BUCKLING UNDER AXIAL COMPRESSION OF LONG CYLINDRICAL SHELLS WITH RANDOM AXISYMMETRIC IMPERFECTIONS¹

BY
JOHN C. AMAZIGO
Harvard University

Abstract. The buckling of long cylinders with homogeneous random axisymmetric geometric imperfections under uniform axial compression is studied by means of a modified truncated hierarchy technique. It is found that the buckling load of the cylinder depends only on the spectral density of the random imperfections. In particular, for small values of the standard deviation of the axisymmetric imperfection the buckling load depends only on the value of the spectral density at a specific wave number.

1. Introduction. It is generally known that the buckling strengths of some structures are greatly reduced by the presence of small imperfections. However, the number of quantitative studies is limited. Furthermore, although it is recognized that these imperfections—geometric or otherwise—are generally random, very few investigators have used a statistical approach to the problem. Two notable exceptions are the works of Bolotin [1] and Fraser [2].

Bolotin assumes that the buckling load λ^* of structures can be determined as a function of a finite number of random variables $\bar{w}_i$ that characterize the initial imperfection in structures:

$$\lambda^* = \lambda^*(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n). \quad (1.1)$$

Then it becomes routine to calculate the probability density of λ^* if a joint probability density is assigned to $\bar{w}_i$. In general, the analytic determination of the relation (1.1) is difficult if not impossible.

Fraser calculates the buckling load of infinitely long imperfect beams on nonlinear foundations by assuming that the imperfections are homogeneous random functions and hence are characterized by their mean and autocorrelation. By means of the method of equivalent linearization he obtains the dependence of the buckling load on the spectral density of the imperfection.

Like Fraser we assume that the cylinder imperfections are homogeneous random functions of the axial coordinate and hence are characterized by their mean and autocorrelation. The buckling of infinitely long imperfect cylindrical shells under axial compression is analyzed on the basis of a modified truncated hierarchy method.

A model problem (the buckling of axially compressed infinite beams on linear stochastic foundations) is used to assess the validity of this technique. This problem is governed by a differential equation which is similar to the Kármán–Donnell equations for cylinders when the latter equations have been linearized for investigating buckling phenomenon.

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In order to demonstrate the features of this technique the model problem is treated in detail in Secs. 2 and 3 of this report. The rest of the report is devoted to the discussion of the main problem; namely, the buckling of axially compressed circular cylindrical shells with random axisymmetric imperfections.

2. Infinite beam on random foundation. Consider a beam loaded at the ends with axial force P , and resting on a foundation of linear springs with variable stiffnesses (Fig. 1). The springs exert a restoring force fu per unit length on the beam, where f is the stochastic stiffness which is assumed homogeneous, and u is the normal deflection from the unloaded position. The differential equation which it must satisfy is

$$EI \frac{d^4 u}{dx^4} + P \frac{d^2 u}{dx^2} + f(x)u = 0 \quad (2.1)$$

where x is the axial coordinate, E is Young's modulus, and I is beam moment of inertia. Let the mean stiffness of the elastic foundation be K , that is

$$\langle f(x) \rangle = K \quad (2.2)$$

where the angular bracket is the average over the length of the beam, namely

$$\langle \cdots \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cdots dx. \quad (2.3)$$

The random stiffness $f(x)$ can then be written as

$$f(x) = K + \bar{g}(x) \quad (2.4)$$

where

$$\langle \bar{g}(x) \rangle = 0. \quad (2.5)$$

2.1 The classical buckling state. It is convenient to introduce the following non-dimensional quantities

$$y = (K/EI)^{1/4} x, \quad \alpha = \frac{P}{2(EIK)^{1/2}}, \quad g = \frac{\bar{g}}{K}. \quad (2.6)$$

Equation (2.1) then becomes

$$\frac{d^4 u}{dy^4} + 2\alpha \frac{d^2 u}{dy^2} + u = -gu. \quad (2.7)$$

The beam is assumed sufficiently long and with sufficiently "strong" end conditions

FIG. 1. Beam geometry.

to justify neglecting the effect of the boundary conditions. We simply demand that u and du/dy be bounded at infinity:

$$|u| < \infty, \quad \left| \frac{du}{dy} \right| < \infty \quad \text{for } |y| \rightarrow \infty. \quad (2.8)$$

Based on the nondimensionalization (2.6), the eigenfunctions and eigenvalues for the homogeneous equation

$$d^4u/dy^4 + 2\alpha d^2u/dy^2 + u = 0 \quad (2.9)$$

are respectively

$$u = \sin \omega y \quad \text{and} \quad \alpha = \frac{1}{2}(\omega^2 + 1/\omega^2).$$

