon_{ij}^{t}} G^{*}(\epsilon_{ij}) d(\epsilon_{ij}\epsilon_{ij}) + \frac{1}{2} \int_{0}^{\epsilon^{t}} \lambda^{*}(e) d(e^{2}) - (3\lambda + 2G)\alpha T e^{t} + \frac{3}{2}(3\lambda + 2G)(\alpha T)^{2}.$$
(13)

Defining

$$\bar{G} = \frac{\int_{0}^{\epsilon_{ij}} G^{*}(\epsilon_{ij}) d(\epsilon_{ij}\epsilon_{ij})}{\epsilon_{kl}^{t} \epsilon_{kl}^{t}} , \qquad \bar{\lambda} = \frac{\int_{0}^{\epsilon_{l}} \lambda^{*}(e) d(e^{2})}{(e^{t})^{2}} , \qquad (14)$$

and substituting for the strains from Eqs. (4), the equation for the total potential energy function in terms of the displacements becomes*

$$\pi = \frac{1}{2} \int_{\mathbb{R}} \left[(\tilde{G})(u_{i,i} + u_{j,i})u_{i,i} + \bar{\lambda}e^2 - 2(3\lambda + 2G)\alpha Te + 3(3\lambda + 2G)(\alpha T)^2 - 2F_i u_i \right] dR - \int_{S_T} X_i u_i \, dS.$$
(15)

From the extremum principle

$$\delta \pi = 0 = \int_{R} \left[\delta W - F_{i} \delta u_{i} \right] dR - \int_{S_{T}} X_{i} \delta u_{i} dS$$
(16)

where, from Eqs. (13), (14), and (15)

$$\delta W(\epsilon_{ij}) = (G^*)(u_{i,j} + u_{j,i})\delta u_{i,j} + \lambda^* e \delta e - (3\lambda + 2G)\alpha T \delta e.$$
(17)

Thus

$$\delta \pi = 0 = \int_{R} \left[\{ G^*(u_{i,i} + u_{j,i}) + \lambda^* e \delta_{ii} - (3\lambda + 2G) \alpha T \delta_{ii} \} \delta u_{i,i} - F_i \delta u_i \right] dR - \int_{R} X_i u_i \, dS \qquad (18)$$

with $\delta u_i \equiv 0$ on S_D . Adding and subtracting terms and using the divergence theorem, the variational equation is expressed in final form as

$$\delta \pi = \int_{\mathcal{S}} \left[\{ G^*(u_{i,i} + u_{j,i}) + \lambda^* e \delta_{ij} - (3\lambda + 2G) \alpha T \delta_{ij} \} n_j - X_i \right] \delta u_i \, dS$$
$$- \int_{\mathcal{R}} \left[(\lambda^* + G^*) e_{,i} + G^* \nabla^2 u_i + G^*_{,i} (u_{i,j} + u_{j,i}) + \lambda^*_{,i} e + F_i - \{ (3\lambda + 2G) \alpha T \}_{,i} \right] \delta u_i \, dR = 0$$
(19)

from which

$$\{G^*(u_{i,i} + u_{j,i}) + \lambda^* e \delta_{ij} - (3\lambda + 2G) \alpha T \delta_{ij}\} n_j = X_i \quad \text{on} \quad S_T$$
(20)

and

$$(\lambda^* + G^*)e_{,i} + G^* \nabla^2 u_i + G^*_{,i}(u_{i,i} + u_{i,i}) + \lambda^*_{,i}e + F_i - \{(3\lambda + 2G)\alpha T\}_{,i} = 0 \text{ in } R.$$
(21)

Eqs. (20) are the natural boundary conditions and Eqs. (21) the governing quasi-linear, elliptic partial differential equations of the small strain plasticity problem as considered herein.

An alternate set of quasi-linear equations, also based upon the Hencky-Nadai law

^{*}The superscript t henceforth is dropped for simplicity.

