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ON THE EXISTENCE OF PERIODIC SOLUTIONS AND NORMAL MODE VIBRATIONS OF NONLINEAR SYSTEMS*

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1. Introduction. Normal mode vibrations of coupled nonlinear systems of two or more degrees of freedom have been the subject of extensive study in recent years [1-11]. Detailed results have generally been limited to the case of *similar* normal mode vibrations where the wave shapes in the various degrees of freedom are proportional and where the existence of these modal vibrations has been established. This case includes the traditional normal mode vibrations of linear systems and also the vibrations of certain highly nonlinear systems [2]. Nevertheless, it is rather special, and it is desirable to have existence theorems for the more general nonsimilar normal mode vibrations [3, 4].

Since normal mode vibrations are select periodic solutions it would appear reasonable to first establish the existence of various families of periodic solutions and then to single out those which correspond to the normal modes. However, this has not been the approach previously adopted, and for a good reason, since it abrogates the principal value derived from the introduction of normal mode vibrations. Normal mode vibrations (in linear or nonlinear systems) are introduced mainly to uncouple the system so as to allow for the *explicit* integration of the equations of motion [4, 5]. Admittedly, only special periodic solutions are obtained, but the method provides for the type of detailed study required in many engineering applications. This, typically, is not the case when more general mathematical theories are applied to these problems. Also, in practice these special periodic solutions depict the most important motions of the systems studied and hence are of prime concern [2]. Aside from all practical considerations, the formulation and treatment of normal mode vibrations as a singular boundary value problem [6] is of genuine theoretical (mathematical) interest.

In this paper we consider a number of existence theorems for periodic solutions of coupled nonlinear systems of two or more degrees of freedom. These include normal mode vibrations as well as other periodic solutions. We establish the existence of periodic solutions of small amplitude which are near to linearized motions and we treat both the critical and the noncritical cases, i.e., cases where the linearized frequencies are commensurable and incommensurable. The method employed here has been used previously [12] and exploits the symmetries of the system together with the classical implicit function techniques. Alternative perturbational methods [13] could also be used but the present one would seem to be very well adapted to this type of problem with its special dynamics features. The results can be considered as contributing to the theory

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of normal mode vibrations only in the somewhat oblique fashion outlined above. It is, perhaps, of some theoretical value to establish indirectly the existence of normal mode vibrations of small amplitude, as done here, but the primary existence question remains an unanswered and a challenging mathematical problem.

2. Preliminary concepts. We shall consider an idealized conservative system consisting of n point masses (executing colinear displacements X_i) which are anchored and coupled by nonlinear springs. The equations of motion will be of the form

$$m_i X_i'' = U_{X_i} = \frac{\partial U}{\partial X_i}, \qquad (i = 1, 2, 3, \cdots, n) \tag{1}$$

where U is a potential function equal to the negative of the strain energy stored in the springs. The system possesses the energy integral

$$T - U = h, \tag{2}$$

where $T = (1/2)\sum_{i=1}^{n} m_i (X_i')^2$ and h is the constant energy level of a given motion. We assume that the potential U is negative definite, is a function of the absolute value of the length changes of the springs, and possesses continuous third order derivatives. The coordinate space $\{(X_1, X_2, \cdots, X_n)\}$ is called the *configuration* space, and for a given energy level $h > 0$, all solution curves of the system in the configuration space are confined to the interior of the surface $-U = h$ (i.e., $T = 0$), called the *bounding* surface. The bounding surface is closed, surrounds the origin, and is symmetric with respect to it. We shall further assume that the linear approximation of the system (1) can be completely uncoupled by a linear transformation and that the motion in each of the uncoupled states is simple harmonic.

Consider now the differential equation (system)

$$\mathbf{z}' = \mathbf{f}(\mathbf{z}), \tag{3}$$

where $\mathbf{z}$ and $\mathbf{f}$ are $2n$-dimensional (row) vectors. The differential equation (3) is said to possess *property E with respect to Q* [13] if Q is an $n \times n$ (constant) matrix satisfying

$$\mathbf{f}(\mathbf{z}Q) = -\mathbf{f}(\mathbf{z})Q \tag{4}$$

for all $\mathbf{z}$. Property E with respect to Q implies that the differential equation (3) remains invariant under the substitution $t \to -t$, $\mathbf{z} \to \mathbf{z}Q$, and depicts certain symmetries of the system. Let l denote the set of all (constant) $2n$-vectors $\mathbf{a}$ satisfying $\mathbf{a}Q = \mathbf{a}$. Then l is called the *lethargic* set (set of fixed points) of Q. If Γ is a trajectory of (3) emanating from the lethargic set at time t_1 which intersects the lethargic set at a future time t_2, then Γ is called an *l-normal* trajectory. It is not difficult to show [12] that if (3) possesses the usual existence-uniqueness properties for differential equations that an l-normal trajectory is a periodic trajectory with half period equal to the elapsed time $t_2 - t_1$. If $\mathbf{z}$ is the usual phase space vector $(X_1, X_2, \cdots, X_n, X_1', X_2', \cdots, X_n')$ it is readily seen that the system (1) possesses property E with respect to the matrix

$$Q_1 = \operatorname{diag}\,(\overbrace{1, 1, \cdots, 1}^{n}, \overbrace{-1, -1, \cdots, -1}^{n})$$

and that the corresponding lethargic set l_1 is merely the bounding surface, i.e., $X_1' = X_2' = \cdots = X_n' = 0$. Thus any solution curve in the configuration space which intercepts the bounding surface twice is a periodic solution.

Suppose there exist two constant matrices Q_1 and Q_2 such that the differential equation (3) satisfies property E with respect to both. Let l_1 and l_2 be the lethargic sets associated with Q_1 and Q_2, respectively, and assume that $(l_1)Q_2$ is a subset of l_1. Then it is only slightly more difficult to show [12] that a trajectory which first intersects l_2 and subsequently intersects l_1 is a periodic solution of (3) with quarter period equal to the elapsed time. Since the potential U is an even function of the spring displacements, it follows that the system (1) also satisfies property E with respect to the matrix

$$Q_2 = \operatorname{diag}(\overbrace{-1, -1, \cdots, -1}^{n}, \overbrace{1, 1, \cdots, 1}^{n})$$

and that the corresponding lethargic set l_2 is the origin of the configuration space, i.e., $X_1 = X_2 = \cdots = X_n = 0$. Further, $(l_1)Q_2 = l_1$ and thus any solution curve in the configuration space which intercepts the bounding surface and the origin is a periodic solution. Of course, any solution in the configuration space which intercepts the origin twice is also a periodic solution.

We note that these results imply that no solution curve in the configuration space can (a) intercept the bounding surface at three distinct points, (b) intercept the bounding surface and the origin without intercepting the bounding surface at two points, (c) intercept the origin so as to form more than two loops. A solution curve which intercepts the bounding surface at two points is a periodic solution which retraces itself each half period. A solution curve (periodic or otherwise) passing through the origin in configuration space is symmetric with respect to the origin. Solutions of (1) in the configuration space which connect the bounding surface with the origin will be called *BOB* (boundary to origin to boundary) periodic solutions. Periodic solutions of (1) which connect two points on the bounding surface without passing through the origin will be called *BB* (boundary to boundary) solutions. Periodic solutions of (1) which pass through the origin twice without intercepting the bounding surface will be called *OO* (origin to origin) solutions.

