UNIFORM ASYMPTOTIC SOLUTIONS OF DISCONTINUOUS INITIAL VALUE PROBLEMS FOR DISPERSIVE HYPERBOLIC EQUATIONS*

BY

J. BOERSMA**

New York University, Courant Institute of Mathematical Sciences

Abstract. This paper is concerned with a discontinuous initial value problem for a dispersive hyperbolic equation where the dispersive term of the equation contains a large parameter.

In the case of an equation with constant coefficients the initial value problem can be solved exactly leading to an integral representation which is expanded asymptotically. Nonuniform and uniform asymptotic expansions are obtained. The nonuniform expansion is singular on a certain space-time line. This singularity is similar to the anomaly occurring at a shadow-boundary in geometrical optics.

For the equation with nonconstant coefficients nonuniform and uniform asymptotic solutions of the initial value problem are derived by ray methods. It is verified that these solutions are in perfect agreement with the nonuniform and uniform asymptotic expansions of the exact solution of the problem.

1. Introduction. In his paper [2] Lewis presents a general method for finding asymptotic solutions of dispersive hyperbolic equations. The dispersive term contains a parameter λ which is assumed to be large. This parameter may also appear in the initial data and in the inhomogeneous (source) term in a variety of ways. Lewis' method is a "ray method" i.e. all functions which appear in the asymptotic expansions satisfy ordinary differential equations along certain space-time curves called rays. These differential equations can be solved explicitly. The starting Ansatz for the ray method is motivated by the form of the asymptotic expansions of certain exact solutions e.g. solutions to problems with constant coefficients. Certain undetermined coefficients which may occur in the ray method solution are obtained by comparing this solution with the asymptotic expansion of the solution of a "canonical problem" i.e., a problem with the same local features but sufficiently simplified to be solved exactly. As a simple example of a dispersive hyperbolic equation that conserves energy Lewis [2] treats a number of initial-boundary value problems for the Klein-Gordon equation. The latter equation is also of interest in itself. As Lewis [2] has pointed out it describes the propagation of electromagnetic waves in certain plasmas.

Bleistein and Lewis [1] considered the same initial-boundary value problems for the Klein-Gordon equation with variable coefficients of special type. Using the tools of Fourier transformation, the WKB method and the method of stationary phase they derived asymptotic expansions of the exact solutions of these problems. It turned out that these expansions were in perfect agreement with the asymptotic solutions derived by means of the ray method.

*Received November 15, 1965; revised manuscript received June 6, 1966. This research was supported by the U. S. Air Force Office of Scientific Research under Grant No. AFOSR-537-64. Reproduction in whole or in part is permitted for any purposes of the U. S. Government.

**On leave of absence from the University of Groningen, The Netherlands. Present address: Department of Mathematics, Technological University, Eindhoven, The Netherlands.

Recently, Lewis [3] discussed another initial value problem for the inhomogeneous Klein-Gordon equation with a source term due to a fixed oscillatory point source. In the case of an equation with constant coefficients this problem can be solved exactly. The asymptotic expansion of this exact solution has an anomaly i.e., a singularity at the points of a particular space-time curve. Lewis calls this curve a shadow-boundary because the anomaly is of the same type as the anomaly occurring at shadow-boundaries in geometrical optics. The asymptotic expansion is not uniform in a space-time region surrounding this shadow-boundary. Lewis removes the anomaly by expanding the exact solution more carefully (cf. [3, Appendix]); this leads to a uniform asymptotic expansion in terms of Fresnel integrals. Guided by this uniform asymptotic expansion the original Ansatz of the ray method is modified in order to yield uniform asymptotic solutions. A formal series involving Fresnel integrals is substituted into the Klein-Gordon equation. It turns out that again the functions making up the formal expansion satisfy ordinary differential equations along certain systems of rays. In this way a uniform asymptotic solution can be derived for the initial value problem mentioned above in the case of an equation with variable coefficients.

The present paper has points of contact with all these papers [1], [2], [3]. Actually, the main goal of this paper is to verify for a special problem the correctness of the uniform asymptotic solution produced by the uniform ray method as introduced in [3]. This verification runs along the same lines as Bleistein and Lewis' [1] verification of the correctness of the ray method solutions presented in [2]. As a simple example of a dispersive hyperbolic equation we deal with an initial value problem for the homogeneous Klein-Gordon equation with oscillatory initial data. Both the phase and amplitude functions occurring in these data are smooth everywhere except at a certain point where these functions and their derivatives show jump discontinuities. A simpler version of this problem was treated in [2, Sec. 3.6].

In the case of an equation with constant coefficients the initial value problem can be solved exactly (Sec. 2). Two different asymptotic expansions are obtained for this exact solution: a nonuniform asymptotic expansion showing a shadow-boundary-type anomaly and a uniform asymptotic expansion in terms of Fresnel integrals.

In Sec. 3 the same initial value problem is considered in the case of an equation with variable coefficients of special type (actually, the coefficients are of the same type as chosen by Bleistein and Lewis [1]). Nonuniform and uniform asymptotic solutions are derived by means of ray methods. The choice of the formal series Ansatz to be substituted into the Klein-Gordon equation is motivated by the exact asymptotic expansions derived in Sec. 2.

Finally, in Sec. 4 we present an alternative treatment of the initial value problem of Sec. 3, leading to nonuniform and uniform asymptotic expansions of the exact solution. These expansions are derived along similar lines as pointed out by Bleistein and Lewis [1]. It turns out that these expansions are in perfect agreement with the previous results derived by means of ray methods. Thus we obtain an independent check on the validity of both nonuniform and uniform ray methods.

2. Asymptotic expansions of the solution of the problem with constant coefficients. We consider the following initial value problem for the homogeneous Klein-Gordon equation in one space dimension

$$c^2 u_{xx} - u_{tt} - \lambda^2 b^2(x)u = 0 \quad (2.1)$$

with oscillatory initial data

$$u(0, x) \sim z_1(x) \exp(i\lambda s_1(x)), \quad u_t(0, x) \sim i\lambda z_2(x) \exp(i\lambda s_2(x)). \quad (2.2)$$

Here c is a positive constant, λ is a large positive parameter. The functions $z_i(x)$, $s_i(x)$ ($i = 1, 2$) are smooth except at $x = x_0$ where these functions and their derivatives possess (finite) jump discontinuities.