The classical buckling load α_c , obtained by minimizing α with respect to ω , is

$$\alpha_c = 1 \quad (2.10)$$

and the corresponding unique buckling mode $\sin y$, hereafter called the classical buckling mode, is given by the wave number

$$\omega_c = 1. \quad (2.11)$$

2.2 Hierarchy equations for the infinite beam. Hierarchy equation techniques have been used widely in theoretical physics particularly in random wave propagation problems [3], [4], in random initial value problems [5],² and more recently in an eigenvalue problem [6]. The method consists, in general, of multiplying the stochastic differential equation an arbitrary number of times by the stochastic function or by the dependent variable evaluated at different values of its argument. The system or hierarchy of equations is then averaged. The hierarchy of equations may be truncated at some finite (preferably low) order and rendered determinate by suitably approximating the highest order averages by products of lower order averages. This will become apparent in the subsequent analysis.

Equation (2.7) can be written for the point $y + \zeta$ as

$$u''''(y + \zeta) + 2\alpha u''(y + \zeta) + u(y + \zeta) = -g(y + \zeta)u(y + \zeta) \quad (2.12)$$

where prime denotes differentiation with respect to the arguments. The autocorrelations $R_u(\eta)$ and $R_g(\eta)$ of $u(y)$ and $g(y)$ which are assumed homogeneous are defined respectively by

$$R_u(\eta) = \langle u(y + \eta)u(y) \rangle, \quad R_g(\eta) = \langle g(y + \eta)g(y) \rangle$$

where the averages are taken over y .

Now

$$u'(y + \zeta)u(y) = \frac{du(y + \zeta)}{d\zeta} u(y) = \frac{d}{d\zeta} [u(y + \zeta)u(y)];$$

hence by interchanging the orders of averaging over y and differentiation it follows that

$$\langle u'(y + \zeta)u(y) \rangle = \frac{d}{d\zeta} \langle u(y + \zeta)u(y) \rangle. \quad (2.13)$$

Results similar to (2.13) will be used in subsequent derivations.

²Richardson [5] discusses various related truncated hierarchy methods and cites references to other applications of these techniques.

Thus multiplying Eq. (2.12) by $u(y)$ and averaging over y gives

$$R_u''''(\zeta) + 2\alpha R_u''(\zeta) + R_u(\zeta) = -R_1(\zeta, \zeta) \quad (2.14)$$

where prime now denotes differentiation with respect to ζ , and R_1 is defined by

$$R_1(\zeta, \mu) \equiv \langle g(y + \mu)u(y + \zeta)u(y) \rangle. \quad (2.15)$$

We now have one equation (2.14) for the two variables R_u and R_1 . For closure, that is in order to render the system determinate, it is necessary to approximate R_1 by known quantities.

There are various techniques for obtaining this approximation. However, in this paper only the correlation discard method will be used. The correlation discard method is *equivalent* to assuming only in the highest order average that the stochastic coefficient g and the dependent variable u are statistically independent. Making this approximation in R_1 leads to

$$R_1(\zeta, \zeta) \cong \langle g(y + \zeta) \rangle \langle u(y + \zeta)u(y) \rangle. \quad (2.16)$$

Since $\langle g(y) \rangle = 0$, then $R_1(\zeta, \zeta) = 0$. Making this approximation in (2.14) gives an equation for R_u that is similar to the homogeneous equation (2.9) for u and will lead to the classical buckling load $\alpha_c = 1$. The approximation (2.16) will therefore not be made.

Consider the next higher order equation which is obtained by multiplying (2.12) by $u(y)g(y + \mu)$ and averaging over y :

$$R_1''''(\zeta, \mu) + 2\alpha R_1''(\zeta, \mu) + R_1(\zeta, \mu) = -\langle g(y + \zeta)g(y + \mu)u(y + \zeta)u(y) \rangle. \quad (2.17)$$

Equations (2.14) and (2.17) are coupled equations for R_u and R_1 but the right-hand side of (2.17) is unknown. In order to close the hierarchy it is necessary to make the correlation discard approximation

$$\langle g(y + \zeta)g(y + \mu)u(y + \zeta)u(y) \rangle \cong \langle g(y + \zeta)g(y + \mu) \rangle \langle u(y + \zeta)u(y) \rangle \quad (2.18)$$

where, as usual, the averages are taken over y . Note that the right-hand side of (2.18) is $R_u(\zeta)R_g(\zeta - \mu)$. Since this is the lowest order truncation and closure that does not yield a zero average it is convenient to call it the first order hierarchy approximation. As will be shown in Sec. 3, this level of approximation gives sufficiently good results.