Normal mode vibrations have been characterized [4] as vibrations-in-unison of the physical system. A spring-mass system is said to execute vibrations-in-unison if the motion satisfies the conditions

(i) the mass points vibrate periodically with common period,
(ii) there exists a time t_1 at which all mass points simultaneously pass through the origin,
(iii) there exists a time $t_2 \neq t_1$ at which all velocities simultaneously vanish, and
(iv) the position of every mass point at any instant of time is uniquely determined by that of any one of them at the same instant.

Clearly, any *BOB* solution of (1) satisfies the first three of these conditions and may or may not satisfy the fourth. However, every normal mode vibration is necessarily a *BOB* solution.

3. General existence theorems. We assume a value for the energy constant $h > 0$ and consider all the motions of the system (1) corresponding to this (fixed) energy level. Introducing the scaled variables

$$u_i = h^{-1/2} m_i^{1/2} X_i ,$$

and a suitable linear transformation

$$x_i = F_i(u_1 , u_2 , \cdots , u_n),$$

we may write the equations of motion in the form

$$x_i'' = -\lambda_i^2 x_i + h^{\sigma/2} P_i(x_1, x_2, \cdots, x_n, h), \qquad (i = 1, 2, \cdots, n) \tag{5}$$

where $\lambda_i > 0$ are the linearized frequencies and P_i are continuously differentiable and at least second order in the x_i variables. The exponent $\sigma \geq 1$ is chosen so that not all P_i vanish identically in $x_1, x_2, \cdots, x_n$ for $h = 0$. The energy integral may be written in the form

$$\tfrac{1}{2} \sum_{i=1}^{n} (x_i')^2 + \tfrac{1}{2} \sum_{i=1}^{n} \lambda_i^2 x_i^2 + h^{\sigma/2} V(x_1, x_2, \cdots, x_n, h) = 1, \tag{6}$$

where V is at least second order in the x_i variables.

For $h = 0$, the bounding surface becomes the ellipsoid

$$\tfrac{1}{2} \sum_{i=1}^{n} \lambda_i^2 \xi_i^2 = 1, \tag{7}$$

and the system (5) possesses n periodic solutions

$$\begin{aligned} x_\nu &= \frac{2^{1/2}}{\lambda_\nu} \cos \lambda_\nu t, \\ x_i &= 0, \qquad i \neq \nu \end{aligned} \tag{8}$$

which generate the axes of this ellipsoid ($\nu = 1, 2, \cdots, n$). If the frequencies λ_ν are incommensurable then these are the only periodic solutions. If the frequencies λ_ν, $\nu = 1, 2, \cdots, k \leq n$, are commensurable, then all solutions lying in the $\{(x_1, x_2, \cdots, x_k)\}$ hyperplane are periodic. The n periodic solutions (8) represent the linearized normal mode vibrations of the system and we shall first determine conditions under which these are members of a continuous family of periodic solutions for small values of h.

For ν fixed, consider the ratios of the linearized frequencies λ_i/λ_ν, where $i \neq \nu$, $1 \leq i \leq n$. If no one of these ratios is an odd integer we shall say that the system (5) is *non-degenerate* with respect to the frequency λ_ν. Whenever one or more of these ratios is an odd integer, system (5) will be said to be *degenerate* with respect to the frequency λ_ν. System (5) is further classified as *partially* or *totally* degenerate with respect to λ_ν depending upon whether some, but not all, or all of these ratios are odd integers.

THEOREM 1. If the system (5) is non-degenerate with respect to the linearized frequency λ_ν, then the periodic solution

$$\begin{cases} x_\nu = \dfrac{2^{1/2}}{\lambda_\nu} \cos \lambda_\nu t, & 0 \leq t \leq \pi/2\lambda_\nu \\ x_i = 0, & i \neq \nu \end{cases} \tag{9}$$

of (5) for $h = 0$ is the generator of a (unique) continuous family of *BOB* periodic solutions for all sufficiently small values of the energy constant h.

Proof: In general, any solution of (5) for $h = 0$ which originates on the bounding ellipsoid is of the form

$$x_i = \xi_i \cos \lambda_i t, \qquad (i = 1, 2, \cdots, n) \tag{10}$$

where the ξ_i satisfy (7). For $h > 0$, any solution of (5) which originates on the bounding surface may be expressed in the form [14]

$$x_i = \xi_i \cos \lambda_i t + \frac{h^{\sigma/2}}{\lambda_i} \int_0^t \sin \lambda_i(t - s) P_i(x_1(\mathbf{y}, s), x_2(\mathbf{y}, s), \cdots, x_n(\mathbf{y}, s), h)\, ds,$$

$$(i = 1, 2, \cdots, n) \tag{11}$$

where the components of the initial vector $\mathbf{y} = (\xi_1, \xi_2, \cdots, \xi_n)$ satisfy the equation of the bounding surface. The solution (11) will be periodic if an initial vector can be found such that for some time t^* the equations

$$x_i(t^*, \mathbf{y}) = 0, \qquad (i = 1, 2, \cdots, n)$$

are satisfied. Consider the $n + 1$ equations

$$W(\mathbf{y}) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2 \xi_i^2 + h^{\sigma/2} V(\xi_1, \xi_2, \cdots, \xi_n, h) = 1,$$

$$x_\nu = \xi_\nu \cos \lambda_\nu t^* + \frac{h^{\sigma/2}}{\lambda_\nu} \int_0^{t^*} \sin \lambda_\nu(t^* - s) P_\nu(x_1(\mathbf{y}, s), x_2(\mathbf{y}, s), \cdots, x_n(\mathbf{y}, s), h)\, ds = 0, \tag{12}$$

$$x_i = \xi_i \cos \lambda_i t^* + \frac{h^{\sigma/2}}{\lambda_i} \int_0^{t^*} \sin \lambda_i(t^* - s) P_i(x_1(\mathbf{y}, s), x_2(\mathbf{y}, s), \cdots, x_n(\mathbf{y}, s), h)\, ds = 0,$$

$$(i = 1, 2, \cdots, \nu - 1, \nu + 1, \cdots, n)$$

in the $n + 1$ variables $t^*, \xi_\nu, \xi_1, \xi_2, \cdots, \xi_{\nu-1}, \xi_{\nu+1}, \cdots, \xi_n$. For $h = 0$, these equations are satisfied by the initial vector $\mathbf{y}_0 = (0, 0, \cdots, 2^{1/2}/\lambda_\nu, 0, \cdots, 0)$ and the time $t^* = \pi/2\lambda_\nu$. Further, the Jacobian matrix of (12) for $h = 0$, $\mathbf{y} = \mathbf{y}_0$, and $t^* = \pi/2\lambda$ becomes

$$J = \operatorname{diag}\left(2^{1/2}\lambda_\nu, -2^{1/2}, \cos \frac{\lambda_1}{\lambda_\nu}\frac{\pi}{2}, \cdots, \cos \frac{\lambda_{\nu-1}}{\lambda_\nu}\frac{\pi}{2}, \cos \frac{\lambda_{\nu+1}}{\lambda_\nu}\frac{\pi}{2}, \cdots, \cos \frac{\lambda_n}{\lambda_\nu}\frac{\pi}{2}\right)$$

and its determinant becomes

$$|J| = -2\lambda_\nu \prod_{\substack{i=1 \\ i \neq \nu}}^{n} \cos \frac{\lambda_i}{\lambda_\nu}\frac{\pi}{2}, \tag{13}$$

and is non-zero since (5) is non-degenerate with respect to the frequency λ_ν, i.e., the ratios λ_i/λ_ν are not odd integers. Therefore, by the implicit function theorem, for all sufficiently small values of h there exists an initial vector $\mathbf{y} = \mathbf{y}(h)$ and a time $t^* = t^*(h)$ (unique for small h), which satisfy Eqs. (12) and for which we have

$$\mathbf{y}(h) \to \mathbf{y}_0, \qquad t^*(h) \to \pi/2\lambda_\nu \quad \text{as} \quad h \to 0.$$

