A DIFFERENCE METHOD FOR PLANE PROBLEMS IN DYNAMIC ELASTICITY*

BY

R. J. CLIFTON

Brown University

Abstract. The equations governing dynamic elastic deformation under conditions of plane strain are written as a system of symmetric, hyperbolic, first order, partial differential equations with constant coefficients. A system of explicit difference equations with second order accuracy are presented for the solution of mixed initial and boundary value problems for regions composed of rectangles. At interior points the difference scheme is the same as the scheme proposed by Lax and Wendroff. They established the schemes conditional stability for initial value problems. At boundary points Butler's procedure based on integration along bicharacteristics is used to derive the appropriate difference equations. The method is applied to a problem for which the exact solution is known. Numerical evidence indicates that the method is stable for a mesh ratio k/h approximately twice as large as required by the Lax-Wendroff condition. The error in total energy is $O(k^3)$ and increases linearly with increasing t .

1. Introduction. The purpose of this investigation was to develop a difference method for the solution of plane problems in dynamic elasticity. The need for a numerical method for solving such problems is dictated by the well-known difficulty of obtaining exact solutions. The usefulness of such a difference method is necessarily limited by the speed and memory of available computing machines. For the present generation of computers difference solutions of problems involving three independent variables are possible; although, for the case of the equations of elastodynamics these solutions must be limited to early times. The next generation of computers (e.g. the CDC 6600 and the IBM 360 System) will increase considerably the range of application of difference methods for the equations of elastodynamics.

The equations governing dynamic elastic deformation for conditions of plane strain are a system of hyperbolic partial differential equations with two real wave speeds. These two wave speeds, which are usually denoted by c_1 and c_2 , are respectively the wave velocities for irrotational and equivoluminal waves. The governing system of partial differential equations can be formulated in several alternative ways. These equations can be written as a pair of coupled second order equations for the displacements in the x and y directions or as a pair of uncoupled second order equations for the dilatation and the rotation. They can also be written as a system of symmetric first order equations for two velocities, the dilatation, and the rotation; or for two velocities, and three stresses. The latter formulation was chosen for this investigation because stress and velocity are the quantities of physical importance which the difference solution is ultimately to provide and because the choice of stresses and velocities as dependent variables avoids boundary conditions involving derivatives.

*Received August 31, 1965. This work is based in part on results presented in a thesis submitted to the Department of Civil Engineering, Carnegie Institute of Technology, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering.

Difference methods for first order symmetric hyperbolic systems of partial differential equations have been studied intensively in recent years. For the case of three independent variables the mesh employed is necessarily coarse so that difference equations with a high order of accuracy are required in order to obtain accurate results. In this respect second order accurate methods proposed by Butler [1] and by Lax and Wendroff [2] for the case of three independent variables have particular significance. Butler presented an explicit method, based on integration along bicharacteristics, for symmetric hyperbolic systems for which the characteristic equation can be factored into a quadratic factor and a repeated linear factor. He applied the method to the equations governing the plane unsteady flow, and three-dimensional steady supersonic flow, of an inviscid nonconducting fluid. Lax and Wendroff presented an explicit method for symmetric hyperbolic systems with constant coefficients based on a straightforward differencing of the original system of equations. They were able to find a restriction on the mesh ratio which insured the stability of their difference scheme for initial value problems.

Butler's method must be extended if it is to apply to the equations of elastodynamics since in this case the characteristic equation has two quadratic factors. When this extension (which is not difficult) is made, and proper consideration is given to the difference formulas to be used for spatial derivatives, it turns out that the resulting difference equations for the dynamic elastic equations are identical with those obtained by using the method of Lax and Wendroff. These are the difference equations used for interior points in the method presented herein. Butler's approach has the advantage that it leads in a straightforward way to the appropriate difference equations at boundary points. For this reason his approach is followed in this paper.

2. Characteristic properties of the equations of elastodynamics. The equations governing the small dynamic deformation of an isotropic linear elastic solid under conditions of plane strain can be written in the following dimensionless form

$$u_t - q_x - p_x - \tau_y = 0, \quad (1a)$$

$$v_t - p_y + q_y - \tau_x = 0, \quad (1b)$$

$$(\gamma^2/(\gamma^2 - 1))p_t - u_x - v_y = 0, \quad (1c)$$

$$\gamma^2 q_t - u_x + v_y = 0, \quad (1d)$$

$$\gamma^2 \tau_t - u_y - v_x = 0, \quad (1e)$$

where subscripts denote partial differentiation with respect to the subscript variable. u and v denote dimensionless velocities in the x and y directions respectively. The dimensionless velocities u and v , the dimensionless time t , and the dimensionless Cartesian coordinate x and y are defined in terms of their dimensional counterparts by

$$u = \hat{u}/c_1; \quad v = \hat{v}/c_1; \quad t = \hat{t}c_1/b; \quad x = \hat{x}/b; \quad y = \hat{y}/b$$

where the hat symbol denotes the corresponding dimensional quantity; c_1 is the dilatational wave velocity and b is a characteristic length for the problem being considered. The dimensionless stresses p , q , and τ are defined by

$$p = \frac{1}{2} \frac{(\sigma_{xx} + \sigma_{yy})}{\rho c_1^2}, \quad q = \frac{1}{2} \frac{(\sigma_{xx} - \sigma_{yy})}{\rho c_1^2}, \quad \tau = \frac{\sigma_{xy}}{\rho c_1^2}$$

where ρ is the density; σ_{xx} , σ_{yy} and σ_{xy} are the Cartesian stress components. In Eqs. (1)

$$\gamma = c_1/c_2$$

in which c_2 is the shear wave velocity. c_1 and c_2 are defined by

$$c_1 = [(\lambda + 2\mu)/\rho]^{1/2}; \quad c_2 = [\mu/\rho]^{1/2}$$

where λ and μ are Lamé's constants.

In matrix notation Eqs. (1) become

$$L(w) \equiv A^t w_t + A^x w_x + A^y w_y = 0 \quad (2)$$

where w denotes the vector

$$w = \begin{bmatrix} u \\ v \\ p \\ q \\ \tau \end{bmatrix}$$

and A^t , A^x , and A^y denote the following matrices.

$$A^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma^2}{(\gamma^2 - 1)} & 0 & 0 \\ 0 & 0 & 0 & \gamma^2 & 0 \\ 0 & 0 & 0 & 0 & \gamma^2 \end{bmatrix}$$

$$A^x = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$A^y = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices A^t , A^x and A^y are symmetric; A^t is symmetric positive definite. Eq. (2) is immediately recognized as a linear homogeneous symmetric hyperbolic system of first order partial differential equations with constant coefficients.

