

SYSTEM IDENTIFICATION AND PREDICTION—AN ALGORITHM USING A NEWTONIAN ITERATION PROCEDURE*

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Abstract. A mathematical technique, especially suitable for programming on a high speed digital computer, is presented for identifying a complete dynamic system having unknown parameters when data concerning one variable of the system is available.

I. Introduction It often occurs that certain state variables of a dynamic system are unmeasurable, while one particular state variable can be measured either continuously or discretely as frequently as desired. These measurements, together with an assumed form for the dynamic equations, should be sufficient to determine the system for all time. The significance of the problem lies in the fact that identifying a system is a necessary prerequisite to controlling it, and so, by solving the posed problem, there is given an effective procedure for the design of a class of adaptive controls. A typical application is in the determination of aerodynamic stability derivatives from flight test measurements. A complete literature on this application exists [1]–[8], but generally the analyses presented in these references depend on the system being linear. The analog matching method [6], [7] and the equations of motion method [8] do not require linear equations; but, in the case of [6], [7], a trial and error matching of function is used, and consequently the method is somewhat subjective depending on the skill of the operator; in the case of [8], the equations of motion are treated as algebraic equations and, as a consequence, data concerning every state variable and its time derivative is required regardless of the convenience or accuracy of such data. Another application is in the determination of satellite orbits where data is obtained from a network of radar tracking stations. With this as motivation Kahne [9] has developed a method for identification whereby the solution curve is made to pass through the given data points. In this method the number of data points and the number of unknowns must be equal. Convergence to the solution is achieved by a method especially devised for solving two point boundary value problems [10]. Kumar and Sridhar [11] essentially solve the same problem using the method of quasi-linearization. These problems have been generalized by Bellman, Kagiwada, and Kalaba [12] in that the number of data points can be unlimited and the curve fit is accomplished in the least square sense. Convergence to the solution is achieved by the method of quasi-linearization.

An iterative method will here be developed which does not suffer from subjectivity and which achieves the curves fit in the least square sense. Convergence to the solution is achieved (though not proved) by a method which arises naturally out of the method of least squares. Two cases will be considered: in the first case the measured data will be assumed to be specified at discrete time intervals; in the second case the measured data will be assumed to be specified continuously as a function of time. In both cases

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the dynamical equations are assumed to be nonlinear and are given as:

$$\dot{y}_i = g_i(y_1, \dots, y_n, \alpha, t), \quad y_i(0) = c_i \quad (1)$$

where the parameter vector α and the initial vector c are both unknown.

II. The discrete case. Suppose measurements to have been made on, say, the first state variable $y_1(t)$ at times t_m ,

$$y_1(t_m) = b_m, \quad m = 1, 2, \dots, M \quad (2)$$

where b_m is the measurement at time t_m . It is required to find an initial vector c and a parameter vector α which, together, minimize the sum of the squares of the deviations:

$$\epsilon = \sum_{m=1}^M \{y_1(t_m) - b_m\}^2. \quad (3)$$

Thus, the solution of (1) is sought which is in best agreement with the measurements in a least square sense.

It has been pointed out by Bellman, Kagiwada, and Kalaba [12] that this problem can be reduced to a nonlinear multi-point boundary value problem. These authors present a method of solution based on the technique of quasi-linearization. Their method is iterative and consists essentially in finding neighboring solutions to the quasilinearized equations while holding the boundary conditions fixed. In this paper, the alternative approach of holding the differential equations fixed and finding solutions which satisfy neighboring boundary conditions will be developed.

As pointed out in [9], [11], [12] the components of the parameter vector α can be considered to be additional state components subject to the equation

$$\dot{\alpha} = 0. \quad (4)$$

In this case, the parameter vector can be suppressed in (1) and the number n increased to include the additional state components. The analysis will proceed on this basis.

It will be supposed that the initial point, $t = 0$, is not necessarily one at which a measurement has been made, which is the most general case. The system will be determined by iteration as follows: Values of the components of the initial vector c are estimated, in which case (1) can be integrated. The estimated initial vector will be denoted by c^* , and the resulting solution vector by y^* ; furthermore, the deviation can then be calculated and its value denoted by ϵ^* . Suppose the initial vector to be changed by an increment δc ; this would cause the solution vector to be changed by an increment δy and the deviation by an increment $\delta \epsilon$. From (3) it is seen that

$$\delta \epsilon = 2 \sum_{m=1}^M \{y_1(t_m) - b_m\} \delta y_1(t_m). \quad (5)$$