For each h, the corresponding trajectory emanating from $\mathbf{y}(h)$ on the bounding surface passes through the origin at time $t^*(h)$. These trajectories constitute the desired continuous family of *BOB* periodic solutions with generator (9).

We remark that for sufficiently small values of h, the periodic solutions of Theorem 1 will also represent normal mode vibrations, since as $h \to 0$ they converge uniformly for all t to the linearized normal mode vibration represented by the generator (9). Also we note that Theorem 1 includes many critical cases where some of the linearized frequencies are commensurable.

Consider a case for which the system (5) is either partially or totally degenerate with respect to some frequency λ_ν of the linearized system. Then it may not be possible, under all circumstances, to obtain a proof of the continuation of the corresponding linear normal mode vibration. This is due to the fact that the determinant (13) vanishes for $h = 0$ whenever any one of the frequency ratios, λ_i/λ_ν, $(i \neq \nu)$ is an odd integer. However, it is possible that other periodic solutions may be continued, provided the perturbational parts of (5) satisfy certain conditions.

THEOREM 2. Assume that the frequency ratios λ_i/λ_ν, $(i = 1, 2, \cdots, \mu < \nu)$ are not odd integers, but that each of the frequency ratios λ_j/λ_ν for $j = \mu + 1, \cdots, n$ is an odd integer. Suppose further that there exists an initial vector $\mathbf{y}_0 = (0, 0, \cdots, 0, \xi_{\mu+1}, \cdots, \xi_n)$ with $\xi_\nu \neq 0$ whose components satisfy the equations

$$\frac{1}{2} \sum_{j=\mu+1}^{n} \lambda_j^2 \xi_j^2 = 1, \tag{14a}$$

$$x_j^* = -\lambda_j \xi_j T_1 + \frac{1}{\lambda_j} \int_0^{t^*} \cos \lambda_j s P_j(0, \cdots, 0, \xi_{\mu+1} \cos \lambda_{\mu+1} s, \cdots, \xi_n \cos \lambda_n s, 0)\, ds = 0,$$

$$(j \neq \nu;\ j = \mu + 1, \cdots, n) \tag{14b}$$

where $t^* = \pi/2\lambda_\nu$ and

$$T_1 = \frac{1}{\lambda_\nu^2 \xi_\nu} \int_0^{t^*} \cos \lambda_\nu s P_\nu(0, \cdots, 0, \xi_{\mu+1} \cos \lambda_{\mu+1} s, \cdots, \xi_n \cos \lambda_n s, 0)\, ds. \tag{14c}$$

If the determinant of the matrix

$$J_1 = \left[\frac{\partial x_j^*}{\partial \xi_k}\right], \qquad (j, k = \mu + 1, \cdots, n;\ j, k \neq \nu) \tag{15}$$

is non-zero then the periodic solution

$$\begin{aligned} x_i &= 0, \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \xi_j \cos \lambda_j t, \qquad (j = \mu + 1, \cdots, n), \qquad 0 \le t \le \pi/2\lambda_\nu \end{aligned} \tag{16}$$

of (5) for $h = 0$ is the generator of a (unique) continuous family of *BOB* periodic solutions for all sufficiently small values of the energy constant h.

Proof: As in the proof of Theorem 1, it is sufficient to establish the existence of an initial vector $\mathbf{y}_1$ near $\mathbf{y}_0$ (on the bounding surface) and a time t^* near $\pi/2\lambda_\nu$ which are solutions of the $n + 1$ equations (12). For $h = 0$, any trajectory emanating from the bounding surface intersects the $x_\nu = 0$ hyperplane at the time $t = \pi/2\lambda_\nu$. Further, the derivative $x_\nu' = dx_\nu/dt$ for $h = 0$, $t = \pi/2\lambda_\nu$ becomes $-\lambda_\nu \xi_\nu$ and is not zero if $\xi_\nu \neq 0$. Thus, by the implicit function theorem, for all sufficiently small values of h and for initial vectors $\mathbf{y}_1$ lying on the bounding surface with ξ_ν bounded away from zero, there exists a time $t_1^* = t_1^*(\xi_1, \xi_2, \cdots, \xi_n, h)$, tending to $\pi/2\lambda_\nu$ as h tends to zero and continuously differentiable in $(\xi_1, \xi_2, \cdots, \xi_n)$, at which the corresponding trajectory intersects the $x_\nu = 0$ hyperplane. This time may be expanded in the form

$$t_1^* = \frac{\pi}{2\lambda_\nu} + h^{\sigma/2} T_1(\mathbf{y}_1, h), \qquad \mathbf{y}_1 = (\xi_1, \xi_2, \cdots, \xi_n), \tag{17}$$

and when it is substituted into the equation $x_j = 0$ $(j \neq \nu,\ \mu + 1 \le j \le n)$ of (12) the latter becomes

$$x_j = -T_1\lambda_j\xi_j h^{\sigma/2}\sin\frac{\lambda_j}{\lambda_\nu}\frac{\pi}{2} + \frac{h^{\sigma/2}}{\lambda_j}\int_0^{t_1*}\sin\lambda_j(t_1^* - s)P_j(x_1(\mathbf{y}_1, s), \cdots,$$
$$\cdot x_n(\mathbf{y}_1, s), h)\, ds + h^\sigma Q_j(\mathbf{y}_1, h) = 0. \qquad (18)$$

