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THERMODYNAMICS AND WAVE PROPAGATION*

BY

BERNARD D. COLEMAN (*Mellon Institute, Pittsburgh*)

AND

MORTON E. GURTIN (*Brown University, Providence*)

1. Introduction. In continuum physics the word *wave* is used with several distinct meanings. To some a wave is a *sinusoidal disturbance*, to some it is any *member of a certain class of solutions* to a hyperbolic equation, and to others it is a *propagating singular surface*. Here we follow Christoffel, Hugoniot, Hadamard, and Duhem and use the word in the last sense; thus, we define a *wave* to be a surface, moving with respect to the material, across which some kinematical variable, such as the acceleration or the velocity, suffers a jump discontinuity. In the present age of sonic booms and nuclear explosions, even the layman is familiar with "shock waves." To find, however, applications for a general theory of propagating singular surfaces, one does not have to turn to the latest accomplishments of physics; it suffices to think carefully of the motion of an object struck with a hammer.

Here we shall briefly review some aspects of the classical theory of wave propagation in elastic materials and discuss recent extensions of the classical theory to materials with memory.

2. Wave propagation in elastic materials. Let $x_i(X_j, t)$ give the spatial position at time t of the material point which occupies the position X_j in the reference configuration.** According to the Duhem–Hadamard classification scheme, a surface $\Sigma = \Sigma(t)$ is a wave of order N if the N'th-order derivatives of $x_i(X_j, t)$ exhibit jump discontinuities at Σ, but all lower derivatives are continuous across Σ. A shock is a wave of order 1; that is, $x_i(X_j, t)$ is continuous, but the velocity $x_i^{(1)} = (\partial/\partial t)x_i(X_j, t)$ and the deformation gradient $F_{ij} = (\partial/\partial X_j)x_i(X_k, t)$ show jumps across Σ. At an *acceleration wave*, second derivatives, such as the acceleration $x_i^{(2)} = (\partial^2/\partial t^2)x_i(X_j, t)$, are the first to suffer jumps; hence, an acceleration wave is a singular surface of order two. Our interest here is in waves with $N \geq 2$.

In the terminology of Truesdell and Toupin [8] and Truesdell [10], the *speed of propagation* V of a wave Σ is the rate of advance of Σ along its unit normal relative to the particles instantaneously situated on Σ. A convenient measure of the *amplitude* of a wave of order N is a vector s_i defined by

$$(-V)^N s_i = [x_i^{(N)}], \tag{1}$$

where $[x_i^{(N)}]$ is the jump in $x_i^{(N)} = (\partial^N/\partial t^N)x_i(X_k, t)$ across Σ. For an acceleration wave, $V^2 s_i = [x_i^{(2)}]$.

Let the *Piola–Kirchhoff stress tensor* S_{ij} be defined by the formula†

$$\rho S_{ik}F_{jk} = T_{ij}, \tag{2}$$

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**We use Cartesian tensor notation.

†Whenever an index is repeated in a product of two terms, summation over that index is understood.

where T_{ij} is the familiar stress tensor of Cauchy and ρ is the present mass density. For an elastic material, when thermodynamic influences are ignored, S_{ij} is a function of the deformation gradient:

$$S_{ij} = S_{ij}(F_{kl}). \tag{3}$$

The classical *Fresnel–Hadamard Theorem* asserts that for any material obeying (2) the amplitude s_i and speed of propagation V of an acceleration wave traveling in the direction n_k must obey the *propagation condition*

$$Q_{ij}(n_k)s_j = V^2 s_i , \tag{4}$$

where the tensor $Q_{ij}(n_k)$, called the *acoustic tensor*, is given by*

$$Q_{ij}(n_k) = F_{am}F_{bl}n_a n_b \frac{\partial}{\partial F_{jm}} S_{il}(F_{pq}). \tag{5}$$

Ericksen [3], working in the theory of isotropic incompressible hyperelastic materials, and Truesdell [10], working in the theory of compressible elastic materials, have shown that all elastic waves of order $N > 2$ must also obey the propagation condition (4) with $Q_{ij}(n_k)$ given by (5).