(but restricted to isothermal deformations), was derived by Ilyushin [17] by "lumping" all of the plasticity effects into a fictitious body force term. It is readily demonstrated that Ilyushin's equations and the isothermal form of Eqs. (21) are (as indeed they must be) algebraically equivalent, although they differ in viewpoint in their respective techniques of successive approximation. Ilyushin's formulation is conceptually appropriate when the Green's function of the corresponding isothermal elasticity problem is known, or when the general solution of the homogeneous elastic equations is obtainable in closed form, in which case the successive corrections [17] to the fictitious body force term correspond with successive particular solutions of the elastic equations. (Integral equation solutions to several one-dimensional problems, based essentially upon Ilyushin's concepts, have been given by Mendelson and Manson [18].) In contrast, Eqs. (21) are analogous to the general equations of uncoupled thermoelasticity and are proposed as an appropriate basis for the numerical analysis of problems which do not admit of an "exact" solution even when plasticity effects are negligible. The corresponding method of iteration to obtain successively improved values of the quantities λ^* and G^* will be discussed in Section 4.

3. Variational formulation of difference equations. The variational method of formulating finite-difference equations corresponding to elliptic boundary value problems, suggested as early as 1928 [19], has received attention and widespread application only in recent years (see [20, 21, 22], for example). This method holds two important advantages over the more traditional approach of directly approximating the derivatives in the governing differential equations: (1) the natural boundary conditions arising from the minimization of the potential energy function do not require an additional, special set of difference approximations; (2) the resulting coefficients matrix for a given boundary value problem is symmetric and, in many cases, positive definite, thereby assuring convergence of any one of several iterative techniques. For the small displacement problem treated herein, limiting consideration to the special case of plane strain^{*}, a quadrature formula for the total potential energy function π can be written

$$\bar{\pi} = \sum_{\bar{R}} \left(\bar{W} - F_i u_i \right)_{\mathfrak{a}} \Delta R_{\mathfrak{a}} - \sum_{\bar{S}_{\mathbf{r}}} \left(X_i u_i \right)_{\mathfrak{a}'} \Delta S_{\mathfrak{a}'}$$
(22)

where the integrations over R and S_T have been replaced by summations over a discrete set of points q and q', corresponding to some finite-difference network, with ΔR_q and $\Delta S_{q'}$, representing the associated incremental areas and boundary curve lengths, respectively. (The horizontal bar above a function indicates the corresponding discretized form.) Denoting x_1 , x_2 by x, y, u_1 , u_2 by u, v and X_1 , X_2 by X, Y, the equation for $\bar{\pi}$ becomes

$$\bar{\pi} = \bar{\Omega} - \sum_{\bar{R}} \left[(3\lambda + 2G)\alpha T(\bar{u}_{,x} + \bar{v}_{,y}) + F_{z}u + F_{y}v - \frac{3}{2}(3\lambda + 2G)(\alpha T)^{2} \right]_{q} \triangle R_{q} - \sum_{\bar{S}r} (Xu + Yv)_{q'} \triangle S_{q'} .$$
(23)

 $\bar{\Omega}$ is that part of the total potential energy function that is quadratic in the derivatives of the displacements. By taking the functions G^* and λ^* as constant with respect to differentiation by some u_P , $\bar{\Omega}$ may be written (see Appendix)

$$\bar{\Omega} = \sum_{\bar{R}} \left[G^* \bar{u}_{,x}^2 + \frac{1}{2} (\bar{u}_{,y} + \bar{v}_{,x})^2 + \bar{v}_{,y}^2 + \frac{1}{2} \lambda^* (\bar{u}_{,x} + \bar{v}_{,y})^2 \right]_q \triangle R_q .$$
(24)

^{*}Other two variable displacement problems (i. e. thin plates and axisymmetric solids) can be treated similarly.