As a preliminary to the derivation of the difference equations by integration along bicharacteristics the geometry of the characteristic surfaces associates with Eq. (2) is investigated. The condition that a surface $\Phi(t, x, y) = \text{const.}$ be a characteristic surface of Eq. (2) is the condition that the determinant of the characteristic matrix A defined by

$$A = A' \Phi_t + A^x \Phi_x + A^y \Phi_y \quad (3)$$

be zero. (See e.g. [3]). The characteristic equation $\det A = 0$ for Eqs. (1) can be written as

$$[\Phi_t^2 - \Phi_x^2 - \Phi_y^2] \left[\Phi_t^2 - \frac{1}{\gamma^2} (\Phi_x^2 + \Phi_y^2) \right] \Phi_t = 0. \quad (4)$$

The factors in brackets are identical with the corresponding quantities for the two-dimensional wave equation with wave velocities ± 1 and $\pm 1/\gamma$ respectively. The wave velocities ± 1 and $\pm 1/\gamma$ correspond to the dimensional wave velocities $\pm c_1$ and $\pm c_2$ respectively. The extra factor Φ_t indicates a characteristic surface with zero velocity. It has no physical significance since it is well known that the equations of elastodynamics do not yield a zero wave velocity; it is a consequence of introducing an additional dependent variable by choosing stresses and velocities as the dependent variables. Indeed, if instead of stresses the kinematic quantities, dilatation, and rotation are used as dependent variables, then the characteristic equation is identical with Eq. (4) except that the factor Φ_t is missing so that the only wave velocities are ± 1 and $\pm 1/\gamma$.

The solutions of Eq. (4), regarded as a partial differential equation, which pass through the line

$$(y - y_0) \sin \alpha + (x - x_0) \cos \alpha = 0; \quad 0 \leq \alpha \leq 2\pi \quad (5)$$

on the plane $t = t_0$ are the characteristic planes

$$c(t - t_0) - (x - x_0) \cos \alpha - (y - y_0) \sin \alpha = 0; \quad c = 1, 1/\gamma \quad (6)$$

which correspond to the wave speeds 1 and $1/\gamma$ respectively and the plane given by Eq. (5) which corresponds to the zero wave speed. If α is regarded as a parameter then Eqs. (6) represent one parameter families of characteristic planes through the point (t_0, x_0, y_0) . The envelopes of these one parameter families of planes are also characteristic surfaces and constitute the characteristic cones

$$c^2(t - t_0)^2 = (x - x_0)^2 + (y - y_0)^2; \quad c = 1, 1/\gamma \quad (7)$$

through the point (t_0, x_0, y_0) . The two backward drawn characteristic cones are shown in Fig. 1. The lines of contact of the one parameter families of planes with their envelopes are

$$\begin{aligned} x - x_0 &= c(t - t_0) \cos \alpha; & c &= 1, 1/\gamma, \\ y - y_0 &= c(t - t_0) \sin \alpha \end{aligned} \quad (8)$$

which are the bicharacteristics for Eqs. (1). The lines of contact given by Eqs. (8) in combination with the directions of the normal to the characteristic surface which contains the bicharacteristics

$$\begin{aligned} \Phi_t &= c; & c &= 1, 1/\gamma, \\ \Phi_x &= -\cos \alpha, \\ \Phi_y &= -\sin \alpha, \end{aligned} \quad (9)$$

define the bicharacteristic strips through the point (t_0, x_0, y_0) .

If Φ is a characteristic surface of Eqs. (1) then there exist null vectors l such that

$$lA = 0 \quad (10)$$

where A is the characteristic matrix defined by Eq. (3). The null vectors l associated with the bicharacteristic strips defined by Eqs. (8) and (9) are

$$l = \begin{bmatrix} -\gamma^2 \cos \alpha \\ -\gamma^2 \sin \alpha \\ (\gamma^2 - 1) \\ \cos 2\alpha \\ \sin 2\alpha \end{bmatrix} \quad \text{for } c = 1, \quad (11a)$$

$$l = \begin{bmatrix} \gamma \sin \alpha \\ -\gamma \cos \alpha \\ 0 \\ -\sin 2\alpha \\ \cos 2\alpha \end{bmatrix} \quad \text{for } c = 1/\gamma. \quad (11b)$$

If l is the null vector for a characteristic surface Φ , then the single partial differential equation,

$$l \cdot L[w] = 0 \quad (12)$$

where the dot denotes the inner product, is an interior differential equation on the surface Φ , [3]. Since this equation involves no derivatives normal to the surface Φ it can at each point be written in terms of derivatives in two directions on the surface Φ . These two directions can be taken to be the bicharacteristic direction and the direction of the curve along which t is constant. The derivative of a function f in a bicharacteristic direction is given by

$$df/dt = f_t + f_x dx/dt + f_y dy/dt \quad (13)$$

where dx/dt and dy/dt along bicharacteristics are obtained by differentiation of Eqs. (8). When Eqs. (11) are substituted in Eq. (12) and partial derivatives with respect to t are eliminated by use of Eq. (13) then the equation $l \cdot L[w] = 0$ involves derivatives along bicharacteristics and derivatives in planes $t = \text{constant}$. This equation can be expressed as a differential relation along bicharacteristics.

Before giving the result of these operations, note that for use in a numerical method it is the differential relation along backward drawn bicharacteristics that is important. In the development so far $\alpha = 0$ corresponds to a forward drawn bicharacteristic drawn in the positive x -direction. It is convenient to let $\alpha = 0$ for backward drawn bicharacteristics drawn in the positive x -direction. This requires the replacement of α by $(\alpha + \pi)$ with a resulting change in sign for terms involving $\sin \alpha$ and $\cos \alpha$. After making this change the equation $l \cdot L[w] = 0$ can be written as

$$\cos \alpha du + \sin \alpha dv + dp + \cos 2\alpha dq + \sin 2\alpha d\tau = -S_1(\alpha) dt, \quad (14a)$$

$$-\gamma \sin \alpha du + \gamma \cos \alpha dv - \gamma^2 \sin 2\alpha dq + \gamma^2 \cos 2\alpha d\tau = -S_2(\alpha) dt \quad (14b)$$

where, for example, du denotes the incremental change on u along the bicharacteristic for an incremental change in time dt . $S_1(\alpha)$ and $S_2(\alpha)$ are given by

$$\begin{aligned} S_1(\alpha) = & (\cos 2\alpha - 1) \cos \alpha q_x + (\cos 2\alpha + 1) \sin \alpha q_y \\ & + (\sin 2\alpha \sin \alpha - \cos \alpha) \tau_y + (\sin 2\alpha \cos \alpha - \sin \alpha) \tau_z - \sin^2 \alpha u_x + (1/\gamma^2)(1 - \cos 2\alpha) u_x \\ & - \cos^2 \alpha v_y + (1/\gamma^2)(1 + \cos 2\alpha) v_y + (1/2) \sin 2\alpha (u_y + v_x) - (1/\gamma^2) \sin 2\alpha (u_y + v_x) \end{aligned}$$

and

$$\begin{aligned} S_2(\alpha) = & \gamma(\sin \alpha - \sin 2\alpha \cos \alpha) q_x + \gamma(\cos \alpha - \sin 2\alpha \sin \alpha) q_y + \gamma \sin \alpha p_x - \gamma \cos \alpha p_y \\ & + \gamma(1 + \cos 2\alpha) \sin \alpha \tau_y - \gamma(1 - \cos 2\alpha) \cos \alpha \tau_z \\ & + (1/2) \sin 2\alpha (u_x - v_y) - \cos^2 \alpha u_y + \sin^2 \alpha v_x. \end{aligned}$$

Equations (14) are the desired differential relations along bicharacteristics on irrotational and equivoluminal wave surfaces respectively. It is easily verified that the derivatives with respect to x and y appearing in S_1 and S_2 are so related that they are equivalent to a derivative in the direction of the tangent to the corresponding characteristic surface. In the next section Eqs. (14) are used as the basis of a numerical method employing differences along bicharacteristics.