Even in elasticity theory one should include thermodynamic influences and allow S_{ij} to depend not only on F_{kl} but also on a thermodynamic variable such as the temperature θ or the specific entropy η. In the thermodynamic theory of acceleration waves, $x_i^{(1)}$, F_{ij}, θ, and η are taken to be continuous across the wave:

$$[x_i^{(1)}] = [F_{ik}] = [\theta] = [\eta] = 0. \tag{6}$$

An acceleration wave is said to be *homothermal* if, in addition to (6), one has

$$[\theta^{(1)}] = \left[\frac{\partial\theta}{\partial X}\right] = 0. \tag{7}$$

On the other hand, if, in place of (7),

$$[\eta^{(1)}] = \left[\frac{\partial\eta}{\partial X}\right] = 0, \tag{8}$$

then the wave is called *homentropic*. It is the content of the *first theorem of Duhem*, that homothermal (homentropic) acceleration waves in elastic materials obey (4), provided that the derivative $\partial/\partial F_{jm}$ in (5) is taken at constant temperature (entropy). The *second theorem of Duhem* gives the physical circumstances in which acceleration waves are homothermal or homentropic: Every acceleration wave in an elastic material obeying Fourier's law of heat conduction with positive-definite thermal conductivity is homothermal; every acceleration wave in an elastic material which does not conduct heat is homentropic.**

An elastic material is said to be *hyperelastic* if the stress-strain function of (3) is obtained by differentiating a stored energy function ϕ; i.e., if

$$S_{ij}(F_{kl}) = \frac{\partial}{\partial F_{ij}} \phi(F_{kl}). \tag{9}$$

*For the most general statement of the Fresnel-Hadamard Theorem and for a detailed discussion of its consequences, see Truesdell [10].

**The two theorems of Duhem are explained and given modern proofs by Truesdell [10].

It is a familiar assertion of classical thermodynamics that every elastic material is hyperelastic in the sense that

$$S_{ij}(F_{kl}\ ,\ \theta) = \frac{\partial}{\partial F_{ij}}\ \psi(F_{kl}\ ,\ \theta), \tag{10}$$

$$S_{ij}(F_{kl}\ ,\ \eta) = \frac{\partial}{\partial F_{ij}}\ \epsilon(F_{kl}\ ,\ \eta), \tag{11}$$

where ψ is the specific Helmholz free energy $\psi = \epsilon - \theta\eta$, and ϵ the specific internal energy.

A *theorem of Hadamard* asserts that for a hyperelastic material the acoustic tensor (5) must be symmetric:

$$Q_{ij}(n_k) = Q_{ji}(n_k). \tag{12}$$

It follows from this result, the first theorem of Duhem, and the classical equations (10) and (11), that the acoustic tensor of an elastic material must be symmetric in homothermal and homentropic waves.

3. Materials with memory. The recent years have seen the development of general theories of non-linear materials with memory.* In these studies it is assumed that the present stress depends not only on the present value of the strain but also on the past history of the strain, and one attempts to solve problems without specializing the functional expressing this dependence. Let us define a function $F_{kl}^{(t)}$ over the half-closed interval $[0,\ \infty)$ as follows:

$$F_{kl}^{(t)}(s) = F_{kl}(t - s), \qquad 0 \le s < \infty. \tag{13}$$

This function is called *the history up to time t* of the deformation gradient. The *past history* $F_{(r)kl}^{(t)}$ of the deformation gradient is just the *restriction* of $F_{kl}^{(t)}$ to the open interval $(0,\ \infty)$; i.e., $F_{(r)kl}^{(t)}(s)$ agrees with $F_{kl}^{(t)}(s)$ for all $s > 0$ but is left undefined for $s = 0$. Obviously, a knowledge of the history $F_{kl}^{(t)}$ is equivalent to a knowledge of the *present value* $F_{kl}(t) = F_{kl}^{(t)}(0)$ *and* the past history $F_{(r)kl}^{(t)}$. Following Noll [6], we define a *simple material* to be a material for which the stress is determined when $F_{kl}^{(t)}$ is given. For such a material we can write

$$S_{ij} = \mathsf{S}_{ij}(F_{kl}^{(t)}). \tag{14}$$

Here S_{ij} is a *functional*; i.e., a function whose argument is a function, $F_{kl}^{(t)}$, and whose value is a tensor, S_{ij}. When we wish to emphasize that for a general simple material S_{ij} depends on both the present value and the past history of the deformation gradient, we write (14) in the form**