This initial value problem can be simplified considerably without essentially changing its features. First, according to [2, Sec. 3.1] the solution of the problem (2.1), (2.2) can be reduced to solving initial value problems with initial data of the following type

$$u(0, x) \sim z_0(x) \exp(i\lambda s_0(x)), \quad u_t(0, x) \sim -i\lambda h_0[s'_0(x), x]z_0(x) \exp(i\lambda s_0(x)) \quad (2.3)$$

where $h_0(k, x) = (c^2 k^2 + b^2(x))^{1/2}$. Secondly, after translating the point of discontinuity x_0 to the origin, the discontinuous functions $z_0(x)$, $s_0(x)$ can be split into functions $z_{0i}(x)$, $s_{0i}(x)$ ($i = 1, 2$) where z_{01} , s_{01} vanish identically for $x < 0$ and z_{02} , s_{02} vanish identically for $x > 0$. The corresponding initial value problems can be solved in the same manner and it is sufficient to deal with only one of them.

Finally, in this paper we will treat the discontinuous initial value problem (2.1), (2.3) where the functions $z_0(x)$, $s_0(x)$ are smooth for $x > 0$ and vanish identically for $x < 0$. At $x = 0$ the functions $z_0(x)$, $s_0(x)$ and their derivatives possess jump discontinuities.

In this section $b(x) = b$ is assumed to be constant and $h_0(k, x)$ is replaced by $h_0(k) = (c^2 k^2 + b^2)^{1/2}$.

The equation (2.1) which is a linear equation with constant coefficients can be solved exactly yielding (cf. [2, Sec. 3.2]),

$$u(t, x) \sim u_+(t, x) + u_-(t, x), \quad (2.4)$$

where

$$u_{\pm}(t, x) = \frac{\lambda}{4\pi} \int_{-\infty}^{\infty} dk \int_0^{\infty} d\sigma z_0(\sigma) \left\{ 1 \pm \frac{h_0[s'_0(\sigma)]}{h_0(k)} \right\} \exp(i\lambda \phi_{\pm}) \quad (2.5)$$

and

$$\phi_{\pm} = s_0(\sigma) + k(x - \sigma) \mp h_0(k)t. \quad (2.6)$$

We expand the integrals (2.5) asymptotically in two different ways leading to a non-uniform and a uniform asymptotic expansion.

2.1. The nonuniform expansion. The asymptotic expansions of the double integrals (2.5) are obtained by the method of stationary phase. Similar to [3] the leading terms of these expansions are due to interior stationary points of the phase functions $\phi_{\pm}$, whereas the second terms of the expansions are due to so-called critical points on the boundary $\sigma = 0$.

The leading terms of the asymptotic expansions of $u_{\pm}(t, x)$ were already derived in [2, Sec. 3.2]. We quote the following results: The leading term for $u_-(t, x)$ vanishes whereas the leading term for $u_+(t, x)$ is given by

$$u_+(t, x) \sim z \exp(i\lambda s) \quad (2.7)$$

where

$$\begin{aligned} z &= z_0(p) \{ 1 + i h_0''[s'_0(p)] s_0''(p) \}^{-1/2}, \\ s &= s_0(p) + \{ h_0'[s'_0(p)] s'_0(p) - h_0[s'_0(p)] \} t. \end{aligned} \quad (2.8)$$

The parameter p depends on t and x through

$$x = p + h'_0[s'_0(p)]t. \quad (2.9)$$

The stationary point of the phase function ϕ_+ is given by

$$\sigma = p, \quad k = s'_0(p). \quad (2.10)$$

Thus the functions z and s appearing in (2.7) are given parametrically with the parameter p by (2.8), (2.9). Alternatively, the equation (2.9) defines a one-parameter family of straight lines in tx -space. These lines which are called rays are illustrated by the solid lines in Fig. 1. For each fixed p , as t varies, z and s are given along a ray by (2.8). Because $z_0(p)$ vanishes for $p < 0$, it is sufficient to consider only the rays (2.9) with $p > 0$.

The second terms of the asymptotic expansions of $u_{\pm}$ are obtained by evaluating the integrals (2.5) taken over the "boundary strip" $-\infty < k < \infty$, $0 \leq \sigma \leq \epsilon$ viz.

$$u'_{\pm} = \frac{\lambda}{4\pi} \int_0^{\epsilon} d\sigma z_0(\sigma) \exp(i\lambda s_0(\sigma)) \cdot \int_{-\infty}^{\infty} dk \left\{ 1 \pm \frac{h_0[s'_0(\sigma)]}{h_0(k)} \right\} \exp[i\lambda\{k(x - \sigma) \mp h_0(k)t\}]. \quad (2.11)$$

Here ϵ is chosen sufficiently small in order that the stationary point (2.10) lies outside the strip of integration.

The inner integral in (2.11) is expanded asymptotically by the method of stationary phase. The stationary points of the phase function follow from the equation

$$x - \sigma \mp h'_0(k)t = 0. \quad (2.12)$$

This equation determines k as a function of σ ($a_1^{-1}x, t$),

$$k = \pm k_1(\sigma). \quad (2.13)$$

Applying the ordinary stationary phase formula we obtain the asymptotic result

$$u'_{\pm} \sim \frac{\lambda^{1/2} e^{\mp \pi i/4}}{2(2\pi)^{1/2}} \int_0^{\epsilon} d\sigma \frac{z_0(\sigma)}{\{h''_0(k_1)t\}^{1/2}} \left\{ 1 \pm \frac{h_0[s'_0(\sigma)]}{h_0(k_1)} \right\} \cdot \exp\{i\lambda[s_0(\sigma) \pm \{k_1 h'_0(k_1) - h_0(k_1)\}t]\}, \quad (2.14)$$

valid for $t > 0$. The integrals (2.14) are integrated by parts. Only the contribution from the end point $\sigma = 0$ is taken into account. The contribution from the end point $\sigma = \epsilon$ is ignored because this contribution is cancelled by the contribution from $\sigma = \epsilon$ to the integral over the remaining part of the (k, σ) plane. The ultimate result is given by