Here, $\sin(\lambda_j\pi/2\lambda_\nu) = \pm 1$ and Q_j represents the higher order terms in the expansion of $\cos\lambda_j(\pi/2\lambda_\nu + h^{\sigma/2}T_1)$ and is continuously differentiable in the components of $\mathbf{y}_1$. With $x_j = h^{\sigma/2}x_j^*$, (18) may be divided by $h^{\sigma/2}$ and for $\mathbf{y}_1 = \mathbf{y}_0$, $h = 0$ becomes (14)b. From the equation $x_\nu = 0$ it follows that

$$T_1(\mathbf{y}_1, h) = \frac{1}{\lambda_\nu^2\xi_\nu}\int_0^{t_1*}\sin\lambda_\nu(t_1^* - s)P_\nu(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds + \frac{h^{\sigma/2}}{\lambda_\nu\xi_\nu}Q_\nu(\mathbf{y}_1, h) \qquad (19)$$

which reduces to (14)c for $h = 0$ and $\mathbf{y}_1 = \mathbf{y}_0$. Now consider the n equations

$$W(\mathbf{y}_1) = \tfrac{1}{2}\sum_{i=1}^{n}\lambda_i^2\xi_i^2 + h^{\sigma/2}V(\xi_1, \cdots, \xi_n, h) = 1,$$

$$x_i = \xi_i\cos\lambda_i t_1^* + \frac{h^{\sigma/2}}{\lambda_i}\int_0^{t_1*}\sin\lambda_i(t_1^* - s)P_i(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds = 0,$$
$$(i = 1, 2, \cdots, \mu) \qquad (20)$$

$$x_j^* = -T_1\lambda_j\xi_j\sin\frac{\lambda_j}{\lambda_\nu}\frac{\pi}{2} + \frac{1}{\lambda_j}\int_0^{t_1*}\sin\lambda_j(t_1^* - s)P_j(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds$$
$$+ h^{\sigma/2}Q_j(\mathbf{y}_1, h) = 0, \qquad (j \neq \nu;\, j = \mu + 1, \cdots, n)$$

in the n variables $\xi_\nu, \xi_1, \xi_2, \cdots, \xi_{\nu-1}, \xi_{\nu+1}, \cdots, \xi_n$ where because of the choice of $t_1^* = t_1^*(\xi_1, \xi_2, \cdots, \xi_n, h)$ we have $x_\nu(t_1^*) = 0$ identically in $\xi_1, \xi_2, \cdots, \xi_n$ (for $\mathbf{y}_1$ on the bounding surface and away from $\xi_\nu = 0$). For $h = 0$ and $\mathbf{y}_1 = \mathbf{y}_0$ these equations are satisfied, by hypothesis, and the Jacobian matrix of (20) becomes

$$J_2 = \left|\begin{array}{ccccc:c} \lambda_\nu^2\xi_\nu & 0 & 0 & \cdots & 0 & \\ 0 & \cos\frac{\lambda_1}{\lambda_\nu}\frac{\pi}{2} & 0 & \cdots & 0 & \\ 0 & 0 & \ddots & & \vdots & \\ \vdots & \vdots & & \ddots & 0 & \\ 0 & 0 & \cdots & & \cos\frac{\lambda_\mu}{\lambda_\nu}\frac{\pi}{2} & \\ \hdashline & & & & & J_1 \end{array}\right|,$$

with determinant

$$|J_2| = \lambda_\nu^2\xi_\nu\left[\prod_{i=1}^{\mu}\cos\frac{\lambda_i}{\lambda_\nu}\frac{\pi}{2}\right]|J_1|, \qquad (21)$$

which does not vanish, by hypothesis. The implicit function theorem then guarantees that for all sufficiently small values of h, there exists an initial vector $\mathbf{y}_1 = \mathbf{y}_1(h)$ (unique for small h) whose components satisfy (20) with $\lim_{h\to 0}\mathbf{y}_1(h) = \mathbf{y}_0$. For each h, the corresponding trajectory, emanating from $\mathbf{y}_1(h)$ on the bounding surface passes through the origin at time $t = t_1^*(\mathbf{y}_1(h), h)$. These trajectories constitute the desired family of *BOB* periodic solutions with generator (16).

In general, the periodic solutions of Theorem 2 will not represent normal mode vibrations unless the generator (16) does. However, it may happen that the hypotheses of Theorem 2 *are* satisfied by an initial vector $\mathbf{y}_0$, all of whose components vanish except ξ_ν. The generator (16) then represents one of the linearized normal mode vibrations and the corresponding family of periodic solutions represent normal modes for small values of h.

If the system is totally degenerate with respect to λ_ν then *all* of the linearized frequency ratios λ_j/λ_ν, $j = 1, 2, \cdots, n$ are odd integers. In such a case, μ is taken as zero in Theorem 2 and J_1, is an $(n-1) \times (n-1)$ matrix.

An Example. We shall now illustrate Theorems 1 and 2 by applying them to a specific two-degree of freedom system. Consider the system illustrated in Fig. 1 consisting of two mass points anchored and coupled by Duffing type springs with restoring forces S_i.

For $h > 0$, and $u_i = h^{-1/2} m_i^{1/2} X_i$ the equations of motion may be written in the form

$$u_i'' = \frac{\partial U}{\partial u_i}, \qquad (i = 1, 2) \tag{22}$$

where

$$U(u_1, u_2) = -\left[\frac{a_1 + a_3}{2}\frac{u_1^2}{m_1} - a_3\frac{u_1 u_2}{(m_1 m_2)^{1/2}} + \frac{a_2 + a_3}{2}\frac{u_2^2}{m_2}\right] - h\left[\frac{b_1}{4}\frac{u_1^4}{m_1^2} + \frac{b_3}{4}\left(\frac{u_1}{(m_1)^{1/2}} - \frac{u_2}{(m_2)^{1/2}}\right)^4 + \frac{b_2}{4}\frac{u_2^4}{m_2^2}\right].$$

Letting

$$a = \frac{a_1 + a_3}{m_1}, \qquad b = \frac{a_3}{(m_1 m_2)^{1/2}}, \qquad c = \frac{a_2 + a_3}{m_2},$$

we introduce the (linearized, normal) coordinates

$$x_1 = \frac{1}{\gamma - \delta}(\gamma u_1 - u_2),$$

$$x_2 = \frac{1}{\delta - \gamma}(\delta u_1 - u_2),$$

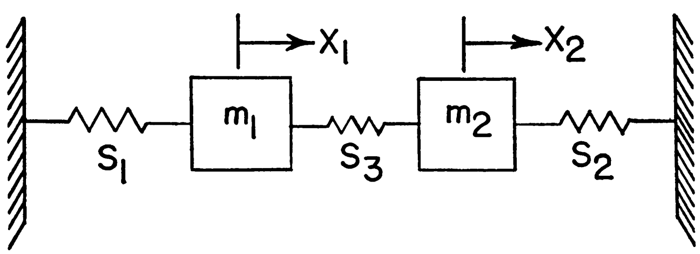

FIG. 1. Spring-Mass System
$S_1(X_1) = a_1 X_1 + b_1 X_1^3$, $\quad S_2(X_2) = a_2 X_2 + b_2 X_2^3$
$S_3(X_1, X_2) = a_3(X_1 - X_2) + b_3(X_1 - X_2)^3$

where

$$\delta = \frac{a - \lambda_1^2}{b}, \qquad \gamma = \frac{a - \lambda_2^2}{b},$$

$$\lambda_1^2 = \frac{a+c}{2} - \left[\left(\frac{a-c}{2}\right)^2 + b^2\right]^{1/2}, \tag{23}$$

$$\lambda_2^2 = \frac{a+c}{2} + \left[\left(\frac{a-c}{2}\right)^2 + b^2\right]^{1/2}.$$

Then the equations of motion become

$$\begin{aligned} x_1'' &= -\lambda_1^2 x_1 + hP_1(x_1, x_2), \\ x_2'' &= -\lambda_2^2 x_2 + hP_2(x_1, x_2), \end{aligned} \tag{24}$$

where

$$\begin{aligned} P_1(x_1, x_2) &= l_1 x_1^3 + n_1 x_1^2 x_2 + p_1 x_1 x_2^2 + q_1 x_2^3 \\ P_2(x_1, x_2) &= l_2 x_2^3 + n_2 x_2^2 x_1 + p_2 x_2 x_1^2 + q_2 x_1^3, \end{aligned}$$

and l_i, n_i, p_i, and q_i are constants. Here we have assumed that the linear coupling coefficient $a_3 \neq 0$. If this is not the case, (22) is already in the normal form (24).