Since G^* and λ^* are always positive (see Sect. 4), $\overline{\Omega}$ is a positive semi-definite quadratic form, expressible in matrix terms as

$$\bar{\Omega} = \frac{1}{2} \{\delta\}^{T} [A] \{\delta\} + \{\delta\}^{T} \{b\} + D_{0} .$$
(25)

The order N of the symmetric coefficients matrix [A] is equal to the total number of unknown displacements u and v in $\overline{R} + \overline{S}_{\tau}$. $\{\delta\}$ is the column vector of these unknown displacements, and the vector $\{b\}$ and scalar D_0 are determined from the prescribed displacements on S_D . From Eq. (23) the potential energy $\overline{\pi}$ can be similarly expressed:

$$\bar{\pi} = \frac{1}{2} \{\delta\}^{T} [A] \{\delta\} + \{\delta\}^{T} \{b\} + \{\delta\}^{T} \{c\} + D.$$
(26)

The vector $\{c\}$ arises from the first temperature term and the body force and prescribed traction terms in Eq. (23), and the scalar D corresponds to D_0 and the second temperature term. The potential energy is minimized by setting the rate of change of $\bar{\pi}$ with respect to each unknown displacement equal to zero. The final matrix equation is

$$\frac{\partial \bar{\pi}}{\partial \delta} = [A]\{\delta\} - \{\delta_0\} = 0 \tag{27}$$

where $\{\delta_0\} = -\{b + c\}$ is the "loads" vector of starting values.

The matrix [A] can be proved positive definite for a network composed of rectangular and triangular grid elements, obtained by covering an arbitrary domain with a large number of irregularly spaced horizontal and vertical grid lines i and j (Fig. 1). Considering a rectangular element ij, i + 1j, i + 1j + 1, ij + 1, appropriate quadrature formulas for evaluation of the terms in Eq. (24) are

$$\begin{split} \iint_{\Delta R} & G^* \bar{u}_{i,z}^2 \, dx \, dy = \frac{1}{4} \{ (G_{i+1i}^* + G_{i,i}^*) (u_{i+1i} - u_{ij})^2 \\ & + (G_{i+1i+1}^* + G_{i+1}^*) (u_{i+1i+1} - u_{ij+1})^2 \} \frac{\eta_i}{\xi_i} \\ \iint_{\Delta R} & \frac{1}{2} G^* (\bar{u}_{i,y} + \bar{v}_{j,z})^2 \, dx \, dy = \frac{1}{8} \left\{ G_{i,i}^* \Big(\frac{u_{ii+1} - u_{ii}}{\eta_i} + \frac{v_{i+1i} - v_{ii}}{\xi_i} \Big)^2 + G_{i+1i}^* \right. \\ & \cdot \Big(\frac{u_{i+1i+1} - u_{i+1i}}{\eta_i} + \frac{v_{i+1i} - v_{ii}}{\xi_i} \Big)^2 + G_{i,i+1}^* \Big(\frac{u_{ii+1} - u_{ii}}{\eta_i} + \frac{v_{i+1i+1} - v_{ii+1}}{\xi_i} \Big)^2 \\ & + G_{i+1i+1}^* \Big(\frac{u_{i+1i+1} - u_{i+1i}}{\eta_i} + \frac{v_{i+1i+1} - v_{ii+1}}{\xi_i} \Big)^2 \Big\} \xi_i \eta_i \\ \iint_{\Delta R} & G^* \bar{v}_{i,y}^2 \, dx \, dy = \frac{1}{4} \{ (G_{i,i+1}^* + G_{i,i}^*) (v_{i+1} - v_{ii})^2 \\ & + (G_{i+1i+1}^* + G_{i+1i}^*) (v_{i+1i+1} - v_{i+1i})^2 \} \frac{\xi_i}{\eta_i} \\ \iint_{\Delta R} & \frac{1}{2} \lambda^* (\bar{u}_{i,z} + \bar{v}_{i,y})^2 \, dx \, dy = \frac{1}{8} \left\{ \lambda_{i,i}^* \Big(\frac{u_{i+1i} - u_{ii}}{\xi_i} + \frac{v_{i+1i+1} - u_{ii+1}}{\eta_i} \Big)^2 + \lambda_{i+1i}^* \Big) \Big(\frac{u_{i+1i} - u_{ii}}{\xi_i} + \frac{v_{i+1i+1} - u_{ii+1}}{\eta_i} \Big)^2 \\ & + \lambda_{i+1i+1}^* \Big(\frac{u_{i+1i+1} - u_{ii+1}}{\xi_i} + \frac{v_{i+1i+1} - v_{i+1i}}{\eta_i} \Big)^2 \Big\} \xi_i \eta_i \, . \end{split}$$



FIG. 1.