3. Derivation of the difference equations. In this section difference equations are derived for computing the solution at a mesh point (t_0, x_0, y_0) (hereafter called simply 0) from known data at neighboring mesh points on the plane $t = t_0 - k$. (See Fig. 1). These equations are obtained by linear combinations of equations resulting from integration of Eqs. (14) along bicharacteristics and integration of Eqs. (1) along the line $x = x_0, y = y_0$.

Integration of Eqs. (14), along the bicharacteristic for which $\alpha = \alpha_i$, from the point 0 to the point of intersection of the bicharacteristic with the plane $t = t_0 - k$ gives

$$\begin{aligned} \cos \alpha_i \delta u + \sin \alpha_i \delta v + \delta p + \cos 2\alpha_i \delta q + \sin 2\alpha_i \delta \tau \\ = -\frac{k}{2} [(S_1(\alpha_i))_i + (S_1(\alpha_i))_0] - W_1(\alpha_i) + O(k^3), \end{aligned} \quad (15a)$$

$$\begin{aligned} -\gamma \sin \alpha_i \delta u + \gamma \cos \alpha_i \delta v - \gamma^2 \sin 2\alpha_i \delta q + \gamma^2 \cos 2\alpha_i \delta \tau \\ = -\frac{k}{2} [(S_2(\alpha_i))_i + (S_2(\alpha_i))_0] - W_2(\alpha_i) + O(k^3) \end{aligned} \quad (15b)$$

for the exterior and interior cones respectively where, for example, δu denotes the increment $u(t_0, x_0, y_0) - u(t_0 - k, x_0, y_0)$ and where

$$\begin{aligned} W_1(\alpha_i) = & \cos \alpha_i (u' - u_i) + \sin \alpha_i (v' - v_i) + (p' - p_i) \\ & + \cos 2\alpha_i (q' - q_i) + \sin 2\alpha_i (\tau' - \tau_i), \\ W_2(\alpha_i) = & -\gamma \sin \alpha_i (u' - u_i) + \gamma \cos \alpha_i (v' - v_i) \\ & - \gamma^2 \sin 2\alpha_i (q' - q_i) + \gamma^2 \cos 2\alpha_i (\tau' - \tau_i). \end{aligned}$$

The subscript i in Eqs. (15a) and (15b) denotes that the associated function is evaluated at the point where the bicharacteristic for which $\alpha = \alpha_i$ on the interior and exterior cones respectively intersects the plane $t = t_0 - k$. The subscript 0 and the prime

denote that the associated function is evaluated at the points (t_0, x_0, y_0) and $(t_0 - k, x_0, y_0)$ respectively.

Additional equations involving the increments $\delta u, \dots, \delta \tau$ can be obtained by integration of Eqs. (1) along the line $x = x_0, y = y_0$ from $t = t_0 - k$. Thus, for example, from Eq. (1a)

$$\delta u = \frac{k}{2} [(q_x + p_x + \tau_y)_0 + (q_x + p_x + \tau_y)'] + O(k^3) \quad (16)$$

where the subscript 0 and the prime have the same meaning as in Eqs. (15).

All the terms on the right hand side of Eqs. (15) and (16) can be evaluated from data on the plane $t = t_0 - k$ except the derivatives at the point 0 which appear in the terms having a subscript 0. A system of five linearly independent equations for the five unknown increments, which do not involve the unknown derivatives at 0, can be obtained by linear combinations of the five equations analogous to Eq. (16) and the eight equations obtained by applying Eqs. (15) to four bicharacteristics on both the exterior and interior ray cones. The use of equations having the form of Eq. (16), in conjunction with the equations along bicharacteristics, in order to eliminate the unknown derivatives was suggested to the author by D. S. Butler in private correspondence. The bicharacteristics corresponding to $\alpha_i = 0, \pi/2, \pi, 3\pi/2$ are used. As a convenience in referring to the resulting equations, Eqs. (15a) and (15b) with

$$\alpha_i = (i - 1)\pi/2 \quad (i = 1, 2, 3, 4)$$

are denoted by a_i and b_i respectively. Also, the equations analogous to Eq. (16) for Eqs. (1a), $\dots$, (1e) are denoted by (1a), $\dots$, (1e). The final system of equations for the increments $\delta u, \dots, \delta \tau$, for which the derivatives at 0 are eliminated, are given by Eqs. (17)

$$a_1 - a_3 - \frac{1}{\gamma} (b_2 - b_4) - 2(1a) \left| 2 \delta u, \quad (17a) \right.$$

$$a_2 - a_4 + \frac{1}{\gamma} (b_1 - b_3) - 2(1b) \left| 2 \delta v, \quad (17b) \right.$$

$$a_1 + a_2 + a_3 + a_4 - 2\left(1 - \frac{2}{\gamma^2}\right)(1c) \left| \frac{2\gamma^2}{(\gamma^2 - 1)} \delta p, \quad (17c) \right.$$

$$a_1 - a_2 + a_3 - a_4 + 2\left(1 - \frac{2}{\gamma^2}\right)(1d) \left| 2\gamma^2 \delta q, \quad (17d) \right.$$

$$b_1 - b_2 + b_3 - b_4 - 2(1e) \left| 2\gamma^2 \delta \tau. \quad (17e) \right.$$

When Eqs. (17) are expanded, the unknown terms in the resulting equations are found to be respectively the quantities to the right of the vertical lines in the above equations. Since only one of the unknown increments appears in each of Eqs. (17), the increments can be computed directly. It is interesting to note that the coefficients of the increments are simply twice the corresponding diagonal element of the matrix A' .

In order to complete the derivation of the difference equations it is necessary to show how the quantities to be evaluated on the plane $t = t_0 - k$ can be evaluated in terms of function values at mesh points. The quantities to be evaluated on the plane $t = t_0 - k$ are of the following four types

$$f(x_0 + ck, y_0) - f(x_0 - ck, y_0), \quad (18a)$$

$$f(x_0 + ck, y_0) + f(x_0 - ck, y_0) - 2f(x_0, y_0), \quad (18b)$$

$$k[f_x(x_0, y_0 + ck) - f_x(x_0, y_0 - ck)], \quad (18c)$$

$$k[f_x(x_0, y_0 + ck) + f_x(x_0, y_0 - ck) - 2f_x(x_0, y_0)] \quad (18d)$$

where $c = 1$ for the exterior cone and $c = 1/\gamma$ for the interior cone. Analogous quantities obtained by interchange of the roles of x and y in Eqs. (18) also occur. In order for the error introduced in one time step to be $O(k^3)$ the error resulting from expressing quantities, having the form shown in Eqs. (18), in terms of values at mesh points must also be $O(k^3)$. A Taylor's expansion of f about the point (x_0, y_0) shows that Eqs. (18) differ respectively from the following quantities by terms which are $O(k^3)$.

$$2ckf_x(x_0, y_0), \quad (19a)$$

$$(ck)^2 f_{xx}(x_0, y_0), \quad (19b)$$

$$2ck^2 f_{xy}(x_0, y_0) \quad (19c)$$

$$0 \quad (19d)$$

Thus the $O(k^3)$ accuracy of Eqs. (17) is retained provided that difference approximations for the first and second partial derivatives at (x_0, y_0) are used which have accuracies $O(k^2)$ and $O(k)$ respectively. For a square mesh, as shown in Fig. 1, centered differ-

FIG. 1. Characteristic cones for the dynamic elastic equations.

ences can be used. The resulting difference scheme is a 9 point scheme since the centered difference formulas for f_x , f_y , f_{xx} , f_{yy} and f_{xy} at (x_0, y_0) involve values for f at the mesh point (x_0, y_0) and the eight neighboring mesh points.