For $h = 0$, the system (24) possesses the periodic solutions

$$x_1 = \frac{2^{1/2}}{\lambda_1} \cos \lambda_1 t, \qquad x_2 = 0, \tag{25a}$$

and

$$x_1 = 0, \qquad x_2 = \frac{2^{1/2}}{\lambda_2} \cos \lambda_2 t. \tag{25b}$$

If the linearized frequencies λ_1 and λ_2 are incommensurable these are the only periodic solutions, while if λ_1 and λ_2 are commensurable all solutions are periodic. If λ_2 ($\lambda_2 \geq \lambda_1$) is not an odd integer multiple of λ_1 then the system (24) is non-degenerate with respect to both λ_1 and λ_2. Thus, by Theorem 1 the periodic solutions (25) are the generators of continuous families of *BOB* periodic solutions for small values of h. These will represent in-phase and out-of-phase [3] normal mode vibrations.

If $\lambda_2 = k\lambda_1$ with k an odd integer greater than one, then the system (24) remains non-degenerate with respect to λ_2 and (25b) is the generator of a continuous family of *BOB* periodic solutions for small values of h. (See Fig. 2a). However, the periodic solution (25a) may or may not be the generator of a family of periodic solutions. We turn to Theorem 2 for further information. In this special case we have $n = 2$, $\mu = 0$, $\nu = 1$, $\sigma = 2$, and (14b) becomes upon integration

$$x_2^* = \frac{\pi \xi_2 \sin \frac{k\pi}{2}}{16k\lambda_1^2} [2p_2 - 3k^2 l_1)\xi_1^2 + (3l_2 - 2k^2 p_1)\xi_2^2] = 0, \quad \text{for} \quad k > 3 \tag{26}$$

or

$$x_2^* = \frac{\pi \sin \frac{3\pi}{2}}{48\lambda_1^2} [(2p_2 - 27l_1)\xi_2\xi_1^2 - 9n_1\xi_1\xi_2^2 + (3l_2 - 18p_1)\xi_2^3 + q_2\xi_1^3] = 0, \quad \text{for} \quad k = 3. \tag{27}$$

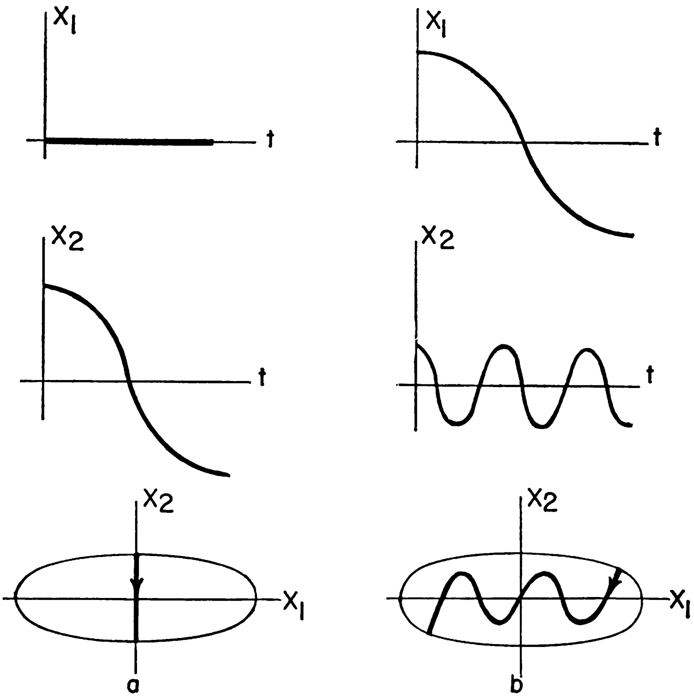

FIG. 2. *BOB* Periodic Solutions, $\lambda_2 = 5\lambda_1$

Equation (26) (as well as (14a)) is satisfied by $\xi_1 = 2^{1/2}/\lambda_1$, $\xi_2 = 0$. Thus if the quantity

$$|J_1| = \left(\frac{\partial x_2^*}{\partial \xi_2}\right)_{\substack{\xi_1 = 2^{1/2}/\lambda_1 \\ \xi_2 = 0}} = \frac{\pi \sin \dfrac{k\pi}{2}}{8k\lambda_1^4}\,(2p_2 - 3k^2 l_1),$$

does not vanish, i.e., if $2p_2 \neq 3k^2 l_1$, then (25a) is the generator of a continuous family of *BOB* periodic solutions for small values of h and we again have both in-phase and out-of-phase normal mode vibrations. Equation (26) will also be satisfied by the components of an initial vector $\mathbf{y}_1 = (\xi_1 , \xi_2)$ whenever $(2p_2 - 3k^2 l_1)\xi_1^2 + (3l_2 - 2k^2 p_1)\xi_2^2 = 0$. If for these components we have, further,

$$|J_1| = \frac{\partial x_2^*}{\partial \xi_2} = \frac{\pi \xi_2^2 \sin \dfrac{k\pi}{2}}{8k\lambda_1^2}\,(3l_2 - 2k^2 p_1) \neq 0,$$

then the corresponding (off-axis) periodic solution is the generator of a continuous family of *BOB* periodic solutions for small values of h. These will not represent normal mode vibrations, however, unless $|\xi_2|$ is small (see Fig. 2b).

If $k = 3$ we conclude that the periodic solution (25a) will be the generator of a continuous family of *BOB* periodic solutions for h sufficiently small if $q_2 = 0$ and $2p_2 - 27l_1 \neq 0$. Further, if $2p_2 - 27l_1 \neq 0$, then for $|q_2|$ small there will be a continuous family of *BOB* periodic solutions for small values of h, with initial vectors tending to the root of (27) which is near the point $(2^{1/2}/\lambda_1 , 0)$. For sufficiently small values of q_2 and h these will also represent normal mode vibrations. Note that (25a) is *not* the generator of a continuous family of these periodic solutions if $q_2 \neq 0$, since then the components of $(2^{1/2}/\lambda_1, 0)$ do not satisfy Eq. (27). More generally, if $\xi_1(\neq 0)$ and ξ_2 satisfy (27) and (14)a and if for these components we have

$$\frac{\partial x_2^*}{\partial \xi_2} = \frac{\pi \sin \dfrac{3\pi}{2}}{48\lambda_1^2} [(2p_2 - 27l_1)\xi_1^2 - 18n_1\xi_1\xi_2 + 3(3l_2 - 18p_1)\xi_2^2] \neq 0,$$

then the periodic solution

$$x_1 = \xi_1 \cos \lambda_1 t, \qquad x_2 = \xi_2 \cos 3\lambda_1 t, \tag{28}$$

is the generator of a continuous family of *BOB* periodic solutions. One would not expect these to correspond to normal mode vibrations unless the generator (28) does so. For example, if $|q_2|$ is not small, the system (24) can *not* vibrate in the in-phase mode for small values of h!