Equation (2.17) then becomes

$$R_1''''(\zeta, \mu) + 2\alpha R_1''(\zeta, \mu) + R_1(\zeta, \mu) = -R_g(\zeta - \mu)R_u(\zeta). \quad (2.19)$$

The solutions to Eqs. (2.14) and (2.19) can be written in integral form as

$$R_u(\zeta) = \int_{-\infty}^{\infty} G(\zeta - \tau)R_1(\tau, \tau) d\tau \quad (2.20)$$

$$R_1(\zeta, \mu) = \int_{-\infty}^{\infty} G(\zeta - \xi)R_g(\xi - \mu)R_u(\xi) d\xi \quad (2.21)$$

where G is the Green's function which satisfies the differential equation

$$G''''(y) + 2\alpha G''(y) + G(y) = -\delta(y)$$

subject to the conditions that $G(y)$ and $G'(y)$ be bounded for $|y| = \infty$. Substituting for $R_1(\tau, \tau)$ in Eq. (2.20) leads to the equation

$$R_u(\zeta) = \int_{-\infty}^{\infty} d\tau G(\zeta - \tau) \int_{-\infty}^{\infty} d\xi G(\tau - \xi) R_g(\xi - \tau) R_u(\xi). \quad (2.22)$$

Taking an inverse Fourier transform reduces Eq. (2.22) to

$$\left[Q(\rho) \int_{-\infty}^{\infty} S_g(\mu) Q(\mu - \rho) d\mu - 1 \right] S_u(\rho) = 0 \quad (2.23)$$

where $Q(k)$, the Fourier transform of $G(x)$, is

$$Q(k) \equiv \int_{-\infty}^{\infty} G(x) e^{ikx} dx = -\frac{1}{k^4 - 2\alpha k^2 + 1} \quad (2.24)$$

and S_g , S_u are the power spectral densities of the foundation stiffness g and of the normal displacement u respectively. The spectral density is defined by the inverse Fourier transform of the autocorrelation:

$$\left[\begin{matrix} S_g(\rho) \\ S_u(\rho) \end{matrix} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\begin{matrix} R_g(\eta) \\ R_u(\eta) \end{matrix} \right] e^{-i\rho\eta} d\eta. \quad (2.25)$$

For nontrivial solution of Eq. (2.23), the spectrum S_u consists of delta functions. Let ω represent the value of ρ for which the spectrum S_u is nonzero; then, since S_u must be an even function,

$$S_u(\rho) = A^2 [\delta(\rho - \omega) + \delta(\rho + \omega)], \quad (2.26)$$

where A is an arbitrary constant. The corresponding autocorrelation is $R_u(\eta) = 2A^2 \cos \omega\eta$. Thus, based on the first order hierarchy approximation, the beam buckles into one sinusoidal mode shape $u(y) = 2A \sin(\omega y + \theta)$ where θ is an arbitrary phase.

For $\rho = \omega$ the quantity in the square bracket in Eq. (2.23) is zero, thus

$$Q(\omega) \int_{-\infty}^{\infty} S_g(\mu) Q(\mu - \omega) d\mu = 1. \quad (2.27)$$

The left-hand side of Eq. (2.27) is a function of the load α , the wave number ω , and a functional of the foundation spectrum S_g . Symbolically, Eq. (2.27) can be written as

$$\Psi(\alpha, \omega; S_g) = 0. \quad (2.28)$$

For a given foundation spectrum Eq. (2.28) defines an implicit relationship between α and ω . The buckling load α^* is obtained by minimizing α with respect to ω . Setting $d\alpha/d\omega = 0$ leads to an additional functional equation

$$\chi(\alpha^*, \omega^*; S_g) = 0. \quad (2.29)$$

At the buckling state Eq. (2.28) is

$$\Psi(\alpha^*, \omega^*; S_g) = 0. \quad (2.30)$$

Equations (2.29) and (2.30) are then solved for α^* .

The buckling load α^* does not depend on the particular realization g but on the spectral density S_g of the random foundation. Thus each member of an ensemble of infinite beams on random foundations described by R_g will have the same buckling load α^* . This buckling load is given by the solution of Eqs. (2.29) and (2.30).

The next three sections will be devoted to the solution of Eqs. (2.29) and (2.30).