Corresponding equations for a triangular element ij, i + 1j, ij + 1 are

$$\iint_{\Delta R} G^* \bar{u}_{,x}^2 \, dx \, dy = \frac{1}{4} (G_{i+1i}^* + G_{ii}^*) (u_{i+1i} - u_{ii})^2 \frac{\eta_i}{\xi_i}$$

$$\iint_{\Delta R} \frac{1}{2} G^* (\bar{u}_{,y} + \bar{v}_{,z})^2 \, dx \, dy = \frac{1}{4} G_{ij}^* \left(\frac{u_{ij+1} - u_{ij}}{\eta_i} + \frac{v_{i+1j} - v_{ij}}{\xi_i} \right)^2 \xi_i \eta_i$$

$$\iint_{\Delta R} G^* \bar{v}_{,y}^2 \, dx \, dy = \frac{1}{4} (G_{ij+1}^* + G_{ij}^*) (v_{ij+1} - v_{ij})^2 \frac{\xi_i}{\eta_i}$$

$$\iint_{\Delta R} \frac{1}{2} \lambda^* (\bar{u}_{,x} + \bar{v}_{,y})^2 \, dx \, dy = \frac{1}{4} \lambda_{ij}^* \left(\frac{u_{i+1j} - u_{ij}}{\xi_i} + \frac{v_{ij+1} - v_{ij}}{\eta_j} \right)^2 \xi_i \eta_j .$$
(29)

Consider now the conditions under which $\overline{\Omega}$ can be zero. The integral approximations of Eqs. (28) and (29) are non-negative and are equal to zero only if the following relations hold for all i, j:

$$u_{ij} = u_{i+1j}, \quad v_{ij} = v_{ij+1},$$

$$(u_{ij+1} - u_{ij}) + \frac{\eta_j}{\xi_i} (v_{i+1j} - v_{ij}) = 0.$$
(30)

These equations are seen to correspond to a rigid body translation of arbitrary magnitude and a very small rigid body rotation. If the numerical solution of the continuous problem is to be unique this rigid body displacement must either be prescribed or eliminated. Thus, it is necessary to specify one u, one v, and at least one additional u or v in the finitedifference discretization. Consider the case where these values are specified to be zero, denoting the corresponding $\overline{\Omega}$ by $\overline{\Omega}_0$. It follows that $\{b\} = 0, D_0 = 0$ and

$$\bar{\Omega}_0 = \frac{1}{2} \{\delta\}^T [A] \{\delta\}.$$
(31)

From Eqs. (30), $\overline{\Omega}_0 = 0$ only if $\{\delta\} \equiv 0$ (i.e. *u* and *v* are everywhere zero). Thus, Eq. (31) is a positive definite quadratic form, and [A] is a positive definite matrix for all physically realizable values of λ^* and G^* corresponding to the Hencky-Nadai theory.

To obtain specific difference equations at a typical interior point ij (Fig. 1), the quadrature formulas of Eqs. (28) may be summed over the four network elements surrounding ij and then differentiated. More conveniently, the potential energy function π of Eq. (23) can be differentiated with respect to a general u_P and quadrature formulas

consistent with Eqs. (28) substituted after differentiation. The resulting equations for u and v are (denoting $A^* = \lambda^* + 2G^*$)