If $k = 1$, i.e., $\lambda_1 = \lambda_2$, then necessarily $a_3 = b = 0$ and $a = c$, which implies that the system is symmetric and possesses no linear coupling. In this case we must revert to the system (22) and it becomes necessary to employ Theorem 2 for both frequencies. However, because of the symmetry we need only consider one degenerate frequency, say λ_1. As before $\mu = 0$, $\nu = 1$, $\sigma = 2$, $n = 2$ and, upon integration, (14)b becomes (in the (u_1, u_2) coordinates)

$$\begin{aligned} u_2^* = \frac{3\pi}{16\lambda_1^2\xi_1} \Bigg[& \frac{b_3}{m_1^{3/2}m_2^{1/2}} \xi_1^4 + \left(\frac{b_1 + b_3}{m_1^2} - \frac{3b_3}{m_1 m_2}\right)\xi_1^3\xi_2 \\ & + 3\left(\frac{b_3}{m_2^{3/2}m_1^{1/2}} - \frac{b_3}{m_1^{3/2}m_2^{1/2}}\right)\xi_1^2\xi_2^2 + \left(\frac{3b_3}{m_1 m_2} - \frac{b_2 + b_3}{m_2^2}\right)\xi_1\xi_2^3 - \frac{b_3}{m_1^{1/2}m_2^{3/2}} \xi_2^4 \Bigg] = 0, \end{aligned} \tag{29}$$

where $\xi_1 \neq 0$. Thus if the components of the initial vector $\mathbf{u}_0 = (\xi_1, \xi_2)$ satisfy (29) and if for these components we have $\partial u_2^*/\partial \xi_2 \neq 0$, then the periodic solution

$$u_1 = \xi_1 \cos \lambda_1 t, \qquad u_2 = \xi_2 \cos \lambda_1 t, \tag{30}$$

is the generator of a continuous family of *BOB* periodic solutions. These will represent normal mode vibrations. In particular, for $b_3 = 0$, (29) is satisfied by the components of the vector $\mathbf{u}_1 = (2^{1/2}/\lambda_1, 0)$ and we have

$$\left(\frac{\partial u_2^*}{\partial \xi_2}\right)_{\substack{\xi_1 = 2^{1/2}/\lambda_1 \\ \xi_2 = 0}} = \frac{3\pi b_1}{8\lambda_1^4 m_1^2} \neq 0,$$

provided $b_1 \neq 0$. If b_1 has the same sign as b_2, (29) is also satisfied by the components of the initial vector

$$\mathbf{u}_2 = \left(\xi_1, \pm\frac{m_2}{m_1}\left|\frac{b_1}{b_2}\right|^{1/2} \xi_1\right) = (\xi_1, \xi_2)$$

where

$$\xi_1^2 = \frac{2}{\lambda_1^2\left[1 + \left(\dfrac{m_2}{m_1}\right)^2 \dfrac{b_1}{b_2}\right]},$$

and we have

$$\frac{\partial u_2^*}{\partial \xi_2} = -\frac{3\pi b_2}{8\lambda_1^2 m_2^2}\xi_2^2 \neq 0,$$

provided $b_2 \neq 0$.

Generalizations. Suppose that the linearized frequencies of system (24) satisfy the relation $\lambda_2 = (p/q)\lambda_1$ where p and q are relatively prime odd integers. Then for $h = 0$, this system possesses the *BOB* periodic solutions

$$x_1 = \xi_1 \cos \lambda_1 t, \qquad x_2 = \xi_2 \cos \frac{p}{q}\lambda_1 t, \qquad \lambda_1^2\xi_1^2 + \lambda_2^2\xi_2^2 = 2 \tag{31}$$

with quarterperiod $q\pi/2\lambda_1$, and it is quite possible that some of these are the generators of continuous families of *BOB* periodic solutions of (24) for small values of h. This suggests the following generalization of Theorem 2 which is proved using precisely the same arguments.

Theorem 3. Assume that the frequency ratios λ_i/λ_ν $(i = 1, 2, \cdots, \mu < \nu)$ of the system (5) are not ratios of odd integers but that for $j = \mu + 1, \cdots, n$ we have $\lambda_j/\lambda_\nu = p_j/q_j$, where p_j and q_j are relatively prime *odd* integers. Let q be the least common multiple of the q_j and assume that the components of the initial vector $\mathbf{y}_0 = (0, 0, \cdots, 0, \xi_{\mu+1}, \cdots, \xi_n)$ with $\xi_\nu \neq 0$ satisfy Eqs. (14) with $t^* = q\pi/2\lambda_\nu$. Then if the determinant of the matrix (15) is non-zero, the periodic solution

$$\begin{aligned} x_i &= 0, \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \xi_j \cos (p_j/q_j)\lambda_\nu t, \qquad (j = \mu + 1, \cdots, n), \qquad 0 \leq t \leq q\pi/2\lambda_\nu \end{aligned} \tag{32}$$

of (5) for $h = 0$ is the generator of a (unique) continuous family of *BOB* periodic solutions for all sufficiently small values of the energy constant h.

These periodic solutions, of course, will not represent normal mode vibrations unless the generator (32) does. Generally, for large integers p_j and q_j, the solution curves for these periodic solutions in the configuration space are very complicated.

Theorem 3 remains valid if the least common multiple q is replaced by any odd integral multiple of q. The corresponding *BOB* periodic solutions could, perhaps, be called sub-harmonic oscillations. Similarly, Theorem 2 may be generalized so as to include subharmonic oscillations merely by replacing the term $\pi/2\lambda_\nu$ in the expansion (17) by any odd integral multiple of itself.

In each of the preceding theorems we have used two lethargic sets to establish *BOB* periodic solutions. In the sequel we shall use a single lethargic set and investigate l-normal periodic solutions which are generally not *BOB* solutions. We shall first consider *BB* solutions, which emerge from and return to the bounding surface. These include additional critical cases.

Theorem 4. Assume that the frequency ratios λ_j/λ_ν $(j \neq \nu; j = \mu + 1, \cdots, n)$ of the system (5) satisfy $\lambda_j/\lambda_\nu = p_j/q_j$, where p_j and q_j are relatively prime integers. (Here both even and odd integers are permitted.) Let q be the least common multiple of the q_j and assume that the frequency ratios λ_i/λ_ν $(i = 1, 2, \cdots, \mu < \nu)$ are not of the form k/q, with k an integer. Further, suppose that the components of the initial vector $\mathbf{y}_0 = (0, 0, \cdots, 0, \xi_{\mu+1}, \cdots, \xi_n)$ with $\xi_\nu \neq 0$ satisfy (14) with $t^* = q\pi/\lambda_\nu$. Then if the determinant of the matrix (15) is non-zero, the periodic solution

$$\begin{aligned} x_i &= 0, \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \xi_j \cos \lambda_j t, \qquad (j = \mu + 1, \cdots, n), \qquad 0 \leq t \leq q\pi/\lambda_\nu \end{aligned} \tag{33}$$

of (5) for $h=0$ is the generator of a (unique) continuous family of *BB* (or *BOB*) periodic solutions for all sufficiently small values of the energy constant h.