An asymptotic expression is sought for α^* for arbitrary spectrum. This is followed by solutions for white noise foundation and for foundation with exponential-cosine auto-correlation.

2.3 *Asymptotic expression for the buckling load α^* for arbitrary foundation spectrum.* The buckling load α^* is determined by minimizing α with respect to ω in Eq. (2.27) which is repeated here in a more convenient form:

$$\omega^4 - 2\alpha\omega^2 + 1 = \Delta^2 \int_{-\infty}^{\infty} \frac{\bar{S}_o(\mu - \omega) d\mu}{\mu^4 - 2\alpha\mu^2 + 1} \quad (2.31)$$

where $\Delta \equiv [R_o(0)]^{1/2}$ is the standard deviation of the foundation stiffness, and

$$S_o(\rho) = \Delta^2 \bar{S}_o(\rho).$$

Setting $d\alpha/d\omega = 0$ in (2.31) leads to

$$4\omega(\omega^2 - \alpha) = -\Delta^2 \int_{-\infty}^{\infty} \frac{\bar{S}'_o(\mu - \omega) d\mu}{\mu^4 - 2\alpha\mu^2 + 1}. \quad (2.32)$$

The equations (2.31) and (2.32) will be solved asymptotically for α^* and ω^* . An asymptotic solution is sought in the form

$$\begin{aligned} \omega^* &= \omega = 1 + \omega_1 \Delta^k + \dots \\ \alpha^* &= \alpha = 1 + \alpha_1 \Delta^k + \dots \end{aligned} \quad (2.33)$$

Substituting the expansions (2.33) into the left-hand side of Eqs. (2.31) and (2.32) leads to

$$\omega^4 - 2\alpha\omega^2 + 1 = -2\alpha_1 \Delta^k + O(\Delta^{2k}) \quad (2.34)$$

and

$$4\omega(\omega^2 - \alpha) = 4(2\omega_1 - \alpha_1)\Delta^k + O(\Delta^{2k}).$$

The integrals in Eqs. (2.31) and (2.32) are evaluated asymptotically in Appendix A. On the basis of the results (A.4) and (A.5) the equations (2.31) and (2.32) simplify respectively to

$$2\alpha_1 \Delta^k = -\frac{\pi}{2(-2\alpha_1)^{1/2}} [\bar{S}_o(0) + \bar{S}_o(2)] \Delta^{2-k/2} \quad (2.35)$$

and

$$4(2\omega_1 - \alpha_1)\Delta^k = \frac{\pi}{2(-2\alpha_1)^{1/2}} [\bar{S}'_o(0) + \bar{S}'_o(2)] \Delta^{2-k/2}$$

where only the lowest order terms are retained, and the fact that the spectral density $\bar{S}_o$ is an even function has been used. Thus, demanding that the lowest order terms on either side of (2.35) be of the same order and solving the resulting equations gives $k = \frac{4}{3}$ along with the values of α_1 and ω_1 . Equations (2.33) then become

$$\alpha^* = 1 - \frac{1}{2}(\pi/2)^{2/3} [\bar{S}_o(0) + \bar{S}_o(2)]^{2/3} \Delta^{4/3} \quad (2.36)$$

and

$$\omega^* = 1 - \frac{1}{4} \left(\frac{\pi}{2} \right)^{2/3} \frac{\bar{S}_o(0) + \bar{S}_o(2) - \bar{S}'_o(0) - \bar{S}'_o(2)}{[S_o(0) + S_o(2)]^{1/3}} \Delta^{4/3}. \quad (2.37)$$

The buckling load is asymptotically given by

$$\alpha^* = 1 - \frac{1}{2}(\pi/2)^{2/3}[S_e(0) + S_e(2)]^{2/3}. \quad (2.38)$$

Thus the buckling strength of an ensemble of infinite beams on infinite stochastic foundation is uniquely determined by the values of the spectrum at two wave numbers if the standard deviation of the foundation is small compared to the mean. This result is valid for continuous spectral densities that do not vanish at both $\rho = 0$ and $\rho = 2$ and that have no delta functions.

2.4 Evaluation of α^ for specific autocorrelation.* The buckling load α^* will now be evaluated for two special autocorrelation functions; namely, delta function autocorrelation (white noise) and exponential-cosine autocorrelation.

2.4.1 White noise foundation. The buckling load of a beam on a purely random (white noise) foundation will be calculated because of the simplicity of the corresponding equations and as a check for the asymptotic formula of the last section.