$$\begin{aligned} \frac{\eta_{j} + \eta_{j-1}}{\xi_{i}} \left(A_{i+1j}^{*} + A_{i}^{*}\right) u_{i+1j} + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i-1}} \left(A_{i-1j}^{*} + A_{ij}^{*}\right) u_{i-1j} + \frac{\xi_{i} + \xi_{i-1}}{\eta_{j}} \right. \\ \left. \cdot \left(G_{ij+1}^{*} + G_{ij}^{*}\right) u_{ij+1} + \frac{\xi_{i} + \xi_{i-1}}{\eta_{j-1}} \left(G_{ij-1}^{*} + G_{ij}^{*}\right) u_{ij-1} + \left(\lambda_{i+1j}^{*} + G_{ij-1}^{*}\right) v_{i+1j+1} \right. \\ \left. - \left(\lambda_{i+1j}^{*} + G_{ij-1}^{*}\right) v_{i+1j-1} - \left(\lambda_{i-1j}^{*} + G_{ij+1}^{*}\right) v_{i-1j+1} + \left(\lambda_{i-1j}^{*} + G_{ij-1}^{*}\right) v_{i-1j-1} \right. \\ \left. - \left[\frac{\eta_{j} + \eta_{j-1}}{\xi_{i}} \left(A_{i+1j}^{*} + A_{ij}^{*}\right) + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i-1}} \left(A_{i-1j}^{*} + A_{ij}^{*}\right) + \frac{\xi_{i} + \xi_{i-1}}{\eta_{j}} \left(G_{ij+1}^{*} + G_{ij}^{*}\right) \right. \\ \left. + \frac{\xi_{i} + \xi_{i-1}}{\eta_{j-1}} \left(G_{ij-1}^{*} + G_{ij}^{*}\right) \right] u_{ij} - \left(\eta_{i} + \eta_{j-1}\right) \left[\left\{ (3\lambda + 2G)\alpha T \right\}_{i+1j} \right. \\ \left. - \left\{ (3\lambda + 2G)\alpha T \right\}_{i-1j} \right] h + \left(\xi_{i} + \xi_{i-1}\right) (\eta_{i} + \eta_{i-1}) \left(F_{z}\right)_{ij} h^{2} = 0, \\ \left. \frac{\xi_{i} + \xi_{i-1}}{\eta_{i}} \left(A_{ij+1}^{*} + A_{ij}^{*}\right) v_{ij+1} + \frac{\xi_{i} + \xi_{i-1}}{\eta_{i-1}} \left(A_{ij-1}^{*} + A_{ij}^{*}\right) v_{ij-1} + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i}} \\ \left. \cdot \left(G_{i+1j}^{*} + G_{ij}^{*}\right) v_{i+1j} + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i-1}} \left(G_{i-1j}^{*} + G_{ij}^{*}\right) v_{i-1j} + \left(\lambda_{ij+1}^{*} + G_{i-1j}^{*}\right) u_{i-1j-1} \right. \\ \left. - \left[\frac{\xi_{i} + \xi_{i-1}}{\eta_{i}} \left(A_{ij+1}^{*} + A_{ij}^{*}\right) + \frac{\xi_{i} + \xi_{i-1}}{\eta_{i-1}} \left(A_{ij-1}^{*} + A_{ij}^{*}\right) + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i}} \left(G_{i+1j}^{*} + G_{ij}^{*}\right) u_{i-1j-1} \right. \\ \left. - \left[\frac{\xi_{i} + \xi_{i-1}}{\eta_{i}} \left(A_{ij+1}^{*} + A_{ij}^{*}\right) + \frac{\xi_{i} + \xi_{i-1}}{\eta_{i-1}} \left(A_{ij-1}^{*} + A_{ij}^{*}\right) + \frac{\eta_{j} + \eta_{j-1}}{\xi_{i}} \left(G_{i+1j}^{*} + G_{ij}^{*}\right) u_{i-1j-1} \right. \\ \left. - \left[\frac{\xi_{i} + \xi_{i-1}}{\eta_{i}} \left(A_{ij+1}^{*} + A_{ij}^{*}\right) + \frac{\xi_{i} + \xi_{i-1}}{\eta_{i-1}} \left(A_{ij-1}^{*} + A_{ij}^{*}\right) + \frac{\eta_{i} + \eta_{i-1}}{\xi_{i}} \left(G_{i+1j}^{*} + G_{ij}^{*}\right) u_{i-1j-1} \right. \\ \left. - \left\{ (3\lambda + 2G)\alpha T \right\}_{ij-1} \right] h + \left(\xi_{i} + \xi_{i-1}\right) \left(\eta_{i} + \eta_{i-1}\right) \left(F_{y}\right)_{ij}h^{2} = 0 \right] \right\}$$