The equation which the incremental solution vector satisfies is obtained by expanding (1) in a Taylor series and retaining only linear terms:

$$\delta \dot{y}_i(t) = \left(\frac{\partial g_i}{\partial y_i} \right)^* \delta y_i, \quad (6)$$

where the repeated suffix implies summation from 1 to n and the asterisk implies that the coefficients are calculated using the estimated solution y^* . Equation (6) must now be integrated n times; the j th time the integration is performed the initial conditions

are that $\delta y_i(0) = 1$ and all the other $\delta y_i(0)$'s vanish. This special solution is denoted by δy_{ii} , and the general solution can then be written, by superposition:

$$\delta y_i = \sum_{j=1}^n \delta c_j \delta y_{ij}(t). \quad (7)$$

In particular, only δy_1 is required.

$$\delta y_1 = \delta c_i \delta y_{1i}(t), \quad (8)$$

where, again, the repeated index implies summation from 1 to n . Upon substituting into (5) and interchanging the order of the summations, the variation of the deviation becomes

$$\delta \epsilon = 2 \delta c_i \sum_{m=1}^M \{y_1(t_m) - b_m\} \delta y_{1i}(t_m). \quad (9)$$

The variation of the deviation has thus been expressed directly in terms of the variation in each of the initial conditions. In order for ϵ to be minimum $\delta \epsilon$ must vanish, which means that if U_i is defined to be

$$U_i = \sum_{m=1}^M \{y_1(t_m) - b_m\} \delta y_{1i}(t_m) \quad (10)$$

the boundary conditions which (1) must satisfy are

$$U_i = 0, \quad i = 1, \dots, n. \quad (11)$$

In general, using the estimated vector c^* and the resulting solution y^* , the values of U_i will not vanish. If the value of U_i as calculated by this procedure is denoted by U_i^* , then it is seen that

$$U_i^* = \sum_{m=1}^M \{y_1^*(t_m) - b_m\} \delta y_{1i}(t_m). \quad (12)$$

The objective is to make the U_i vanish by an iteration procedure. In the special case where $M = n$, (10) and (11) constitute n simultaneous homogeneous linear equation for the n unknowns $y_1(t_m) - b_m$, whose solution is $y_1(t_m) - b_m = 0$, i.e., the solution must pass directly through the measured points, in which case $\epsilon \equiv 0$. Whenever $M > n$, which is generally the case, the procedure is more complex and ϵ can only be minimized. Consider the increment in U_i caused by the increment in y_i . From (10) there is obtained

$$\delta U_i = \sum_{m=1}^M \delta y_1(t_m) \delta y_{1i}(t_m). \quad (13)$$

In order that U_i vanish the condition

$$\delta U_i = -U_i^* \quad (14)$$

must be imposed. Substituting (8) into (13) and interchanging the order of summation there is finally obtained

$$\delta U_i = \delta c_j \sum_{m=1}^M \delta y_{1j}(t_m) \delta y_{1i}(t_m). \quad (15)$$

Equations (12), (14), (15) constitute n simultaneous linear algebraic equations for the n unknowns δc_i . Upon adding the incremental values to the estimated values c_i^* ,

improved estimates of the c_i are obtained. It should be noted that the matrix of coefficients in (15) is symmetrical.

III. The continuous case. In the continuous case a continuous record of $y_1(t)$ is assumed to exist. If the measured value of $y_1(t)$ be denoted by $b(t)$, this can be expressed as

$$y_1(t) = b(t). \quad (16)$$

In this case it is required to minimize the deviation defined as

$$\epsilon = \int_0^T [y_1(t) - b(t)]^2 dt, \quad (17)$$

where T is the length of the record and is assumed to be given.

Once again the parameter vector α can be suppressed and the components of the parameter vector taken to be additional state components subject to (4). The asterisk will once again be used to denote results obtained using an estimated initial vector c^* . The increment in the deviation in this case becomes

$$\delta\epsilon = 2 \int_0^T [y_1(t) - b(t)] \delta y_{1i}(t) dt, \quad (18)$$

where (6), (7) are again the incremental equations and their solution respectively. Upon substituting (8) and interchanging the summation and integration there results

$$\delta\epsilon = 2 \delta c_i \int_0^T [y_1(t) - b(t)] \delta y_{1i}(t) dt. \quad (19)$$

Once again $\delta\epsilon$ must vanish, which means that if U_i is defined to be

$$U_i = \int_0^T [y_1(t) - b(t)] \delta y_{1i}(t) dt \quad (20)$$

the boundary conditions which (1) must satisfy are

$$U_i = 0, \quad i = 1, \dots, n. \quad (21)$$

Denoting the value of U_i as obtained using c^* by U_i^* , then

$$U_i^* = \int_0^T [y_1^*(t) - b(t)] \delta y_{1i}(t) dt. \quad (22)$$

Once again it is required to make the U_i vanish by an iteration procedure. From (20) there is obtained

$$\delta U_i = \int_0^T \delta y_{1i}(t) \delta y_{1i}(t) dt, \quad (23)$$

which becomes, upon substituting (8) and interchanging the order of summation and integration

$$\delta U_i = \delta c_i \int_0^T \delta y_{1i}(t) \delta y_{1i}(t) dt. \quad (24)$$

Equations (22), (14), (24) constitute n simultaneous linear algebraic equations for the n unknowns δc_i .