$$u'_{\pm} \sim \lambda^{-1/2} \hat{z}_{\pm} \exp(i\lambda \hat{s}_{\pm}) \quad (2.15)$$

where

$$\hat{z}_{\pm} = \frac{iz(+0) \exp(\mp \pi i/4)}{2(2\pi)^{1/2}} \frac{\{1 \pm h_0[s'_0(+0)]/h_0(\kappa)\}}{\{h''_0(\kappa)t\}^{1/2} \{s'_0(+0) \mp \kappa\}}, \quad (2.16)$$

$$\hat{s}_{\pm} = s_0(+0) \pm \{ \kappa h'_0(\kappa) - h_0(\kappa) \} t.$$

In (2.16) we introduced the parameter κ defined by

$$\kappa = k_1(0), \quad (2.17)$$

hence, according to (2.12), (2.13) κ is given as a function of t and x by

$$x = h'_0(\kappa)t. \quad (2.18)$$

The functions $\hat{z}_\pm$, $\hat{s}_\pm$ appearing in (2.16) are given parametrically with parameter κ by (2.16), (2.18). Alternatively, the equation (2.18) defines a one-parameter family of rays. For each fixed value of κ , as t varies, $\hat{z}_\pm$ and $\hat{s}_\pm$ are given along a ray by (2.16). These rays are illustrated by the dashed lines in Fig. 1. As κ varies between $-\infty$ and $+\infty$ they fill out the "characteristic cone" of (2.1) with apex at the origin.

The final asymptotic expansion of the solution $u(t, x)$ is obtained by inserting the results (2.7) and (2.15) in (2.4) viz.

$$u(t, x) \sim z \exp(i\lambda s) + \lambda^{-1/2}(\hat{z}_+ \exp(i\lambda \hat{s}_+) + \hat{z}_- \exp(i\lambda \hat{s}_-)). \quad (2.19)$$

The term $z \exp(i\lambda s)$ is called the main term of the asymptotic expansion. The terms of order $\lambda^{-1/2}$ in (2.19) which decay with time (cf. (2.16)) are called transient terms. Similarly, the corresponding rays are denoted by main rays and transient rays.

The asymptotic expansion (2.19) has an anomaly or a defect along the main ray emanating from the origin. This ray which is given by (2.9) with $p = +0$ coincides with the transient ray given by (2.18) with $\kappa = s'_0(+0)$. We call these coinciding rays a shadow-boundary because the anomaly is of the same type as the anomaly occurring at a shadow-boundary in geometrical optics. Across the shadow-boundary the main term $z \exp(i\lambda s)$ is discontinuous (cf. (2.8)). At the shadow-boundary the transient term $\lambda^{-1/2} \hat{z}_+ \exp(i\lambda \hat{s}_+)$ becomes infinite (cf. (2.16)). It is clear that the asymptotic

FIG. 1.

The rays for the asymptotic solution of the initial value problem (2.1), (2.3) with constant b . The main rays are illustrated by solid lines and the transient rays by dashed lines. One ray of each family coincides with the shadow-boundary. For nonconstant $b(x)$ the essential features of the figure are unchanged, but the rays are not straight lines.

expansion (2.19) is not uniform in the neighborhood of the shadow-boundary. It can easily be seen that the transient term $\lambda^{-1/2} \hat{z}_- \exp(i\lambda \hat{s}_-)$ has no anomalies.

2.2. The uniform expansion. In this section we derive an asymptotic expansion of $u(t, x)$ which is uniformly valid near the shadow-boundary. Only the part $u_+(t, x)$ of $u(t, x)$ needs to be considered. The asymptotic expansion of this part as derived in Sec. 2.1 consisted of two terms due to interior stationary points of the phase function and to critical points on the boundary $\sigma = 0$. These terms were calculated separately. However, when the point (t, x) lies in the neighborhood of the shadow-boundary, then the stationary point (2.10) is close to the boundary $\sigma = 0$. Hence, the two terms of the expansion have to be treated simultaneously. The integral (2.5) for $u_+(t, x)$ is replaced by the integral u_+^* introduced in (2.11), but now ϵ is chosen sufficiently large in order that the stationary point lies inside the strip of integration. The inner integral of (2.11) is again expanded asymptotically by the method of stationary phase leading to (2.14). The phase of the integrand in (2.14) has a stationary point $\sigma = p$ where p is given by (2.9).

A uniform asymptotic expansion of the integral (2.14) is obtained by using a result of Lewis presented in the Appendix of [3]. Lewis deals with integrals of the following type

$$u(k, p) = \int_0^d g(x) \exp(ik\phi(x)) dx \quad (2.20)$$

where d is a positive constant and k is a large positive parameter. The functions $g(x)$ and $\phi(x)$ which also depend on a parameter p are assumed to be smooth functions of x in an interval $-\epsilon \leq x \leq d$ where $\epsilon > 0$. If $g(x)$ and $\phi(x)$ are originally defined only for $0 \leq x \leq d$ a smooth continuation to $-\epsilon \leq x \leq d$ is necessary. The function $\phi(x)$ has a stationary point at $x = p$, but otherwise $\phi'(x)$ does not vanish in $-\epsilon \leq x \leq d$. It is assumed that $-\epsilon < p < p_1$ where $p_1 < d$ and that the second derivative of $\phi(x)$ does not vanish at $x = p$. Then the following uniform asymptotic expansion holds for $u(k, p)$,

$$u(k, p) \sim e^{ik\phi(0)} \left[\left\{ \frac{2}{k |\phi''(p)|} \right\}^{1/2} g(p) g_{\pm} [\operatorname{sgn}(-p)(k |\phi(0) - \phi(p)|)^{1/2}] + \frac{ig(0)}{k\phi'(0)} \right], \quad \phi''(p) \gtrless 0. \quad (2.21)$$

The functions $g_{\pm}(x)$ are defined by

$$g_{\pm}(x) = \exp(\mp ix^2) \int_x^{\infty} \exp(\pm it^2) dt \pm \frac{1}{2ix}. \quad (2.22)$$

Strictly speaking, (2.21) is not the complete asymptotic expansion of $u(k, p)$ because the contribution of the end point $x = d$ is ignored.