$$S_{ij} = \mathsf{S}_{ij}(F_{(r)kl}^{(t)}\ ;\ F_{kl}), \tag{15}$$

where, for short, we put $F_{kl} = F_{kl}^{(t)}(0)$. On comparing (3) and (15), we see that the elastic materials considered in the previous section are those simple materials for which the dependence of $\mathsf{S}_{ij}(F_{(r)kl}^{(t)}\ ;\ F_{kl})$ on the past history $F_{(r)ij}^{(t)}$ is negligible. Here we do not assume that the influence of $F_{(r)kl}^{(t)}$ on S_{ij} can be neglected; we assume merely that this

*See, for example, the works of Green and Rivlin [5], Noll [6], Coleman and Noll [7, 9], and Wang [22], which are summarized and extended in the exposition of Truesdell and Noll [20].

**There is no summation over k or l in equations (15) and (16).

influence is compatible with the principle of fading memory; i.e., with a smoothness postulate used by Coleman and Noll [7, 9] to render mathematical the intuitive idea that strains which occurred in the very distant past should have a smaller influence on the present stress than strains which occurred in the recent past.

Let us denote by $D_{F_{mn}}\mathsf{S}_{ij}(F_{kl}^{(t)})$ the fourth-order tensor obtained by differentiating the stress with respect to the present value of deformation gradient holding fixed the past history:

$$D_{F_{mn}}\mathsf{S}_{ij}(F_{kl}^{(t)}) = \frac{\partial}{\partial F_{mn}} \mathsf{S}_{ij}(F_{(r)kl}^{(t)} \; ; F_{kl}). \tag{16}$$

This tensor [12, 13] gives the moduli for instantaneous response to small strain impulses superimposed on $F_{kl}^{(t)}$ at time t.

Equation (16) defines a linear differential operator $D_{F_{mn}}$ mapping functionals into functionals; this operator plays a central role in the theory of wave propagation [15-18] and in the thermodynamics of materials with memory [12, 13].

The theory of singular surfaces propagating in materials with memory is not an empty subject. Among the materials subsumed under the class of simple materials with fading memory are the materials of the theory of linear viscoelasticity. In that theory, Sips [2], Lee and Kanter [4], and Chu [11] have exhibited explicit solutions of the dynamical equations showing shock waves. It is an elementary exercise to construct from these solutions others showing acceleration waves. Further, Pipkin [23] has obtained exact solutions showing shock and acceleration waves for a special simple fluid with fading memory that gives rise to nonlinear field equations.

Recently [15-18], we have been able to extend to general non-linear materials with memory the classical propagation theorems given in the previous section. For example, we have the following extension [18] of the Fresnell–Hadamard and Ericksen–Truesdell theorems. Consider a wave of order 2 or greater traveling in the direction n_k in a general simple material with fading memory; such a wave still obeys the propagation condition (4)*, and the tensor $Q_{ij}(n_k)$, now called the *instantaneous acoustic tensor*, is given by the following remarkably simple generalization of (5):

$$Q_{ij}(n_k) = F_{am}F_{bl}n_a n_b D_{F_{jm}}\mathsf{S}_{il}(F_{pq}^{(t)}). \tag{17}$$

For the validity of this theorem the past history of the material just in front of the wave may be arbitrary, subject only to certain natural tameness hypotheses.

Of course, we know that the stress in a general material with memory depends not only on the history of the deformation gradient but also on the history of a thermodynamic variable. Hence, (14) should be replaced by either

$$S_{ij} = \bar{\mathsf{S}}_{ij}(F_{kl}^{(t)}, \theta^{(t)}) \tag{18}$$

or

$$S_{ij} = \hat{\mathsf{S}}_{ij}(F_{kl}^{(t)}, \eta^{(t)}), \tag{19}$$

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*Propagation conditions of the type (4) are known for acceleration waves in several special materials; for elastic materials (4) was derived by Hadamard [1] and for the theory of linear viscoelasticity by Herrera and Gurtin [19]. We have recently seen a manuscript by Varley [21] in which he arrives at (4) for acceleration waves in materials of integral type.