with difference equations for points on or adjacent to the boundary derivable in similar fashion. For an interior point with equally spaced neighbors ($\xi = \eta = 1$), Eqs. (32) and (33) easily can be shown to correspond to a regular central difference approximation of the governing differential equations (21).

One possible disadvantage of a variational formulation of the difference equations, as opposed to a direct discretization of the field equations and boundary conditions, should be noted. Prescribed traction and "mixed-mixed" boundary conditions are not necessarily satisfied (for a given finite network) by the variational method since they arise as distinct conditions only in the continuous problem and are not established separately from the field equations in minimizing the discretized form of the energy function. Thus, what was cited as one of the advantages of the variational approach could prove, from the standpoint of accuracy within a specific problem, its primary if not sole disadvantage. A direct method of quadratic approximation for first order partial differential boundary conditions has been suggested and applied by Havner [23, 24] and Greenspan [25], with excellent results. Such a direct formulation of the difference equations leads to an unsymmetric matrix, however, for which positive definiteness cannot be readily established nor convergence of interative techniques assured.

4. Method of successive solutions. Consider a general, monotonically increasing



Fig. 2.

stress-strain curve from the tensile test of a Hencky-Nadai material. The equivalent stress $\bar{\sigma}$ is related to an equivalent strain $\bar{\epsilon}$ by the equation

$$\bar{\sigma} = Ef(\bar{\epsilon}) \tag{34}$$

where, in general, both E and the function f depend upon temperature. The iterative solution of the general plasticity problem, represented by Eqs. (20) and (21), is accomplished with the aid of Eq. (34) by determining successively improved values of the strain-dependent functions λ^* and G^* . Assuming that the plastic strains in a given material body can be anticipated as small and contained by regions of elastic deformations, the elastic values λ and G are chosen as the first approximations to λ^* and G^* (hence, the initial solution of the problem is that of the strain distribution in a linearly elastic body). All equations are satisfied, with the exception of the stress-strain relation of Eq. (34). Thus, this relation is used to obtain corrected values of λ^* and G^* , and a second solution is carried out, corresponding to a fictitious, nonhomogeneous linearly elastic material with modulus E_1^* (as depicted in Fig. 2). From Eqs. (4), (8), (10), (11), and (34), the first and all subsequent corrections can be determined from the general equations

$$\epsilon_{n-1}^{*} = \frac{1}{1 + \nu_{n-1}^{*}} \sqrt{\left\{\frac{3}{4}(\bar{u}_{i,i} + \bar{u}_{i,i})\bar{u}_{i,i} - \frac{1}{2}(\bar{u}_{k,k})^{2}\right\}_{n-1}},$$

$$\psi_{n} = \frac{\epsilon_{n-1}^{*}}{f(\epsilon_{n-1}^{*})} - 1 > 0,$$
(35)

and

$$E_{n}^{*} = \frac{E}{1 + \psi_{n}}, \qquad \nu_{n}^{*} = \frac{\nu + 0.5\psi_{n}}{1 + \psi_{n}}, \qquad (36)$$
$$G_{n}^{*} = \frac{E}{2(1 + \nu + 1.5\psi_{n})}, \qquad \lambda_{n}^{*} = \frac{(\nu + 0.5\psi_{n})E}{(1 + \nu + 1.5\psi_{n})(1 - 2\nu)},$$

where *n* indicates the *n*th correction to the elastic solution and $\nu_0^* \equiv \nu$. A relatively simple analytical curve for representing the function *f*, suitable as an approximation to the true stress-strain curve in tension of certain metals at moderate temperature levels, is given by the equation

$$f(\epsilon) = \left(1 - \frac{E_T}{E}\right)\epsilon_0 \tanh \frac{\epsilon}{\epsilon_0} + \frac{E_T}{E}\epsilon$$
(37)

with ϵ_0 and E_r defined in Fig. 3. Eq. (37) is a modification of an analytical curve for non strain-hardening materials suggested by Prager [26].