Application of Lewis' formula (2.21) to the integral (2.14) for u_+^* leads after some evaluations to the following uniform asymptotic expansion,

$$u_+^* \sim \exp(i\lambda \hat{s}_+) \left\{ \frac{\exp(-\pi i/4)}{\pi^{1/2}} z g_+ [\operatorname{sgn}(-p)(\lambda(\hat{s}_+ - s))^{1/2}] + \lambda^{-1/2} \hat{z}_+ \right\}. \quad (2.23)$$

The final uniform asymptotic expansion for $u(t, x)$ reads

$$u(t, x) \sim \frac{\exp(-\pi i/4)}{\pi^{1/2}} z \exp(i\lambda \hat{s}_+) g_+ [\operatorname{sgn}(-p)(\lambda(\hat{s}_+ - s))^{1/2}] + \lambda^{-1/2} (\hat{z}_+ \exp(i\lambda \hat{s}_+) + \hat{z}_- \exp(i\lambda \hat{s}_-)). \quad (2.24)$$

The functions z , s are given parametrically by (2.8), (2.9). The functions $\hat{z}_\pm$, $\hat{s}_\pm$ are given parametrically by (2.16), (2.18). Actually, the functions $z_0(p)$, $s_0(p)$ occurring in (2.8) must be continued smoothly for negative values of p . This continuation which replaces the original definition of these functions for $p < 0$, also involves a continuation of the system of main rays (2.9) for $p < 0$.

An alternative expression for the uniform asymptotic expansion of $u(t, x)$ can be shown to be given by

$$u(t, x) \sim \eta(p)z \exp(i\lambda s) + \lambda^{-1/2}(\hat{z}_+ \exp(i\lambda \hat{s}_+) + \hat{z}_- \exp(i\lambda \hat{s}_-)) + \frac{\exp(-\pi i/4)}{\pi^{1/2}} \operatorname{sgn}(-p)z \exp(i\lambda \hat{s}_+)g_+[(\lambda(\hat{s}_+ - s))^{1/2}] \quad (2.25)$$

where $\eta(p) = 1$ if $p > 0$ and $\eta(p) = 0$ if $p < 0$. This expression shows more clearly the relationship between the nonuniform and the uniform asymptotic expansion. It can easily be verified that the last term of (2.25) is of order $\lambda^{-3/2}$ sufficiently far from the shadow-boundary and hence, can be neglected.

3. Asymptotic solutions of the problem with variable coefficients derived by ray methods. In this section we treat the initial value problem (2.1), (2.3) for the case of an equation with a variable coefficient $b(x)$. The function $b(x)$ is assumed to be smooth, positive and decreasing i.e. $b(x) > 0$, $b'(x) < 0$. The limits $\lim_{x \rightarrow -\infty} b(x) = b_0$, $\lim_{x \rightarrow \infty} b(x) = b_1$ will exist, hence $b_0 > b(x) > b_1$ for each value of x .

Using ray methods we will derive nonuniform and uniform asymptotic solutions for the initial value problem.

3.1. The nonuniform asymptotic solution. Asymptotic solutions of the Klein-Gordon equation derived by the ray method are discussed in [1], [2]. The ray method comes down to substituting a formal expansion

$$u \sim \exp(i\lambda s(t, x)) \sum_{m=0}^{\infty} (i\lambda)^{-m} z^{(m)}(t, x) \quad (3.1)$$

into the Klein-Gordon equation. Then one obtains transport equations for the amplitude functions $z^{(m)}(t, x)$ and a dispersion equation for the phase function $s(t, x)$. The latter first order partial differential equation can be solved by the method of characteristics leading to a system of characteristic equations. Each solution of these equations defines a space-time curve called a ray. Along the rays the transport equations can be written as ordinary differential equations. After supplying initial data for the phase and amplitudes the terms of the formal series (3.1) can be solved recursively. In the following we are only interested in the leading terms of formal expansions like (3.1). We will need two types of solutions of the characteristic equations, dispersion equation and zero-order transport equation corresponding to the initial data being given on a one-dimensional or on a zero-dimensional initial manifold i.e. on a line or at a point. These solutions are quoted from Bleistein and Lewis [1] who studied initial-boundary value problems for the same equation.

Guided by the nonuniform asymptotic expansion (2.19) of $u(t, x)$ in the case of the problem with constant coefficients we state the following asymptotic solution of the initial value problem with variable coefficients,

$$u(t, x) \sim z \exp(i\lambda s) + \lambda^{-1/2}(\hat{z}_+ \exp(i\lambda \hat{s}_+) + \hat{z}_- \exp(i\lambda \hat{s}_-)) \quad (3.2)$$

where z , s , $\hat{z}_\pm$, $\hat{s}_\pm$ are to be determined. The term $z \exp(i\lambda s)$ is again called the main term, the remaining terms of (3.2) are called transient terms. The corresponding rays

are called main rays and transient rays. It is assumed that the main rays emanate from the line $t = 0$ whereas the transient rays emanate from the point $t = 0, x = 0$.

The following initial data for z, s and s_t are to be prescribed in order to satisfy the initial conditions (2.3),

$$z(0, x) = z_0(x), \quad s(0, x) = s_0(x), \quad -s_t(0, x) = h_0[s'_0(x), x]. \quad (3.3)$$

Then according to Bleistein and Lewis [1] the amplitude and phase z, s are given by the following parametric representation with parameter p ,

$$t = \operatorname{sgn} \{s'_0(p)\} \frac{1}{c} \int_p^x \frac{h_0[s'_0(p), p]}{\sqrt{h_0^2[s'_0(p), p] - b^2(\eta)}} d\eta, \quad (3.4)$$

$$z(t, x) = z_0(p)[j_1(t(x))]^{-1/2},$$

$$s(t, x) = s_0(p) + \operatorname{sgn} \{s'_0(p)\} \frac{1}{c} \int_p^x (h_0^2[s'_0(p), p] - b^2(\eta))^{1/2} d\eta - h_0[s'_0(p), p]t. \quad (3.5)$$

The Eq. (3.4) defines the system of main rays and $j_1(t(x))$ is the Jacobian corresponding to these rays viz.

$$\begin{aligned} j_1(t(x)) &= |\partial x / \partial p| \\ &= \left| \frac{(h_0^2[s'_0(p), p] - b^2(x))^{1/2}}{(h_0^2[s'_0(p), p] - b^2(p))^{1/2}} + (h_0^2[s'_0(p), p] - b^2(x))^{1/2} \right. \\ &\quad \left. \cdot \frac{c^2 s'_0(p) s''_0(p) + b(p) b'(p)}{h_0^2[s'_0(p), p]} \int_p^x \frac{b^2(\eta) d\eta}{\{h_0^2[s'_0(p), p] - b^2(\eta)\}^{3/2}} \right|. \end{aligned} \quad (3.6)$$

Because $z_0(p) \equiv 0$ for $p < 0$, it is sufficient to consider only the main rays (3.4) with $p > 0$.