where the function $\theta^{(t)}$ is the *history of the temperature* and the function $\eta^{(t)}$ the *history of the specific entropy*:

$$\theta^{(t)}(s) = \theta(t - s), \qquad \eta^{(t)}(s) = \eta(t - s), \qquad 0 \leq s < \infty. \tag{20}$$

When (18) is assumed, it is also reasonable to assume that the specific Helmholz free energy ψ is given by a functional p of the histories $F_{kl}^{(t)}$, and $\theta^{(t)}$,

$$\psi = \mathsf{p}(F_{kl}^{(t)}, \theta^{(t)}), \tag{21}$$

and that the heat flux vector q_i depends not only on the present temperature gradient $g_m = (\partial/\partial x_m)\theta$ but also on $F_{kl}^{(t)}$ and $\theta^{(t)}$,

$$q_i = \mathsf{q}_i(F_{kl}^{(t)}, \theta^{(t)}; g_m). \tag{22}$$

When (19) is assumed, it is reasonable to postulate that the specific internal energy is given by a functional of $F_{kl}^{(t)}$ and $\eta^{(t)}$:

$$\epsilon = \mathsf{e}(F_{kl}^{(t)}, \eta^{(t)}). \tag{23}$$

Starting from assumptions somewhat more general than these, Coleman [12] has shown that the functionals $\bar{\mathsf{S}}_{ij}$ and $\hat{\mathsf{S}}_{ij}$ are compatible with the principle of fading memory and the second law of thermodynamics only if these functionals are determined by p and e through the relations,

$$\bar{\mathsf{S}}_{ij}(F_{kl}^{(t)}, \theta^{(t)}) = D_{F_{ij}}\mathsf{p}(F_{kl}^{(t)}, \theta^{(t)}), \tag{24a}$$

$$\hat{\mathsf{S}}_{ij}(F_{kl}^{(t)}, \eta^{(t)}) = D_{F_{ij}}\mathsf{e}(F_{kl}^{(t)}, \eta^{(t)}), \tag{24b}$$

where $D_{F_{ij}}$ is the operator defined in (16). These relations generalize to materials with memory the classical relations (10) and (11) for elastic materials.

Even when the history of a thermodynamic variable is brought in, the propagation condition (4) still holds for homothermal and homentropic acceleration waves [17, 18]. For homothermal waves the instantaneous acoustic tensor Q_{ij} is given by (17) with $\mathsf{S}_{ij}(F_{pq}^{(t)})$ replaced by $\bar{\mathsf{S}}_{ij}(F_{pq}^{(t)}, \theta^{(t)})$, and the function $\theta^{(t)}$, the history of the temperature up to the moment of arrival of the wave, is held fixed in the computation of the derivative (16). For homentropic acceleration waves $D_{F_{pq}}$ is applied to $\hat{\mathsf{S}}_{ij}(F_{pq}^{(t)}, \eta^{(t)})$, and the function $\eta^{(t)}$, the history of the entropy at the wave, is held fixed in (16) and (17). These observations extend to materials with memory the first theorem of Duhem. The second theorem of Duhem also has a direct generalization [17, 18]: In a definite conductor of heat, every acceleration wave is homothermal; in a non-conductor, every acceleration wave is homentropic. Here, by a definite conductor, we mean a material for which $-\partial q_i/\partial g_m$, computed using (22), is always a positive-definite tensor. The proof of the theorem is straightforward for a definite conductor; the proof for a non-conductor, i.e., a material with $q_i \equiv 0$, uses a generalization [12] of the relations (24).

Our main purpose in writing the present article is to bring to the attention of experimenters the following extension [18] of the classical symmetry condition (12): *The relations* (24) *imply that, even in a material with memory, the instantaneous acoustic tensors for both homothermal and homentropic waves are always symmetric tensors in the sense of equation* (12). This theorem appears to supply a method of testing the physical appropriateness of the relations (24). Fortunately, Truesdell [10, 14], working in the theory of elastic materials, has found situations in which measurements of wave velocity can test the symmetry of the acoustic tensor. His analyses can be applied with small modifications to the theory of acceleration waves entering a general material with memory which previous to arrival of the wave had always been at rest in a fixed configuration.

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