Proof: According to Eqs. (11), the velocity components of a solution of (5) originating at $\mathbf{y}$ on the bounding surface are of the form

$$x_i' = -\lambda_i \xi_i \sin \lambda_i t + h^{\sigma/2} \int_0^t \cos \lambda_i (t - s) P_i(x_1(\mathbf{y}, s), x_2(\mathbf{y}, s), \cdots, x_n(\mathbf{y}, s), h)\, ds, \qquad (i = 1, 2, \cdots, n).$$

The solution (11) will be periodic if an initial vector $\mathbf{y}_1$ can be found such that for some time $t^* > 0$ the equations

$$x_i'(t^*, \mathbf{y}_1) = 0, \qquad (i = 1, 2, \cdots, n) \tag{34}$$

are satisfied. For $h = 0$ the νth-velocity component x_ν' of any trajectory originating on the bounding surface vanishes at the time $t = q\pi/\lambda_\nu$. Further, the derivative $x_\nu'' = dx_\nu'/dt$ for $h = 0$, $t = q\pi/\lambda_\nu$ becomes $-\lambda_\nu^2 \xi_\nu \cos q\pi$ and is not zero if $\xi_\nu \neq 0$. Thus, by the implicit function theorem, for all sufficiently small values of h and for initial vectors $\mathbf{y}_1$ lying on the bounding surface with ξ_ν bounded away from zero, there exists a time $t_2^* = t_2^*(\xi_1, \xi_2, \cdots, \xi_n, h)$, tending to $q\pi/\lambda_\nu$ as h tends to zero and continuously differentiable in $\xi_1, \xi_2, \cdots, \xi_n$, at which the νth-velocity component of the corresponding trajectory vanishes. Further, this time may be expanded in the form

$$t_2^* = \frac{q\pi}{\lambda_\nu} + h^{\sigma/2} T_2(\mathbf{y}_1, h), \tag{35}$$

and when it is substituted into the equation $x_j' = 0$ ($j \neq \nu$; $\mu + 1 \leq j \leq n$) of (34) the latter equation becomes

$$x_j' = -T_2 \lambda_j^2 \xi_j h^{\sigma/2} \cos (p_j/q_j) q\pi + h^{\sigma/2} \int_0^{t_2^*} \cos \lambda_j (t_2^* - s) P_j(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds + h^\sigma Q_j(\mathbf{y}_1, h) = 0. \tag{36}$$

Equation (36) may be divided by $h^{\sigma/2}$ and for $h = 0$, $\mathbf{y}_1 = \mathbf{y}_0$ becomes (14b). From the equation $x_\nu' = 0$ it follows that

$$T_2(\mathbf{y}_1, h) = \frac{1}{\lambda_\nu^2 \xi_\nu \cos q\pi} \int_0^{t_2^*} \cos \lambda_\nu (t_2^* - s) P_\nu(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds + \frac{h^{\sigma/2}}{\lambda_\nu^2 \xi_\nu \cos q\pi} Q_\nu(\mathbf{y}_1, h) \tag{37}$$

which reduces to (14c) for $h = 0$, $\mathbf{y}_1 = \mathbf{y}_0$. Now consider the n equations

$$W(\mathbf{y}_1) = \tfrac{1}{2} \sum \lambda_i^2 \xi_i^2 + h^{\sigma/2} V(\xi_1, \cdots, \xi_n, h) = 1$$

$$x_i' = -\lambda_i \xi_i \sin \lambda_i t_2^* + h^{\sigma/2} \int_0^{t_2^*} \cos \lambda_i (t_2^* - s) P_i(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}, s), h)\, ds = 0. \qquad (i = 1, 2, \cdots, \mu) \tag{38}$$

$$x_j^* = -T_2 \lambda_j^2 \xi_j \cos (p_j/q_j) q\pi + \int_0^{t_2^*} \cos \lambda_j (t_2^* - s) P_j(x_1(\mathbf{y}_1, s), \cdots, x_n(\mathbf{y}_1, s), h)\, ds + h^{\sigma/2} Q_j(\mathbf{y}_1, h) = 0, \qquad (j \neq \nu; j = \mu + 1, \cdots, n)$$

in the n variables ξ_ν, ξ_1, ξ_2, $\cdots$, $\xi_{\nu-1}$, $\xi_{\nu+1}$, $\cdots$, ξ_n, where because of the choice of t_2^* we have $x_\nu'(t_2^*) = 0$ identically in these variables (for $\mathbf{y}_1$ on the bounding surface and away from $\xi_\nu = 0$). For $h = 0$ and $\mathbf{y}_1 = \mathbf{y}_0$ these equations are satisfied, by hypothesis, and the determinant of the corresponding Jacobian matrix becomes

$$|J_2'| = (-1)^\mu \lambda_\nu^2 \xi_\nu \left[\prod_{i=1}^{\mu} \lambda_i \sin \frac{\lambda_i}{\lambda_\nu} q\pi \right] |J_1|,$$

which does not vanish, by hypothesis. The implicit function theorem then guarantees that for all sufficiently small values of h, there exists an initial vector $\mathbf{y}_1 = \mathbf{y}_1(h)$ (unique for small h) whose components satisfy (38) with $\lim_{h\to 0} \mathbf{y}_1(h) = \mathbf{y}_0$. For each h, the corresponding trajectory emanating from $\mathbf{y}_1(h)$ on the bounding surface returns to the bounding surface at time $t = t_2^*(\mathbf{y}_1(h), h)$. These trajectories constitute the desired family of *BB* (or *BOB*) periodic solutions with generator (33). It is clear that this result is also valid for a system whose potential U may not be an even function of the spring displacements.

The following theorem is concerned with the existence of 00 periodic solutions of system (5) which emerge from and return to the origin. The proof is similar to that of Theorem 4 and will be omitted.

THEOREM 5. Let the frequency ratios λ_j/λ_ν $(j = 1, 2, \cdots, n)$ of system (5) satisfy the hypotheses of Theorem 4. Further, suppose that the components of the initial velocity vector $\mathbf{v}_0 = (0, 0, \cdots, 0, v_{\mu+1}, \cdots, v_n)$ with $v_\nu \neq 0$ satisfy the equations

$$1 = \sum_{j=\mu+1}^{n} v_j^2 \tag{39a}$$

$$x_j^* = v_j T_3 - \frac{1}{\lambda_j} \int_0^{q\pi/\lambda_\nu} \sin \lambda_j s P_j\left(0, 0, \cdots, 0, \frac{v_{\mu+1}}{\lambda_{\mu+1}} \sin \lambda_{\mu+1} s, \cdots, \frac{v_n}{\lambda_n} \sin \lambda_n s, 0\right) ds = 0$$

$$(j \neq \nu;\ j = \mu + 1, \cdots, n) \tag{39b}$$

where

$$T_3 = \frac{1}{\lambda_\nu v_\nu} \int_0^{q\pi/\lambda_\nu} \sin \lambda_\nu s P_\nu\left(0, \cdots, 0, \frac{v_{\mu+1}}{\lambda_{\mu+1}} \sin \lambda_{\mu+1} s, \cdots, \frac{v_n}{\lambda_n} \sin \lambda_n s, 0\right) ds. \tag{39c}$$

Then if the determinant of the matrix

$$J_3 = \left[\frac{\partial x_j^*}{\partial v_k} \right], \qquad (j, k = \mu + 1, \cdots, n;\ j, k \neq \nu)$$

is non-zero, the periodic solution

$$\begin{aligned} x_i &= 0, \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \frac{v_j}{\lambda_j} \sin \lambda_j t, \qquad (j = \mu + 1, \cdots, n), \qquad 0 \leq t \leq q\pi/\lambda_\nu \end{aligned} \tag{40}$$

of (5) for $h = 0$ is the generator of a (unique) continuous family of *OO* (or *BOB*) periodic solutions for all sufficiently small values of the energy constant h.