IV. An example. Suppose a freely falling body to be observed in vacuo near an earthlike planet. The position is observed at four equal increments in time according to the following table:

TABLE I

t_m	b_m
0	0
1	15
2	65
3	145

The question is: what is the best estimate of the acceleration due to gravity, α , based on this data. This problem was selected to illustrate the method because the exact solution can be obtained by more conventional means as a check, and because the arithmetical steps can readily be performed by hand. Furthermore, the problem serves to demonstrate, in a simpleminded way, that the method can be used to analyze data obtained from an aerospace probe in order to deduce geophysical or meteorological parameters.

The dynamical equations are

$$\frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = y_3, \quad \frac{dy_3}{dt} = 0, \quad (25)$$

where y_1 is the displacement, y_2 the velocity, and y_3 ($\equiv \alpha$) the acceleration due to gravity.

The general solution to this system of equations is

$$y_1 = \frac{\alpha t^2}{2} + At + B, \quad (26)$$

where A and B are constants of integration. Upon substituting into Eq. (3) the following expression for ϵ is obtained

$$\begin{aligned} \epsilon = B^2 + \left[\frac{\alpha}{2} + A + B - 15 \right]^2 + \left[2\alpha + 2A + B - 65 \right]^2 \\ + \left[\frac{9}{2}\alpha + 3A + B - 145 \right]^2. \end{aligned} \quad (27)$$

In order to minimize ϵ , it is necessary to set the three partial derivatives $\partial\epsilon/\partial\alpha$, $\partial\epsilon/\partial A$, $\partial\epsilon/\partial B$ equal to zero. This provides three simultaneous equations for α , A , and B whose solution is

$$A = -.25, \quad B = -.25, \quad \alpha = 32.5. \quad (28)$$

This constitutes the conventional solution which is exact. Now consider the same problem from the point of view of the Newtonian iteration procedure. The estimate of the initial vector will be taken to be

$$c^* = \{0, 0, 32\}. \quad (29)$$

In this case y^* is

$$y^* = \{16t^2, 32t, 32\}. \quad (30)$$

The perturbation equations are

$$\delta\dot{y}_1 = \delta y_2, \quad \delta\dot{y}_2 = \delta y_3, \quad \delta\dot{y}_3 = 0, \quad (31)$$

which are exact because the original equations are linear. Because these equations are exact it can be anticipated that the corrections as calculated by the Newtonian method will be exact in one iteration. The solution matrix is

TABLE II

	$i = 1$	$i = 2$	$i = 3$
$j = 1$	1	0	0
$j = 2$	t	1	0
$j = 3$	$t^2/2$	t	1

Only $i = 1$ is of interest. From Table II and the estimated solution (30), the following table of calculated data can be constructed

TABLE III

t_m	δy_{11}	δy_{12}	δy_{13}	$y_1^* - b_m$
0	1	0	0	0
1	1	1	1/2	1
2	1	2	2	-1
3	1	3	9/2	-1

With the aid of Table III, the quantities U_1^* can be calculated according to (12):

$$U_1^* = -1, \quad U_2^* = -4, \quad U_3^* = -6.$$

Furthermore, the matrix of coefficients can be calculated according to (15):

$$\begin{vmatrix} 4 & 6 & 7 \\ 6 & 14 & 18 \\ 7 & 18 & 49/2 \end{vmatrix}.$$

The solution concurs precisely with the results previously obtained.

V. *Final remarks.* The iteration technique presented herein can be generalized to cases where more than one variable is measured. The deviation can then be generalized (in the discrete case) to

$$\epsilon = w_1 \sum_{m=1}^M \{y_1(t_m) - b_{1m}\}^2 + w_2 \sum_{m=1}^M \{y_2(t_m) - b_{2m}\}^2, \quad (32)$$

where the weighting constants w are a measure of the relative confidence one has in the measurement of each variable, and reflects this confidence directly by being the reciprocal of the mean square deviation of the measurement with respect to random errors (see [4] for further discussion of this point).

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