Consider now the question of convergence of the method of successive solutions defined by Eqs. (35) and (36). Restricting the Poisson's ratio of the material to the physically significant range $0 < \nu < 0.5$, λ^* and G^* are positive and nonsingular, approaching finite limits as ϵ increases unbounded. For the particular stress-strain curve of Eq. (37), the limits are

$$\lim_{\epsilon \to \infty} \psi = \frac{E}{E_T} - 1$$

$$\lim_{\epsilon \to \infty} \lambda^* = \frac{\{1 - (1 - 2\nu)E_T/E\}E}{\{3 - (1 - 2\nu)E_T/E\}(1 - 2\nu)}$$

$$\lim_{\epsilon \to \infty} G^* = \frac{E_T}{3 - (1 - 2\nu)E_T/E}.$$
(38)

Even for the most general monotonically increasing stress-strain curve, it is evident from Eqs. (36) that λ^* and G^* are always bounded:

$$\lambda \le \lambda^* < \frac{1+\nu}{3\nu} \lambda$$

$$0 < G^* \le G.$$
(39)

It follows that the coefficients in the general difference equations of the plane strain problem (Eqs. 32 and 33) are bounded above by the corresponding elastic values and below by the limiting values related to infinite strain. The ranges are $2(1 - \nu)\lambda/\nu \rightarrow 2(1 + \nu)\lambda/(3\nu)$, $E/(1 + \nu) \rightarrow 0$, and $\lambda/2\nu \rightarrow (1 + \nu)\lambda/(3\nu)$ for $(A^* + A^*)$, $(G^* + G^*)$, and $(\lambda^* + G^*)$, respectively. (For a specific stress-strain curve with a finite upper limit on ψ , such as Eq. (37), the lower bounds are increased slightly and the range is narrowed.) Thus, the elements in the coefficients matrix [A] of Eq. (27) are bounded; and since this matrix is symmetric and positive definite for any set of values λ^* and G^* within the physically attainable range above (Eq. 39), it has a unique inverse whose elements also are bounded. Therefore, the displacement vector $\{\delta\}$ is finite for any set of λ^* , G^* evolved from the elastic values through successive application of Eqs. (35) and (36), from which it is concluded that the iteration cannot diverge unbounded.

To further consider the question of convergence, Eqs. (27), (35) and (36) can be written in the functionally descriptive form

$$[A(\psi_n(x, y))]\{\delta_n\} = \{\delta^0\}$$

$$\psi_n(x, y) = \psi\{\delta_{n-1}\}.$$
(40)

The elements of $[A(\psi)]$ are bounded for all ψ and, from Eqs. (32), (33) and (36), decrease linearly with the function

$$\beta_n = \frac{\psi_n - \psi_{n-1}}{(1 + \nu + 1.5\psi_n)(1 + \nu + 1.5\psi_{n-1})}.$$
(41)

The dimensionless rates of change with respect to ψ of the functions λ^* and G^* are, from Eqs. (36),

$$\frac{1}{E}\frac{d\lambda^*}{d\psi} = \frac{1}{2(1+\nu+1.5\psi)^2}, \qquad \frac{1}{E}\frac{dG^*}{d\psi} = -\frac{3}{4(1+\nu+1.5\psi)^2}, \qquad (42)$$

which approach zero with increasing ψ . (The incremental changes in λ^* and G^* are proportional to β_n .) Thus, if it is hypothesized that within a given problem ψ increases monotonically from zero over all the network points, the iteration will obviously converge monotonically, following some curve of successive solutions (Fig. 2) for each unknown displacement. However, it is certainly possible that, as the iteration approaches the true solution, the values of ψ will oscillate at certain points of the network. The method of successive solutions will then either converge in an oscillatory manner, or the displacement values will continue to oscillate within finite bounds, resulting in oscillatory divergence. Although such oscillation of solutions may not seem probable, it is not obvious that a proof of convergence can be advanced in the general case.