4. Treatment of boundary points. Appropriate boundary conditions for Eqs. (1) have the form

$$au_N + b\sigma_N = g_1, \quad (20a)$$

$$cu_T + d\sigma_T = g_2 \quad (20b)$$

where a , b , c , d , g_1 and g_2 are functions defined on the boundary which satisfy $ab \geq 0$ and $cd \geq 0$. Both a and b or c and d cannot vanish simultaneously since then the number of boundary conditions would be reduced. u_N and u_T are respectively the normal and tangential components of velocity at the boundary; σ_N and σ_T are the analogous components of the stress vector. u_N , u_T , σ_N and σ_T can be written as

$$u_N = u \cos \theta + v \sin \theta, \quad (21a)$$

$$u_T = v \cos \theta - u \sin \theta, \quad (21b)$$

$$\sigma_N = p + q \cos 2\theta + \tau \sin 2\theta, \quad (21c)$$

$$\sigma_T = \tau \cos 2\theta - q \sin 2\theta \quad (21d)$$

where θ is the counterclockwise angle from the x -axis to the exterior normal to the boundary at the point considered. If the boundary of the region is the curve $\Phi(x, y) = 0$ then the conditions $ab \geq 0$ and $cd \geq 0$ insure that, for the homogeneous problem with $g_1 = g_2 = 0$, the quadratic form $w \cdot Aw$ satisfies the following condition on the boundary.

$$w \cdot Aw \geq 0 \quad (22)$$

w is the solution vector; the matrix A is defined by Eq. (3). The condition given in Eq. (22) can be interpreted as a statement that for homogeneous boundary conditions the region of interest does not gain energy from its surroundings. A solution of Eqs. (1) which satisfies boundary conditions of the type given in Eqs. (20), with the previously specified restrictions on a , b , c , and d , is a unique solution of the respective mixed initial and boundary value problem for Eqs. (1). (See e.g. [3]).

Consideration is now given to the computation of the difference solution at a plane boundary parallel to the x or y axis. It is assumed that the boundary coincides with a line of mesh points of the square mesh which covers the region of interest. In order to calculate the difference solution at mesh points on the boundary the difference method used for interior points must be modified in two respects.

First, centered differences cannot be used at boundary points for approximating derivatives with respect to the coordinate normal to the boundary. In these cases backward or forward differences must be used. For example, if the region $x \geq 0$ is the region of interest the following differences can be used for approximating the derivatives in Eqs. (1) at the boundary $x = 0$.

$$2ckf_x(0, y_0) = 2c \frac{k}{h} [2f(h, y_0) - \frac{1}{2}f(2h, y_0) - \frac{3}{2}f(0, y_0)] + O(k^3), \quad (23a)$$

$$(ck)^2 f_{xx}(0, y_0) = \left(c \frac{k}{h}\right)^2 [f(0, y_0) - 2f(h, y_0) + f(2h, y_0)] + O(k^3), \quad (23b)$$

$$2ck^2 f_{xv}(0, y_0) = \left(c \frac{k}{h}\right)^2 [2f(h, y_0 + h) - \frac{1}{2}f(2h, y_0 + h) - \frac{3}{2}f(0, y_0 + h) \\ - 2f(h, y_0 - h) + \frac{1}{2}f(2h, y_0 - h) + \frac{3}{2}f(0, y_0 - h)] + O(k^3). \quad (20c)$$

Second, a choice must be made as to what conditions are to be satisfied at boundary points since it is generally impossible for the five unknown increments to satisfy five equations obtained from Eqs. (17) by using the appropriate differences for spatial derivatives and two equations obtained from the boundary conditions, Eqs. (20). Adoption of Butler's procedure for boundary points provides a straightforward solution to this difficulty. The idea is to compute the unknown increments from the two boundary conditions and the three linear combinations of Eqs. (17) for which the equations along backward drawn bicharacteristics extending outside of the region of interest are eliminated. For example, if $x \geq 0$ is the region of interest and the solution is to be computed at a point 0 on the boundary $x = 0$, then a_3 and b_3 in Eqs. (17) are equations along bicharacteristics extending outside the boundary. Elimination of a_3 and b_3 from Eqs. (17) yields the following equations

$$2a_1 + a_2 + a_4 - \frac{1}{\gamma}(b_2 - b_4) - 2(1a) - 2\left(1 - \frac{2}{\gamma^2}\right)(1c) \left| 2 \delta u + \frac{2\gamma^2}{(\gamma^2 - 1)} \delta p, \quad (24a)$$

$$2a_1 - a_2 - a_4 - \frac{1}{\gamma}(b_2 - b_4) - 2(1a) + 2\left(1 - \frac{2}{\gamma^2}\right)(1d) \left| 2 \delta u + 2\gamma^2 \delta q, \quad (24b)$$

$$a_2 - a_4 + \frac{1}{\gamma}(2b_1 - b_2 - b_4) - 2(1b) - \frac{2}{\gamma}(1e) \left| 2 \delta v + 2\gamma \delta \tau. \quad (24c)$$

The unknown terms in each equation are again indicated to the right of the vertical line. Two additional equations are obtained from the boundary conditions which result when Eqs. (21) with $\theta = \pi$ are substituted in Eq. (20). It is easy to verify that, for the restrictions previously imposed on the coefficients a , b , c , and d in Eqs. (20), Eqs. (24) and the two boundary conditions uniquely determine the solution at points on the boundary $x = 0$.

It is interesting to compare the method just described for the computation of the difference solution at boundary points with the method used by Duff [4] for constructing the solution near a boundary in his proof of the existence of solutions for linear symmetric hyperbolic systems for mixed initial and boundary value problems. Duff considered the symmetric hyperbolic system of equations written in canonical form relative to the time coordinate and the coordinate normal to the boundary. In the case of Eqs. (1) for the region $x \geq 0$ the appropriate canonical form of the equations has the form

$$U_t + DU_x + HU_v = 0 \quad (25)$$

where D is a diagonal matrix having as its diagonal elements the values $(1, 1/\gamma, 0, -1, -1/\gamma)$ and H is a symmetric matrix. The components U_k , $k = 1, \dots, 5$, of the vector U can be written in terms of the variables u , v , p , q , and τ as follows:

$$U_1 = \gamma(-u + p + q), \quad (26a)$$

$$U_2 = -v + \gamma\tau, \quad (26b)$$

$$U_3 = \frac{-\gamma}{(\gamma^2 - 1)^{1/2}} p + \gamma(\gamma^2 - 1)^{1/2} q, \quad (26c)$$

$$U_4 = \gamma(u + p + q), \quad (26d)$$

$$U_5 = v + \gamma\tau. \quad (26e)$$

The i th equation in Eq. (25) is an interior differential equation on the characteristic surface Φ^i associated with the characteristic velocity which appears as the diagonal element in the i th row of the matrix D . If the vector U is continuous across the characteristic surface Φ^i then the jump in the normal derivative of U_k across Φ^i is zero unless $i = k$. These special features of the canonical form, Eqs. (25), makes the construction of the solution near the boundary $x = 0$ more transparent.