Similarly according to [1] the amplitudes and phases $\hat{z}_\pm, \hat{s}_\pm$ are given by a parametric representation with parameter κ ,

$$t = \operatorname{sgn}(\kappa) \frac{1}{c} \int_0^x \frac{h_0(\kappa, 0)}{(h_0^2(\kappa, 0) - b^2(\eta))^{1/2}} d\eta, \quad (3.7)$$

$$\hat{z}_\pm(t, x) = \tilde{z}_\pm \left\{ \frac{c^2 b^2(0)}{h_0^3(\kappa, 0) j_2(t(x))} \right\}^{1/2}, \quad (3.8)$$

$$\hat{s}_\pm(t, x) = \hat{s}_\pm(0, 0) \pm \operatorname{sgn}(\kappa) \frac{1}{c} \int_0^x (h_0^2(\kappa, 0) - b^2(\eta))^{1/2} d\eta \mp h_0(\kappa, 0)t.$$

The Eq. (3.7) defines the system of transient rays and $j_2(t(x))$ is the Jacobian corresponding to these rays viz.

$$j_2(t(x)) = \left| \frac{\partial x}{\partial \kappa} \right| = \left| \frac{c^2 \kappa (h_0^2(\kappa, 0) - b^2(x))^{1/2}}{h_0^2(\kappa, 0)} \int_0^x \frac{b^2(\eta) d\eta}{\{h_0^2(\kappa, 0) - b^2(\eta)\}^{3/2}} \right| \quad (3.9)$$

In (3.8) the initial values $\hat{s}_\pm(0, 0)$ and the limits $\tilde{z}_\pm = \lim_{t \rightarrow 0} \hat{z}_\pm t^{1/2}$ are still undetermined. It is assumed that these values only depend on the local value $b(0)$ of $b(x)$. Therefore we apply the "indirect" method and compare the preceding results specialized for the problem with constant coefficients $b(x) = b(0)$ (the "canonical" problem) with the exact asymptotic result for this problem as derived in Sec. 2.1. In this way $\tilde{z}_\pm$ and $\hat{s}_\pm(0, 0)$ can be determined and substituted into (3.8). We only give the ultimate result,

$$\hat{z}_{\pm}(t, x) = \frac{iz_0(+0) \exp(\mp \pi i/4)}{2(2\pi)^{1/2}} \frac{\{1 \pm h_0[s'_0(x_0 + 0), x_0]/h_0(\kappa, 0)\}}{s'_0(x_0 + 0) \mp \kappa} \{j_2(t(x))\}^{-1/2}, \quad (3.10)$$

$$\hat{s}_{\pm}(t, x) = s_0(+0) \pm \operatorname{sgn}(\kappa) \frac{1}{c} \int_0^x (h_0^2(\kappa, 0) - b^2(\eta))^{1/2} d\eta \mp h_0(\kappa, 0)t.$$

Actually, the asymptotic solution (3.2), (3.5), (3.10) is not yet complete. If $s'_0(p) < 0$, $h_0[s'_0(p), p] < b_0$ the main ray (3.4) has a turning point $x = x_p$, $b(x_p) = h_0[s'_0(p), p]$. Similarly, if $\kappa < 0$, $h_0(\kappa, 0) < b_0$ the transient ray (3.7) has a turning point $x = x_{\kappa}$, $b(x_{\kappa}) = h_0(\kappa, 0)$. In these cases the formulae (3.5), (3.10) only hold up to the turning points. In this paper we do not present additional formulae for the amplitudes and phases of the main term and of the transient terms holding beyond the turning points along the turned rays. These formulae can easily be quoted from Bleistein and Lewis [1].

The asymptotic solution has again a shadow-boundary-type anomaly. The "shadow-boundary" is formed by the main ray (3.4) with $p = +0$ which coincides with the transient ray (3.7) with $\kappa = s'_0(+0)$. Again the main term $z \exp(i\lambda s)$ is discontinuous across the shadow-boundary whereas the transient term $\hat{z}_{+} \exp(i\lambda \hat{s}_{+})$ becomes infinite at the shadow-boundary. Hence, the asymptotic solution (3.2) is not uniform in the neighborhood of the shadow-boundary.

3.2. The uniform asymptotic solution. A uniform asymptotic solution of our initial value problem will be derived by means of a uniform ray method as introduced by Lewis [3]. The method comes down to substituting a formal expansion of the following type,

$$u(t, x) \sim \exp(i\lambda \hat{s}(t, x)) \left[g_{+}(\lambda^{1/2} \theta) \sum_{m=0}^{\infty} (i\lambda)^{-m} z^{(m)}(t, x) + \lambda^{-1/2} \sum_{m=0}^{\infty} (i\lambda)^{-m} \hat{z}^{(m)}(t, x) \right] \quad (3.11)$$

into the Klein-Gordon equation. The function g_{+} is given by (2.22) and $\theta^2 = \hat{s}(t, x) - s(t, x)$. One obtains again transport equations for the amplitude functions $z^{(m)}(t, x)$, $\hat{z}^{(m)}(t, x)$, and a dispersion equation for the phase functions $s(t, x)$, $\hat{s}(t, x)$. Confining ourselves to the leading terms of the expansion (3.11) it turns out (cf. [3]) that the transport equation for $z^{(0)}(t, x)$, $\hat{z}^{(0)}(t, x)$ and the dispersion equation for $s(t, x)$, $\hat{s}(t, x)$ are identical with the corresponding equations arising at the ordinary ray method. Hence, the same two types of solutions of the characteristic equations, the dispersion equation and the transport equation can be used as before in Sec. 3.1.