5. Conclusion. From the minimum potential energy principle for small thermalmechanical strains of a Hencky-Nadai material, general quasi-linear field equations and natural boundary conditions have been formulated, and a proposed method of successive solutions has been presented. For plane strain problems, quasi-linear difference equations have been obtained through minimization of the discretized total potential energy function. It has been proved that, for a general boundary configuration and arbitrary loading, the resulting coefficients matrix is positive definite for all physically realizable values of the pseudo "material properties" λ^* , G^* . The question of convergence of the successive solutions method has been discussed, with a proof given that the iteration cannot diverge unbounded and an argument for convergence set forth.

Appendix

From Eq. (15) the integral expression for Ω is

$$\Omega = \frac{1}{2} \int_{R} \left[(\bar{G}) \langle u_{i,j} + u_{j,i} \rangle u_{i,j} + \bar{\lambda} e^2 \right] dR.$$
(43)

Introducing a quadrature formula, as in Eq. (22), and differentiating with respect to some u_P :

$$\frac{\partial \bar{\Omega}}{\partial u_P} = \frac{1}{2} \sum_{R_P} \left[\frac{\partial}{\partial u_P} \left\{ (\bar{G}) (\bar{u}_{i,i} + \bar{u}_{i,i}) \bar{u}_{i,i} \right\} + \frac{\partial}{\partial u_P} \left\{ \bar{\lambda} \bar{e}^2 \right\} \right]_q \Delta R_q$$
(44)

where R_P represents the set of network elements surrounding point P.

From Eqs. (14) and (4) (again introducing t to denote the true state),

 $\delta[\bar{G}_{i}(u_{i,j}^{t}+u_{j,i}^{t})u_{i,j}^{t}] \equiv G_{i}^{*}\delta[(u_{i,j}^{t}+u_{j,i}^{t})u_{i,j}^{t}]$

and

$$\delta[\bar{\lambda}_{\iota}(u_{k,k}^{\iota})^{2}] \equiv \lambda_{\iota}^{*}\delta[u_{k,k}^{\iota}]^{2}, \qquad (45)$$

with the corresponding equations from the finite-difference discretization expressible as

$$\frac{\partial}{\partial u_P} \left[\bar{G}_i(\bar{u}_{i,i}^t + \bar{u}_{j,i}^t) = G_i^* \frac{\partial}{\partial u_P} \left[(\bar{u}_{i,i}^t + \bar{u}_{j,i}^t) \bar{u}_{i,j}^t \right]$$

and

$$\frac{\partial}{\partial u_P} \left[\bar{\lambda}_t (\bar{u}_{k,k}^t)^2 \right] = \lambda_t^* \frac{\partial}{\partial u_P} \left[\bar{u}_{k,k}^t \right]^2.$$
(46)

Thus, substituting into Eq. (44) (dropping the notation t),

$$\frac{\partial \bar{\Omega}}{\partial u_P} = \frac{1}{2} \sum_{R_P} \left[G^* \frac{\partial}{\partial u_P} \left\{ (\bar{u}_{i,i} + \bar{u}_{j,i}) \bar{u}_{i,j} \right\} + \lambda^* \frac{\partial}{\partial u_P} (\bar{u}_{k,k}^2) \right]_a \Delta R_a$$
(47)

from which it is seen that $\overline{\Omega}$ may be written

$$\bar{\Omega} = \frac{1}{2} \sum_{R} \left[G^* \{ (\bar{u}_{i,j} + \bar{u}_{j,i}) \bar{u}_{i,j} \} + \lambda^* \bar{u}_{k,k}^2 \right]_q \triangle R_q$$
(48)

if the functions G^* and λ^* are taken as constant with respect to differentiation by u_P .

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