The characteristic surfaces Φ^1 and Φ^2 which extend from the line $x = 0, t = t_0 - k$ into the first quadrant of the x - t plane are shown in Fig. 2. These are the characteristic surfaces associated with the positive roots 1, and $1/\gamma$ respectively of the characteristic equation. The boundary conditions, Eqs. (20), for the boundary $x = 0$ become the following conditions for the components of the vector U .

$$(a + b)U_1 = 2\gamma g_1 + (a - b)U_4, \quad (27a)$$

$$(c\gamma + d)U_2 = 2\gamma g_2 + (c\gamma - d)U_5. \quad (27b)$$

Following Duff [4], the solution in region D_2 of Fig. 2, for the case of zero initial data at $t = t_0 - k$, can be expanded as the following infinite series

$$U_k = \sum_{m=0}^{\infty} [U_k^{(m)}]_1 (t - x)^m + \sum_{m=0}^{\infty} [U_k^{(m)}]_2 (t - \gamma x)^m \quad (28)$$

where $[U_k^{(m)}]_i$ denotes the jump in the m th time derivative of U_k across the characteristic surface Φ^i . The terms $[U_k^{(m)}]_i$ are determined recursively. For $k \neq i$ these jumps can be computed from jumps of order $m - 1$ or less; for $k = i$ they can be obtained by solving an initial value problem on the surface Φ^i . The initial data for the initial value problem on Φ^i is obtained by differentiating the boundary condition for U_i m times with respect to t and taking jumps. It is assumed that a, b, c , and d are analytic functions of t and that g_1 and g_2 are analytic functions of t except that discontinuities in the functions and/or the derivatives can occur at $t = t_0 - k$. Duff's existence proof follows by showing that the two infinite series in Eq. (28) are convergent in some neighborhood of $x = 0, t = t_0 - k$.

FIG. 2. Characteristic surfaces through the intersection of a boundary surface and an initial surface.

Equation (28) provides valuable insight into the question of which conditions should be used for computing the difference solution at the boundary point $x = 0, t = t_0$ in Fig. 2. Observe that if g_1 and g_2 are zero for $t \leq t_0 - k$ but $\partial g_1/\partial t$ and $\partial g_2/\partial t$ are non-zero at $t = t_0 - k$ then the terms $[U_k^{(0)}]_i$ are all zero and $[U_k^{(1)}]_i$ are all zero except $[U_1^{(1)}]_1$ and $[U_2^{(1)}]_2$. The solution at $x = 0, t = t_0$ is therefore $O(k)$ for U_1 and U_2 and at most $O(k^2)$ for U_3, U_4 , and U_5 . If in a difference solution for this situation U_1 and U_2 at $x = 0, t = t_0$ were computed from the initial data by differencing the first two of Eqs. (25) and U_4 and U_5 were computed from the boundary conditions, Eqs. (27), the resulting difference solution would bear no resemblance to the exact solution. Indeed, U_1 and U_2 would be found to be zero instead of $O(k)$; U_4 and U_5 would be found to be $O(k)$ (i.e. provided $a - b \neq 0, c\gamma - d \neq 0$) instead of $O(k^2)$. The correct procedure is clearly to compute U_3, U_4 , and U_5 at $x = 0, t = t_0$ from the initial data at $t = t_0 - k$ by using difference approximations of the last three of Eqs. (25) and substitute the resulting values for U_4 and U_5 into Eqs. (27) to determine U_1 and U_2 . This procedure is analogous to the procedure used at boundaries by Courant, Isaacson and Rees [5] for the case of two independent variables. It has been used by Thomee [6] for systems involving more than two independent variables. Finally, the proposed difference method for boundary points based on Butler's idea of eliminating equations along bicharacteristics which extend outside of the region of interest also results in a procedure for which the increments $\delta U_3, \delta U_4$, and δU_5 are computed from initial data on the plane $t = t_0 - k$ and δU_1 , and δU_2 are computed from the boundary conditions, Eqs. (27). This feature of the proposed difference methods is obvious from the following equations relating the unknown quantities in Eqs. (24) to the unknown increments $\delta U_3, \delta U_4$, and δU_5 .

$$2\left(\delta u + \frac{\gamma^2}{(\gamma^2 - 1)} \delta p\right) = \frac{2}{\gamma} \left(\delta U_4 - \frac{1}{(\gamma^2 - 1)^{1/2}} \delta U_3\right), \quad (29a)$$

$$2(\delta u + \gamma^2 \delta q) = \frac{2}{\gamma} ((\gamma^2 - 1)^{1/2} \delta U_3 + \delta U_4), \quad (29b)$$

$$2(\delta v + \gamma \delta \tau) = 2 \delta U_5. \quad (29c)$$

So far, only difference equations for plane boundaries have been presented. In order for the difference method to be applicable to regions built up of rectangles it is necessary to consider what happens at corner points. If the region $x \geq 0, y \leq 1$ is the region of interest, then at the corner point $x = 0, y = 1$ a_2, a_3, b_2 , and b_3 in Eqs. (17) are equations along bicharacteristics which extend outside the boundary. Elimination of a_2, a_3, b_2 , and b_3 from Eqs. (17) yields the equation

$$a_1 + a_4 - \frac{1}{\gamma} (b_1 - b_4) - (1a) - \left(1 - \frac{2}{\gamma^2}\right)(1c) + (1b) + \frac{1}{\gamma} (1e) \left| \delta u + \frac{\gamma^2}{(\gamma^2 - 1)} \delta p - \delta v - \gamma \delta \tau \right| \quad (30)$$

for the corner point $x = 0, y = 1$. Two boundary conditions of the form given in Eqs. (20) are to be satisfied on both $x = 0$ and on $y = 1$. The solution at the corner point $x = 0, y = 1$ is uniquely determined by Eq. (30) and the four boundary conditions provided that the coefficients in the boundary conditions satisfy

$$(ac\hat{b}\hat{d} + \hat{a}\hat{c}b\hat{d}) \neq 0 \quad (31)$$

where a, b, c, d are the coefficients in Eqs. (20) for the boundary $x = 0$ and $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are the corresponding coefficients for the boundary $y = 1$. Because of the previous restrictions on admissible coefficients for Eqs. (20) the two terms in Eq. (31) are of like sign. Thus the condition of Eq. (31) is violated only if both terms vanish.

Usually the boundary conditions are conditions on either stress or velocity rather than a linear combination of stress and velocity so that only one of the coefficients in each of Eqs. (20) is nonzero. In such cases condition (31) is satisfied only when stresses are prescribed on one of the plane boundaries and velocities are prescribed on the other. When condition (31) is violated one additional equation is needed except for the case when velocities are specified on both $x = 0$, and $y = 1$. In the latter case two additional equations are needed. A straightforward method of selecting the additional equations to be used has not been found. However, it has been found that the following equation

$$a_1 - a_4 + \frac{1}{\gamma}(b_1 + b_4) - (1a) - (1b) + \left(1 - \frac{2}{\gamma^2}\right)(1d) \left| \delta u + \delta v + \gamma^2 \delta q \right. \quad (32)$$

in conjunction with Eq. (30) and three of the boundary conditions uniquely determines the solution at the corner point $x = 0, y = 1$ in all the cases for which condition (31) is violated except the case where velocities are prescribed on both $x = 0$ and $y = 1$. Eq. (32) has the property that all the unknown spatial derivatives at the point $x = 0, y = 1, t = t_0$ are eliminated except for the quantity $(u_y - v_x)$ which is the rate of rotation of the corner element. Eq. (32) has been used, with no apparent difficulty, for calculation of the case when the boundary conditions at $x = 0$ have the form $u = f(t)$, $\tau = 0$ and the boundary $y = 1$ is stress free. For these boundary conditions $(u_y - v_x)$ vanishes identically at the corner $x = 0, y = 1$. It seems reasonable to expect that Eq. (32) can be used for other situations in which $(u_y - v_x)$ at the corner, although not zero, is expected to vary slowly with increasing t . In such cases the value of $(u_y - v_x)$ at the corner point at $t = t_0$ which appears in Eq. (32) could be replaced by its value at $t = t_0 - k$.