Further Illustrations. We again consider the two-dimensional system (24) in order to illustrate Theorems 4 and 5. If $\lambda_2 = \frac{3}{2}\lambda_1$, for example, (14b) becomes upon integration

$$x_2^* = -\frac{\pi \xi_2}{16\lambda_1} [(27 l_1 - 8 p_2)\xi_1^2 + (18 p_1 - 12 l_2)\xi_2^2] = 0. \tag{41}$$

Thus, if $\xi_1(\neq 0)$ and ξ_2 satisfy (41) and $\lambda_1^2(\xi_1^2 + \frac{9}{4}\xi_2^2) = 2$ and if for these components we have

$$\frac{\partial x_2^*}{\partial \xi_2} = -\frac{\pi}{16\lambda_1}[(27l_1 - 8p_2)\xi_1^2 + (54p_1 - 36l_2)\xi_2^2] \neq 0.$$

then the periodic solution

$$x_1 = \xi_1 \cos \lambda_1 t$$
$$x_2 = \xi_2 \cos \tfrac{3}{2}\lambda_1 t \tag{42}$$

of (24) for $h = 0$ is the generator of a continuous family of *BB* (or *BOB*) periodic solutions for all sufficiently small values of the energy constant h. In particular, (41) is satisfied by the components $\xi_1 = 2^{1/2}/\lambda_1$, $\xi_2 = 0$, and the corresponding partial derivative $\partial x_2^*/\partial \xi_2$ is non-zero if $(27l_1 - 8p_2) \neq 0$. If the quantities $(27l_1 - 8p_2)$ and $(54p_1 - 36l_2)$ differ in sign then there will be two additional off-axis generators (42), (see Fig. 3a). Similarly, in this case (39b) becomes

$$x_2^* = \frac{\pi v_2}{\lambda_1^5}\left[\left(\frac{3}{4}l_1 - \frac{2}{9}p_2\right)v_1^2 + \left(\frac{2}{9}p_1 - \frac{4}{27}l_2\right)v_2^2\right] = 0. \tag{43}$$

Thus, if $v_1(\neq 0)$ and v_2 satisfy (43) and $v_1^2 + v_2^2 = 1$ and if for these components we have

$$\frac{\partial x_2^*}{\partial v_2} = \frac{\pi}{\lambda_1^5}\left[\left(\frac{3}{4}l_1 - \frac{2}{9}p_2\right)v_1^2 + \left(\frac{2}{3}p_1 - \frac{4}{9}\right)l_2v_2^2\right] \neq 0,$$

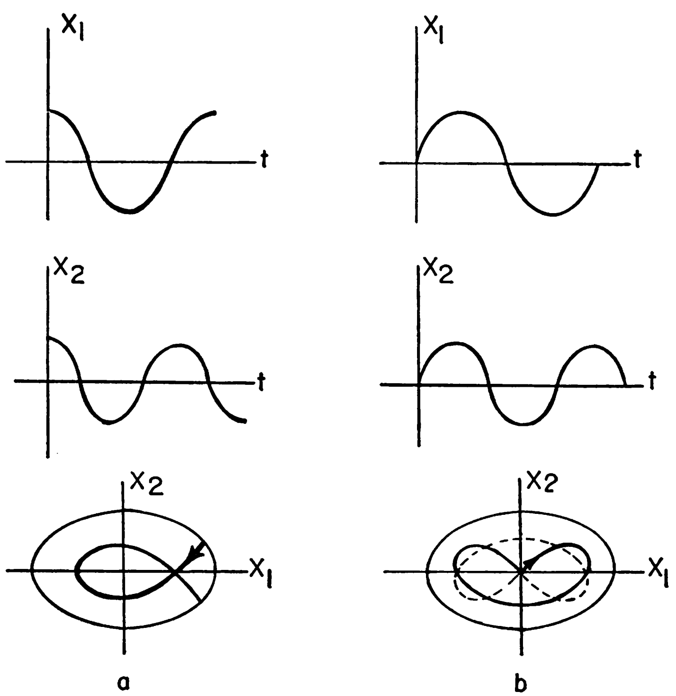

FIG. 3. *BB* and *OO* Periodic Solutions, $2\lambda_2 = 3\lambda_1$

then the periodic solution

$$x_1 = \frac{v_1}{\lambda_1} \sin \lambda_1 t, \qquad x_2 = \frac{2v_2}{3\lambda_1} \sin \tfrac{3}{2}\lambda_1 t \tag{44}$$

of (24) for $h = 0$ is the generator of a continuous family of *OO* (or *BOB*) periodic solutions for all sufficiently small values of the energy constant h. In particular, (43) is satisfied by the components $v_1 = 1$, $v_2 = 0$ and the corresponding partial derivative $\partial x_2^*/\partial v_2$ is non-zero if $(27l_1 - 8p_2) \neq 0$. If the quantities $(27l_1 - 8p_2)$ and $(3p_1 - 2l_1)$ differ in sign, then there will be two additional off-axis generators (44), (see Fig. 3b).

A NON-EXISTENCE THEOREM. As the two-dimensional example illustrates, each of the four existence theorems 2 to 5 leads to a corresponding non-existence theorem. These are given collectively below and follow readily from the earlier proofs.

THEOREM 6. If the linearized frequency ratios λ_j/λ_ν $(j = \mu + 1, \cdots, n; \mu < \lambda)$ of the system (5) are rational numbers, then (a) the periodic solution

$$\begin{aligned} x_i &= 0 \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \xi_j \cos \lambda_j t, \qquad (j = \mu + 1, \cdots, n) \end{aligned} \tag{45}$$

of (5) for $h = 0$ can be the generator of the continuous family of *BOB* or *BB* periodic solutions given in Theorem 2, 3, or 4 *only* if the components ξ_j of the initial vector satisfy (14)b with the appropriate t^*; (b) the periodic solution

$$\begin{aligned} x_i &= 0, \qquad (i = 1, 2, \cdots, \mu) \\ x_j &= \frac{v_j}{\lambda_j} \sin \lambda_j t, \qquad (j = \mu + 1, \cdots, n) \end{aligned} \tag{46}$$

of (5) for $h = 0$ can be the generator of the continuous family of *OO* periodic solutions given in Theorem 5 *only* if the components v_j of the initial velocity vector satisfy (39)b.

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