Guided by the uniform asymptotic expansion (2.24) of $u(t, x)$ in the case of the problem with constant coefficients we state the following uniform asymptotic solution of the initial value problem with variable coefficients,

$$u(t, x) \sim \frac{\exp(-\pi i/4)}{\pi^{1/2}} z \exp(i\lambda \hat{s}_{+}) g_{+}[\delta(\lambda(\hat{s}_{+} - s))^{1/2}] \\ + \lambda^{-1/2} (\hat{z}_{+} \exp(i\lambda \hat{s}_{+}) + \hat{z}_{-} \exp(i\lambda \hat{s}_{-})), \quad (3.12)$$

where z , s , $\hat{z}_{\pm}$, $\hat{s}_{\pm}$, $\delta = \pm 1$, are to be determined. It is assumed that the rays corresponding to the amplitude z and the phase s emanate from the line $t = 0$. Similarly, the rays corresponding to the amplitudes $\hat{z}_{\pm}$ and the phases $\hat{s}_{\pm}$ are assumed to emanate from the point $t = 0$, $x = 0$. These rays are called main rays and transient rays.

Expanding the function g_{+} asymptotically the Ansatz (3.12) satisfies the initial conditions (2.3) provided that the initial data (3.3) are prescribed for z , s and s_t . Hence

the amplitude and phase z, s are again given by the parametric representation (3.4), (3.5). We remark that in the present case the functions $z_0(p), s_0(p)$ as occurring in (3.5) must be continued smoothly for negative values of p . This continuation which replaces the original definition of these functions for $p < 0$, also involves a continuation of the system of main rays (3.4) for $p < 0$.

The amplitudes and phases $\hat{z}_\pm, \hat{s}_\pm$ of the transient terms are again given by the parametric representation (3.7), (3.8). The undetermined coefficients follow by the indirect method. The ultimate result is again given by (3.10) as can easily be verified. Moreover, the indirect method yields $\delta = \text{sgn}(-p)$ where p is the parameter entering in (3.4).

4. Asymptotic expansions of the exact solution of the problem with variable coefficients. Using a method due to Bleistein and Lewis [1] we derive a nonuniform and a uniform asymptotic expansion of the exact solution of the initial value problem studied in Sec. 3. First, we apply Fourier transformation to $u(t, x)$ with respect to the time t . The transformed function is denoted by $v(x, \omega)$,

$$v(x, \omega) = \int_0^\infty u(t, x) \exp(i\lambda\omega t) dt. \quad (4.1)$$

It is assumed that the integral (4.1) is absolutely convergent when $\text{Im } \omega > 0$ and that the absolute integral is uniformly bounded with respect to x . The Fourier inversion theorem yields under suitable conditions on $v(x, \omega)$,

$$u(t, x) = \frac{\lambda}{2\pi} \sum_{\pm} \int_0^\infty v(x, \pm\omega) \exp(\mp i\lambda\omega t) d\omega. \quad (4.2)$$

Taking the Fourier transform of the Klein-Gordon equation (2.1) we obtain the following ordinary differential equation for $v(x, \omega)$,

$$v''(x, \omega) + \lambda^2 \{\omega^2 - b^2(x)\} v(x, \omega) = r(x, \omega; \lambda), \quad (4.3)$$

where

$$r(x, \omega; \lambda) = i\lambda \{\omega + h_0[s'_0(x), x]\} z_0(x) \exp(i\lambda s_0(x)) \quad (4.4)$$

and a prime denotes differentiation with respect to x . Note, that $r(x, \omega; \lambda)$ vanishes identically for $x < 0$. According to Bleistein and Lewis [1] $v(x, \omega)$ has to satisfy the "radiation conditions,"

$$\lim_{x \rightarrow -\infty} v(x, \omega) = 0, \quad \lim_{x \rightarrow -\infty} v(x, \omega) = 0, \quad \text{when } \text{Im } \omega > 0. \quad (4.5)$$

We introduce two linearly independent solutions $v_1(x, \omega), v_2(x, \omega)$ of the homogeneous form of the equation (4.3). Moreover, these solutions satisfy the first and second of the radiation conditions (4.5). Then the solution of (4.3) subject to the conditions (4.5) is given by

$$v(x, \omega) = -\frac{1}{c^2 W(v_1, v_2)} \left[v_1(x, \omega) \int_{-\infty}^x v_2(\xi, \omega) r(\xi, \omega; \lambda) d\xi + v_2(x, \omega) \int_x^\infty v_1(\xi, \omega) r(\xi, \omega; \lambda) d\xi \right], \quad (4.6)$$

where the Wronskian $W(v_1, v_2) = v_1 v_2' - v_2 v_1'$ is independent of x .

The solutions $v_1(x, \omega), v_2(x, \omega)$ are expanded asymptotically for large values of λ . Two regions of the (x, ω) plane are of interest.

Region I. $\omega > b_0$, $-\infty < x < \infty$. A solution of the form $a(x, \omega) \exp(i\lambda s(x, \omega))$ is formally substituted into (4.3) leading to differential equations for a and s which can be solved explicitly. Similar to Bleistein and Lewis [1] we obtain the asymptotic results,

$$\begin{aligned} v_1(x, \pm\omega) &\sim \{\omega^2 - b^2(x)\}^{-1/4} \exp\left[\pm i \frac{\lambda}{c} \int_{x_1}^x (\omega^2 - b^2(\eta))^{1/2} d\eta\right], \\ v_2(x, \pm\omega) &\sim \{\omega^2 - b^2(x)\}^{-1/4} \exp\left[\pm i \frac{\lambda}{c} \int_x^{x_1} (\omega^2 - b^2(\eta))^{1/2} d\eta\right], \end{aligned} \quad (4.7)$$

where x_1 is arbitrary but fixed. The Wronskian is given by

$$W(v_1, v_2) = \mp \frac{2i\lambda}{c}. \quad (4.8)$$

Region II. $b_0 > \omega > b_1$, $x_\omega < x < \infty$ where $b(x_\omega) = \omega$. The function $v_1(x, \omega)$ which satisfies the radiation condition (4.5) for $x \rightarrow \infty$ is again given by (4.7) where we replace x_1 by x_ω for convenience,

$$v_1(x, \pm\omega) \sim \{\omega^2 - b^2(x)\}^{-1/4} \exp\left[\pm i \frac{\lambda}{c} \int_{x_\omega}^x (\omega^2 - b^2(\eta))^{1/2} d\eta\right]. \quad (4.9)$$