If the region of interest is the region $-\infty < y < \infty$ for $x \leq 0$ and $y \geq 1$ for $x \geq 0$, then new considerations must be used in the formulation of difference equations for the re-entrant corner $x = 0, y = 1$. In this case it seems reasonable to treat the corner $x = 0, y = 1$ as a special case of an interior point. The only modification necessary is that centered differences cannot be used for terms having the form of Eq. (19c). The resulting solution will in general not satisfy the boundary conditions at $x = 0, y = 1$. This objection is eliminated by regarding the re-entrant corner as having a fillet of radius h so that the point $x = 0, y = 1$ is indeed an interior point. This proposed treatment of re-entrant corners has not been tested by actual calculations.

5. Remarks on stability of the difference equations. It is well known (see e.g. [7]) that stability in the L_2 norm (i.e. mean square sense) of difference methods for initial value problems for hyperbolic systems with constant coefficients is equivalent to the condition that all powers of the amplification matrix associated with the difference equations be bounded. A necessary condition for all powers of a matrix to be bounded is the condition that the spectral radius of the matrix be not greater than unity (von Neumann condition). A sufficient condition is the condition that the spectral norm of the matrix be not greater than unity. If the amplification matrix is a normal matrix then these two conditions are equivalent. However, in general, and in particular for the difference method presented in the previous section, the amplification matrix is not a

normal matrix so that the two preceding conditions are not equivalent. Kreiss [8] has proved the equivalence of four conditions which are both necessary and sufficient for all powers of a matrix to be bounded.

Because of the complexity of the difference equations derived in the previous section it has not been possible to establish stability of the difference equations for initial value problems by verifying either the previously stated sufficient condition or any of the four conditions which appear in Kreiss' theorem. In fact, satisfaction of the von Neumann necessary conditions has not even been verified. It has however been possible to verify that if the mesh ratio k/h satisfies the condition

$$k/h \leq 1/(2(2)^{1/2}c_{\max}) \quad (33)$$

where $c_{\max}$ is the maximum wave speed, then a new sufficient condition due to Lax and Wendroff is satisfied. The Lax-Wendroff stability condition is based on their result that all the powers of a matrix are bounded if the absolute values of the field of values of the matrix are not greater than unity. The absolute values of the field of values of a matrix are not greater than the spectral norm of the matrix so that the Lax-Wendroff stability condition imposes a no stronger, and perhaps weaker, restriction on the mesh ratio than is imposed by the condition that the spectral norm not exceed unity.

Verification of stability of the difference equations derived in Sec. (2) is achieved by demonstrating that these difference equations are a special case of the difference equations for symmetric hyperbolic systems with constant coefficients proposed by Lax and Wendroff [2] for which stability was proved provided that inequality (33) holds. The Lax-Wendroff difference equations for Eqs. (2) are based on a Taylor's expansion of the solution about the point (nk, x_0, y_0) . Thus, from Taylor's theorem

$$A'w^{n+1} = A'w^n + kA'w_t^n + \frac{k^2}{2} A'w_{tt}^n + O(k^3) \quad (34)$$

where $w^n = w(nk, x_0, y_0)$ and $w^{n+1} = w((n+1)k, x_0, y_0)$. From Eqs. (2)

$$A'w_t^n = -A^xw_x^n - A^yw_y^n \quad (35)$$

and

$$A'w_{tt}^n = -A^xw_{xt}^n - A^yw_{yt}^n. \quad (36)$$

Differentiation of Eq. (35) with respect to x and with respect to y and substitution in Eq. (36) gives

$$A'w_{tt}^n = A^z(A')^{-1}(A^xw_{xx} + A^yw_{xy}) + A^y(A')^{-1}(A^xw_{xy} + A^yw_{yy}). \quad (37)$$

Substitution of Eqs. (35) and (37) in Eq. (34) gives an equation for $\delta w = w^{n+1} - w^n$ which depends only on first and second spatial derivatives at the point (nk, x_0, y_0) . It is easily verified that the resulting equations are identical with Eqs. (17) once the terms in Eqs. (17) having the form given in Eq. (18) are replaced by the corresponding derivatives, Eqs. (19). Lax and Wendroff's difference equations are obtained by replacing the first and second spatial derivatives at the point (nk, x_0, y_0) by centered differences. Since the same difference approximation was used in the previous section the resulting difference equations for interior points are the same as obtained by applying Lax and Wendroff's method to Eqs. (1).

Strictly speaking, the Lax-Wendroff difference method is for systems of equations

having the form

$$\tilde{w}_t + \tilde{A}^x \tilde{w}_x + \tilde{A}^y \tilde{w}_y = 0 \quad (38)$$

instead of the form of Eqs. (2). Eqs. (2) can be written in the form of Eqs. (38) by the substitution $w = T\tilde{w}$ where $T^2 = (A')^{-1}$. The matrices $\tilde{A}^x$ and $\tilde{A}^y$ are given by $\tilde{A}^x = TA^xT$ and $\tilde{A}^y = TA^yT$. If the amplification matrices for the difference equations for Eqs. (2) and Eqs. (38) are denoted by S and $\tilde{S}$ respectively, then it is easily shown that $\tilde{S} = T^{-1}ST$. Since S and $\tilde{S}$ are similar, the stability of S and the stability of $\tilde{S}$ are equivalent.

Inequality (33) is a sufficient condition for stability of the Lax-Wendroff difference equations for initial value problems, but it is not in general a necessary condition. For example, in the case of the hyperbolic system

$$u_t = u_x + v_y, \quad (39a)$$

$$v_t = -v_x + u_y \quad (39b)$$

which corresponds to the two dimensional wave equation the condition

$$k/h \leq 2^{-1/2} \quad (40)$$

is both necessary and sufficient for stability of the Lax-Wendroff difference equations for initial value problems. Inequality (40) is readily obtained by examining the amplification matrix for the Lax-Wendroff difference equations for Eqs. (39). (The same difference equations are obtained by integration along bicharacteristics as in the previous section.) The amplification matrix, which will be denoted by $S(\theta_1, \theta_2)$, can be written as (see [2])

$$S(\theta_1, \theta_2) = R + iJ \quad (41)$$

where R and J are real symmetric matrices defined by

$$R = \left\{ 1 - \left(\frac{k}{h} \right)^2 (2 - \cos \theta_1 - \cos \theta_2) \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \frac{k}{h} \begin{bmatrix} \sin \theta_1 & \sin \theta_2 \\ \sin \theta_2 & -\sin \theta_1 \end{bmatrix}$$

in which θ_1 and θ_2 are arbitrary parameters corresponding to arbitrary wave lengths in the x and y directions respectively. If the spectral radius of $S(\theta_1, \theta_2)$ is denoted by $r_1(S)$ then

$$r_1^2(S) = \left[1 - \left(\frac{k}{h} \right)^2 (2 - \cos \theta_1 - \cos \theta_2) \right]^2 + \left(\frac{k}{h} \right)^2 (\sin^2 \theta_1 + \sin^2 \theta_2). \quad (42)$$

It is easily shown that $r_1(S)$ does not exceed unity when the mesh ratio satisfies inequality (40). Hence, the von Neumann necessary condition is satisfied. Since $RJ = JR$ the matrix $S(\theta_1, \theta_2)$ is a normal matrix so that the von Neumann condition is also a sufficient condition.