The function $v_2(x, \omega)$ which is determined by its behaviour when $x \rightarrow -\infty$ assumes the following asymptotic value when $x < x_\omega$,

$$v_2(x, \omega) \sim \{b^2(x) - \omega^2\}^{-1/4} \exp\left[-\frac{\lambda}{c} \int_x^{x_\omega} (b^2(\eta) - \omega^2)^{1/2} d\eta\right]. \quad (4.10)$$

Using the connection formulae of the WKB method (cf. [1]) we obtain for $x > x_\omega$,

$$v_2(x, \pm\omega) \sim 2\{\omega^2 - b^2(x)\}^{-1/4} \sin\left[\frac{\lambda}{c} \int_{x_\omega}^x (\omega^2 - b^2(\eta))^{1/2} d\eta + \frac{\pi}{4}\right]. \quad (4.11)$$

The Wronskian of the two solutions $v_1(x, \pm\omega)$, $v_2(x, \pm\omega)$ is given by

$$W(v_1, v_2) = \frac{2\lambda}{c} \exp(\mp \pi i/4). \quad (4.12)$$

Actually, we still have to consider the regions, $b_0 > \omega > b_1$, $-\infty < x < x_\omega$ and $0 < \omega < b_1$, $-\infty < x < \infty$ of the (x, ω) plane but in these regions $v(x, \omega)$ is exponentially small as Bleistein and Lewis [1] have shown.

Substitution of these asymptotic results into (4.6) leads after some reductions to the following asymptotic expansion for $v(x, \pm\omega)$,

$$\begin{aligned} v(x, \pm\omega) &\sim v_A(x, \pm\omega) \quad \text{when } \omega > b_0, \quad -\infty < x < \infty, \\ v(x, \pm\omega) &\sim v_A(x, \pm\omega) + v_B(x, \pm\omega) \quad \text{when } b_0 > \omega > b_1, \quad x_\omega < x < \infty, \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} v_A(x, \pm\omega) &= \pm \frac{1}{2i\lambda c} \int_{-\infty, x_\omega}^{\infty} r(\xi, \pm\omega; \lambda) \{\omega^2 - b^2(x)\}^{-1/4} \{\omega^2 - b^2(\xi)\}^{-1/4} \\ &\quad \cdot \exp\left\{\pm i \frac{\lambda}{c} \int_{x_\omega}^{x_\omega} (\omega^2 - b^2(\eta))^{1/2} d\eta\right\} d\xi, \end{aligned} \quad (4.14)$$

$$v_B(x, \pm\omega) = \mp \frac{\exp(\pm\pi i/2)}{2i\lambda c} \int_{x_w}^{\infty} r(\xi, \pm\omega; \lambda) \{\omega^2 - b^2(x)\}^{-1/4} \{\omega^2 - b^2(\xi)\}^{-1/4} \\ \cdot \exp \left\{ \pm i \frac{\lambda}{c} \left(\int_{x_w}^x + \int_{x_w}^{\xi} \right) (\omega^2 - b^2(\eta))^{1/2} d\eta \right\} d\xi. \quad (4.15)$$

In (4.14) $x_< = \min(x, \xi)$, $x_> = \max(x, \xi)$.

The asymptotic expansion (4.13) is substituted into (4.2) yielding,

$$u(t, x) \sim u_A(t, x) + u_B(t, x), \quad (4.16)$$

where

$$u_A(t, x) = \frac{\lambda}{2\pi} \sum_{\pm} \int_{b_1}^{\infty} v_A(x, \pm\omega) \exp(\mp i\lambda\omega t) d\omega, \\ u_B(t, x) = \frac{\lambda}{2\pi} \sum_{\pm} \int_{b_1}^{b_0} v_B(x, \pm\omega) \exp(\mp i\lambda\omega t) d\omega. \quad (4.17)$$

In the following sections we derive a nonuniform and a uniform asymptotic expansion for $u_A(t, x)$ which can be represented by (cf. (4.4), (4.14), (4.17))

$$u_A(t, x) = u_{A+}(t, x) + u_{A-}(t, x), \quad (4.18)$$

$$u_{A\pm}(t, x) = \frac{\lambda}{4\pi c} \int_0^{\infty} d\xi \int_{b(\xi)}^{\infty} d\omega z_0(\xi) \{\omega \pm h_0[s'_0(\xi), \xi]\} \{\omega^2 - b^2(x)\}^{-1/4} \\ \cdot \{\omega^2 - b^2(\xi)\}^{-1/4} \exp \left[i\lambda \left\{ s_0(\xi) \pm \frac{1}{c} \int_{x_<}^{x_>} (\omega^2 - b^2(\eta))^{1/2} d\eta \mp \omega t \right\} \right]. \quad (4.19)$$

It turns out that these expansions agree with the asymptotic solutions derived in Secs. 3.1 and 3.2 by ray methods.

Similar expansions were obtained for $u_B(t, x)$ and it was verified that these expansions were in agreement with the ray method solutions as "propagated" along the turned rays. No details of these expansions are given here.

4.1. The nonuniform asymptotic expansion. The asymptotic expansions of the double integrals (4.10) are obtained by the method of stationary phase. The leading terms of these expansions are again due to interior stationary points of the phase, whereas the second terms of the expansions are due to critical points on the boundary $\xi = 0$.