In view of the weaker restriction, Eq. (40), found to be sufficient for stability when the difference method is applied to Eqs. (39) it is not unreasonable to expect that the difference equations for Eqs. (1) are stable for larger values of the mesh ratio than satisfy inequality (33). Although the amplification matrix for the difference equations for Eqs. (1) is not simple enough to permit an analytical determination of the restriction on k/h imposed by the von Neumann necessary condition it is possible to find the

restriction imposed for certain special choices of the parameters θ_1 and θ_2 . It is helpful to note that for Eqs. (39) the strongest restriction on (k/h) which is imposed by the requirement $r_1^2(S) \leq 1$ occurs for $\theta_1 = \theta_2 = \pi$. For the special case $\theta_1 = \theta_2 = \pi$ the requirement that the spectral radius of the amplification matrix for Eqs. (1) not exceed unity yields the following restriction on the mesh ratio.

$$\left(\frac{k}{h}\right)^2 \leq \min \left\{ \frac{\gamma^2}{\gamma^2 + 1}, \frac{\gamma^2}{2(\gamma^2 - 1)} \right\}. \quad (43)$$

This condition is the strongest restriction on k/h that was obtained by choosing values of θ_1 and θ_2 for which explicit relation for the spectral radius could be obtained. The numerical example gives some indication that the condition in Eq. (43) may also be a sufficient condition for stability since for the example Eq. (43) becomes the condition $k/h \leq 0.837$ and there was no indication of instability even for $k/h = 0.8$. These mesh ratios are more than twice as large as the maximum value which satisfies inequality (33).

6. A numerical example. In order to investigate the accuracy of the difference method for Eqs. (1) the difference method was applied to a mixed initial and boundary value problem for which the exact solution is known. The problem considered is the classical Rayleigh-Lamb problem (see e.g. [9]) of an infinite train of sinusoidal waves in a plate which extends indefinitely in the x -direction and is bounded by stress-free boundaries at $y = \pm b$. The plate is regarded as constrained in the z -direction so that plane strain conditions exist. The geometry and boundary conditions are shown in Fig. 3 in terms of the dimensionless variables used in Eqs. (1).

The exact solution for velocities for symmetric modes (i.e. modes for which the velocity u is symmetric with respect to $y = 0$) having wave length Λ is (see e.g. [9 pg. 241]).

$$u = B \frac{\bar{c}}{\gamma \Lambda} \left[\xi \cos \frac{\pi}{2} \alpha y + \frac{(\beta^2 - \xi^2) \cos(\pi/2) \alpha}{2\xi \cos(\pi/2) \beta} \cos \frac{\pi}{2} \beta y \right] \cos 2\pi \left(\frac{x}{\Lambda} - \frac{t}{T} \right), \quad (44a)$$

$$v = -B \frac{\bar{c}}{\gamma \Lambda} \left[\alpha \sin \frac{\pi}{2} \alpha y - \frac{(\beta^2 - \xi^2) \cos(\pi/2) \alpha}{2\beta \cos(\pi/2) \beta} \sin \frac{\pi}{2} \beta y \right] \sin 2\pi \left(\frac{x}{\Lambda} - \frac{t}{T} \right) \quad (44b)$$

where the amplitude B is an arbitrary constant and the dimensionless phase velocity $\bar{c}$ is a solution of the frequency equation

FIG. 3. Boundary conditions for the numerical example.

$$\frac{\tan \frac{2\pi}{\Lambda} (\bar{c}^2 - 1)^{1/2}}{\tan \frac{2\pi}{\Lambda} \left(\frac{\bar{c}^2}{\gamma^2} - 1 \right)^{1/2}} + \frac{4(\bar{c}^2 - 1)^{1/2} \left(\frac{\bar{c}^2}{\gamma^2} - 1 \right)^{1/2}}{(2 - \bar{c}^2)^2} = 0. \quad (45)$$

The dimensionless phase velocity $\bar{c}$, wave numbers ξ , α , β , and period T are defined by

$$\bar{c} = c_p/c_2; \quad \xi = 4/\Lambda; \quad \alpha = \xi(\bar{c}^2/\gamma^2 - 1)^{1/2}; \quad \beta = \xi(\bar{c}^2 - 1)^{1/2}; \quad T = \gamma\Lambda/\bar{c}$$

where c_p denotes the dimensional phase velocity. The exact solution for stresses is easily obtained from Eqs. (44) by use of Eqs. (1c), (1d), and (1e).

Because the exact solution is periodic in x with wave length Λ it suffices to consider the difference solution for the rectangular region $0 \leq x \leq \Lambda$; $0 \leq y \leq 1$. This rectangular region, which is henceforth called the region R , is regarded as covered by a square mesh with mesh size h as shown in Fig. 3. Initial values for u , v , p , q , and τ are obtained for all mesh points from the exact solution evaluated at $t = 0$. The difference equations are then used to obtain a difference solution for later times.

The difference equations for mesh points on the boundary $y = 0$ are obtained from the difference equations for interior points by making use of the symmetry of u , p , q and the anti-symmetry of v , τ with respect to $y = 0$. Mesh points on the boundaries $x = 0$ and $x = \Lambda$ can be treated as interior points by using the periodicity of the solution to supply data at mesh points on the lines $x = -h$ and $x = \Lambda + h$ respectively. The difference equations for the boundary $y = 1$ are obtained by eliminating a_2 and b_2 from Eqs. (17). Thus

$$a_1 + a_3 - 2a_4 + \frac{1}{\gamma} (b_1 - b_3) - 2(1b) + 2\left(1 - \frac{2}{\gamma^2}\right)(1d) \left| 2 \delta v + 2\gamma^2 \delta q, \quad (46a) \right.$$

$$\tau_1 + a_3 + 2a_4 - \frac{1}{\gamma} (b_1 - b_3) + 2(1b) - 2\left(1 - \frac{2}{\gamma^2}\right)(1c) \left| \frac{2\gamma^2}{(\gamma^2 - 1)} \delta p - 2 \delta i \quad (46b) \right.$$

$$a_1 - a_3 - \frac{1}{\gamma} (b_1 + b_3 - 2b_4) - 2(1a) + \frac{2}{\gamma} (1e) \left| 2 \delta u - 2\gamma \delta \tau. \quad (46c) \right.$$

Equations (46) and the boundary conditions $\delta\tau = 0$ and $\delta p - \delta q = 0$ uniquely determine δu , δv , $\dots$, $\delta\tau$ at $y = 1$. The corner points of the region R pose no difficulty because they can be regarded simply as boundary points on $y = 0$ and $y = 1$ for which a periodicity condition holds.