It can easily be checked that the leading term corresponding to $u_{A-}(t, x)$ vanishes. However, $u_{A+}(t, x)$ yields a leading term $z \exp(i\lambda s)$ where z and s are represented parametrically by (3.4), (3.5) with parameter p . Expressed into this same parameter the stationary point of the phase of the latter integral is given by

$$\xi = p, \quad \omega = h_0[s'_0(p), p]. \quad (4.20)$$

The second term of the asymptotic expansion of $u_A(t, x)$ is obtained by investigating the integrals (4.19) taken over the strip $0 \leq \xi \leq \epsilon$, $b(\xi) \leq \omega < \infty$,

$$u_{A+}^* = \frac{\lambda}{4\pi c} \int_0^{\epsilon} d\xi z_0(\xi) \exp(i\lambda s_0(\xi)) \int_{b(\xi)}^{\infty} d\omega \{\omega \pm h_0[s'_0(\xi), \xi]\} \{\omega^2 - b^2(x)\}^{-1/4} \\ \cdot \{\omega^2 - b^2(\xi)\}^{-1/4} \exp \left[i\lambda \left\{ \pm \frac{1}{c} \int_{x_<}^{x_>} (\omega^2 - b^2(\eta))^{1/2} d\eta \mp \omega t \right\} \right]. \quad (4.21)$$

Here ϵ is chosen sufficiently small in order that the stationary point (4.20) lies outside the strip of integration. The inner integral is expanded asymptotically by the method of stationary phase. The stationary points of the phase function follow from the equation

$$t = \frac{1}{c} \int_{x<}^{x>} \frac{\omega}{(\omega^2 - b^2(\eta))^{1/2}} d\eta = \operatorname{sgn}(x - \xi) \frac{1}{c} \int_{\xi}^x \frac{\omega}{(\omega^2 - b^2(\eta))^{1/2}} d\eta, \quad (4.22)$$

assuming $t > 0$. This equation determines ω as a function of ξ (and x, t),

$$\omega = \omega_1(\xi). \quad (4.23)$$

Applying the ordinary stationary phase formula we obtain the asymptotic result,

$$\begin{aligned} u_{A\pm}^* \sim & \frac{\lambda^{1/2}}{2c(2\pi)^{1/2}} \int_0^\epsilon d\xi z_0(\xi) \{\omega_1(\xi) \pm h_0[s'_0(\xi), \xi]\} \{\omega_1^2(\xi) - b^2(x)\}^{-1/4} \\ & \cdot \{\omega_1^2(\xi) - b^2(\xi)\}^{-1/4} \left| \frac{1}{c} \int_{x<}^{x>} \frac{b^2(\eta) d\eta}{\{\omega_1^2(\xi) - b^2(\eta)\}^{3/2}} \right|^{-1/2} \\ & \cdot \exp \left[i\lambda \left\{ s_0(\xi) \pm \frac{1}{c} \int_{x<}^{x>} (\omega_1^2(\xi) - b^2(\eta))^{1/2} d\eta \mp \omega_1(\xi)t \right\} \mp \frac{\pi i}{4} \right]. \end{aligned} \quad (4.24)$$

The integrals (4.24) are integrated by parts. Only the contribution from the end point $\xi = 0$ is taken into account. After introducing the parameter κ according to

$$\omega_1(0) = (c^2\kappa^2 + b^2(0))^{1/2} = h_0(\kappa, 0), \quad \operatorname{sgn}(\kappa) = \operatorname{sgn}(x) \quad (4.25)$$

it can be verified that the asymptotic expansion of $u_{A\pm}^*$ is given by $\lambda^{-1/2} \hat{z}_\pm \exp(i\lambda \hat{s}_\pm)$ where $\hat{z}_\pm, \hat{s}_\pm$ are represented parametrically by (3.7), (3.10) with parameter κ .

Concluding, the asymptotic expansion of $u_A(t, x)$ is in perfect agreement with the asymptotic solution derived in Sec. 3.1.

4.2. The uniform asymptotic expansion. In this section we derive an asymptotic expansion of $u_A(t, x)$ which is uniformly valid near the shadow-boundary. The asymptotic expansion of $u_{A-}(t, x)$ is again given by $\lambda^{-1/2} \hat{z}_- \exp(i\lambda \hat{s}_-)$, this term being finite and continuous near the shadow-boundary. The asymptotic expansion of $u_{A+}(t, x)$ as derived in Sec. 4.1 consisted of two terms due to interior stationary points of the phase and to critical points on the boundary $\xi = 0$. In the present case where the point (t, x) lies in the neighborhood of the shadow-boundary, these terms cannot be separated any more but they have to be treated simultaneously. It is allowed to replace the integral (4.19) for u_{A+} by the integral u_{A+}^* as introduced in (4.21) but now ϵ is chosen sufficiently large in order that the stationary point (4.20) lies inside the strip of integration. Expanding the inner integral of (4.21) asymptotically by the method of stationary phase we are again led to (4.24). A uniform asymptotic expansion of u_{A+}^* as represented by (4.24) is obtained by using Lewis' result as presented in the Appendix of [3] (cf. (2.20), (2.21)). After some evaluations we get

$$u_{A+}(t, x) \sim \exp(i\lambda \hat{s}_+) \left\{ \frac{\exp(-\pi i/4)}{\pi^{1/2}} z g_+ [\operatorname{sgn}(-p)(\lambda(\hat{s}_+ - s))^{1/2}] + \lambda^{-1/2} \hat{z}_+ \right\} \quad (4.26)$$

where z, s and $\hat{z}_+, \hat{s}_+$ are given by the parametric representations (3.4), (3.5), and (3.7), (3.10).

In conclusion, the uniform asymptotic expansion of $u_A(t, x)$ is in perfect agreement with the uniform asymptotic solution derived in Section 3.2.

Acknowledgement. The author wishes to express his gratitude to Professor R. M. Lewis for suggesting the problem considered in this paper and for his helpful interest in the work.

REFERENCES

1. N. Bleistein and R. M. Lewis, *Asymptotic solution of a dispersive hyperbolic equation with variable coefficients*, New York University, Courant Institute of Mathematical Sciences, Div. Electromagnetic Res., Res. Rep. No. EMP-206, 1965
2. R. M. Lewis, *Asymptotic methods for the solution of dispersive hyperbolic equations*, New York University, Courant Institute of Mathematical Sciences, Div. Electromagnetic Res., Res. Rep. No. EM-197, 1964 = *Asymptotic solutions of differential equations and their applications*, C. H. Wilcox, ed., University of Wisconsin Symposium Proceedings, pp. 53-107, Wiley, N. Y., 1964
3. R. M. Lewis, *Asymptotic theory of transients*, presented at U.R.S.I. Symposium, Delft, 1965