For the exact solution the total energy (i.e. the sum of the kinetic energy and the elastic strain energy) in the region R is independent of the time t . Thus a measure of the stability and convergence of the difference method in the mean square sense can be obtained by examining the change in the total energy of the difference solution as t increases. The dimensionless total energy density W is given by

$$W = \frac{\hat{W}}{\rho c_1^2} = \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2} \frac{\gamma^2}{(\gamma^2 - 1)} p^2 + \frac{1}{2}\gamma^2 q^2 + \frac{1}{2}\gamma^2 \tau^2 \quad (47)$$

where $\hat{W}$ is the usual dimensional counterpart of W . The sum of the first two terms is the kinetic energy density; the sum of the last three terms is the elastic strain energy density. In terms of the solution vector w and the matrix A' of Eqs. (2) W can be written as

$$W = \frac{1}{2}(w \cdot A' w) \quad (48)$$

where the dot denotes the Euclidean inner product. The total energy of the difference solution for the region R is obtained by multiplying the total energy density at each mesh point by the associated area (i.e. h^2 for interior points, $\frac{1}{2}h^2$ for regular boundary points, and $\frac{1}{4}h^2$ for corner points) and taking the sum over all mesh points.

The error in the total energy of the difference solution at time t is equal to the change in the total energy of the difference solution at time t from its value at time $t = 0$ plus (or minus) the error in the total energy of the difference solution at $t = 0$ which results from replacing integration by summation over mesh points. The latter contribution to the error is not important in studying accuracy of the difference method and is henceforth disregarded. That is, the per cent change in the total energy of the difference solution at time t from the total energy of the difference solution at time $t = 0$ will be referred to as the per cent error in the total energy at time t .

The growth of the per cent error in the total energy of the difference solution is shown in Fig. 4 for various mesh sizes and mesh ratios for the case of the first symmetric mode with $\gamma^2 = 3.5$, $\Lambda = 5$. The sign of the per cent error is such that the total energy decreases monotonically with increasing t . In all cases the increase in error is essentially linear in t . For both $k/h = 1/2$ and $k/h = 1/2^{-1/2}$ halving the mesh size reduces the error by a factor of approximately eight. Thus it appears that the error in the total energy is $O(k^3)$ for all t in the range considered. Numerical results for $h = 1/8$ and k/h varying from 0.3 to 0.8 showed that the per cent error decreases with decreasing mesh ratio k/h . The per cent error for the cases $k/h = 0.3$ and $k/h = 0.8$ are shown in Fig. 4. There was no sign of instability for $h = 1/8$, $k/h = 0.8$ even after 96 time steps.

FIG. 4. Per cent error in the energy balance for the difference solution of the Rayleigh-Lamb problem. ($\gamma^2 = 3.5$, $\Lambda = 5$, $c_p/c_2 = 1.544575$)

For the numerical computations the constant B in Eqs. (44) was taken as 4. Then at $x = 0$, $t = 0$ the velocity u varied from $u = 1.9355$ at $y = 0$ to $u = 0.7930$ at $y = 1$. At $x = 1.25$, $t = 0$ the velocity v varied from $v = 0$ at $y = 0$ to $v = 1.4199$ at $y = 1$. These values are given to indicate the variation in the initial data and thereby demonstrate that the very slow growth in the per cent error in total energy is not a meaningless consequence of assuming essentially uniform initial data. Comparison of the difference solution with the known exact solution revealed that not only was the error in total energy small but also that the local error was small. For example, for $h = 1/16$, $k/h = 1/2$ the maximum error in any of the five functions at any mesh point at time $t = 3$ was approximately .002.

In order to better understand how soon an unstable system would begin giving meaningless results computations were carried out using a mesh ratio k/h for which the inequality in Eq. (43) is violated. For $\gamma^2 = 3.5$ Eq. (43) becomes the condition $k/h < 0.837$; $k/h = 0.95$ and $h = 1/8$ were chosen for the computations. The per cent errors after 16, 32, 48, and 64 time steps were respectively 0.133, 0.263, -39.1 , and -54×10^6 . The instability was obvious after 48 time steps. There was however an indication of instability earlier because the per cent error after 16 and 32 time steps were slightly less than at the corresponding times for $k/h = 0.8$. That is, in contrast with the behavior for k/h less than 0.8, the per cent error decreased with the increase in k/h from 0.8 to 0.95.

The calculations were carried out on a Bendix G-20 digital computer. The time required for 96 time steps with $h = 1/16$ was approximately one hour.

7. Concluding remarks. It is believed that the difference method presented here can be used successfully for many plane problems in dynamic elasticity. However, at the present stage of development the method is far from being completely general. For example, so far the method has been developed only for regions built up of rectangles and even for such regions there are some boundary conditions for which the treatment of corner points is unclear. The treatment of general curved boundaries remains a challenge. The solution of problems involving discontinuous initial and/or boundary data has not been attempted.

Although the difference method is derived for conditions of plane strain it is also applicable to the case of plane stress. The plane stress case is obtained by replacing the dilatational wave velocity c_1 throughout by the velocity of plane longitudinal waves c_L given by

$$c_L = [2\mu/\rho(1 - \nu)]^{1/2} \quad (49)$$

where ν is Poisson's ratio.

In the course of this investigation it has become evident that a theoretical analysis of stability, convergence, and error estimates for mixed initial and boundary value problems for equations as complex as the equations of elastodynamics presents major, if not insuperable, difficulties. The initial value problem does lend itself to theoretical analysis. However, even in this case it is difficult to obtain sharp results. For example, for general symmetric hyperbolic systems with constant coefficients Lax and Wendroff found that their difference method was stable for $k/h \leq 1/(2(2)^{1/2}c_{\max})$. That this condition is not in general necessary has been demonstrated by pointing out that, for the special case of a system which corresponds to the two dimensional wave equation, the condition $k/h \leq 1/2^{1/2}$ is both necessary and sufficient. This sharp result for a

special case does not indicate a weakness in the Lax-Wendroff proof, but rather illustrates the necessity of taking into account the special features of the system considered. It is precisely at this point that the major difficulties emerge for the case of the dynamic elastic equations. In this case the amplification matrix is a 5×5 matrix depending on two parameters (i.e. wave numbers in the x and y directions) in addition to the mesh ratio. As stated before none of the necessary and sufficient conditions for all powers of the amplification matrix to be bounded seem to be simple enough to yield an explicit restriction on the mesh ratio. For problems involving boundary conditions a theoretical investigation of the question of stability is even more difficult.

In view of the difficulty involved in a theoretical examination of a difference method for the equations of elastodynamics, actual numerical calculations must play an important role in the evaluation of the method. The numerical example presented here permits a comparison of the exact solution and the difference solution. This example has given an indication of the stability and rate of convergence of the proposed difference method as well as the magnitude and rate of growth of the error. It is hoped that when new difference methods for the equations of elastodynamics are proposed that numerical results for the same example will be obtained so that a meaningful comparison of the various difference methods can be made.

Acknowledgment. It is a pleasure to acknowledge the assistance of W. Recker, a graduate student in Civil Engineering at Carnegie Tech, in programming the difference method and carrying out the calculations for the numerical example. The author is grateful for the guidance provided by his thesis advisor, Professor T. Au.

While at Carnegie Tech the author was supported in the early stages of this investigation by an NDEA Fellowship and in the final six months by an NSF Contract.

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