For white noise foundation the spectrum can be written as

$$S_e(\rho) = (1/2\pi)A_0^2$$

where A_0 is a constant. Equation (2.27) then reduces to

$$\omega^4 - 2\alpha\omega^2 + 1 = A_0^2/4(\frac{1}{2}(1 - \alpha))^{1/2}$$

which leads to a buckling load α^* governed by the equation

$$(1 + \alpha^*)(1 - \alpha^*)^{3/2} = A_0^2 2^{1/2}/4. \quad (2.39)$$

For small values of A_0 , a series expansion of α^* is easily obtained:

$$\alpha^* = 1 - \frac{1}{2^{5/3}} A_0^{4/3} - \frac{1}{3 \cdot 2^{10/3}} A_0^{8/3} \dots \quad (2.40)$$

As expected, this is in agreement with Eq. (2.38) with $A_0 = \Delta$.

2.4.2 Exponential-cosine autocorrelation. In Sec. 2.3 a simple expression was derived for the buckling strength of a beam when the standard deviation of the elastic foundation is small compared to its mean. It should prove useful to determine not only the range of validity of this formula but also the dependence of the buckling strength upon other parameters for a specific autocorrelation.

With this in mind, the autocorrelation

$$R_e(\eta) = \Delta^2 e^{-\beta|\eta|} \cos \gamma\eta \quad (2.41)$$

is chosen. This autocorrelation occurs frequently in random noise and communication theory (for example see Bendat [7]) and was used in a buckling problem by Fraser [2]. The significant properties of this autocorrelation can best be seen by examining its spectral density

$$S_e(\rho) = \frac{\Delta^2 \beta(\rho^2 + \beta^2 + \gamma^2)}{\pi[\rho^4 + 2(\beta^2 - \gamma^2)\rho^2 + (\beta^2 + \gamma^2)^2]}. \quad (2.42)$$

Two distinct ranges of parameters β and γ arise depending on whether $S_e(\rho)$ has one maximum at $\rho = 0$ or two symmetric maxima at $\rho = \pm \bar{\rho}$, ($\bar{\rho} > 0$).

Range I. If $\beta^2 > 3\gamma^2$, $S_e(\rho)$ has a maximum at the origin $\rho = 0$ and decreases monotonically for $|\rho| > 0$. At the origin

$$S_o(0) = \Delta^2 \beta / \pi (\beta^2 + \gamma^2). \quad (2.43)$$

Range II. If $\beta^2 < 3\gamma^2$, $S_o(\rho)$ has two symmetric maxima at $\rho = \pm \bar{\rho}$, where

$$\bar{\rho} = (\beta^2 + \gamma^2)^{1/4} \{2\gamma - (\beta^2 + \gamma^2)^{1/2}\}^{1/2} \quad (2.44)$$

and the value of the maxima is

$$S_o(\bar{\rho}) = \Delta^2 \beta / 4\pi \gamma \{(\beta^2 + \gamma^2)^{1/2} - \gamma\}. \quad (2.45)$$

It is noted that for $\beta \ll 1$ and $\beta \ll \gamma$ the spectrum is strongly peaked at its maxima. Typical graphs of $\bar{S}_o(\rho) = S_o(\rho)/\Delta^2$ versus ρ in the two ranges of values of β and γ are plotted in Fig. 2.

Substituting the spectral density (2.42) and the transform (2.24) into Eq. (2.27) and integrating gives, with $a = (\frac{1}{2}(1 + \alpha))^{1/2}$, $b = (\frac{1}{2}(1 - \alpha))^{1/2}$

$$\begin{aligned} \omega^4 - 2\alpha\omega^2 + 1 = \frac{\Delta^2}{8ab} & \left\{ \frac{b(\omega - \gamma + a) + a(b + \beta)}{(\omega - \gamma + a)^2 + (b + \beta)^2} + \frac{b(\omega + \gamma + a) + a(b + \beta)}{(\omega + \gamma + a)^2 + (b + \beta)^2} \right. \\ & \left. + \frac{b(\omega - \gamma - a) - a(b + \beta)}{(\omega - \gamma - a)^2 + (b + \beta)^2} + \frac{b(\omega + \gamma - a) - a(b + \beta)}{(\omega + \gamma - a)^2 + (b + \beta)^2} \right\}. \quad (2.46) \end{aligned}$$

FIG. 2. Typical spectral density curves.

FIG. 3. Infinite beam: buckling loads for range I.

This algebraic equation gives the load parameter α as a function of ω . The buckling load α^* is determined by minimizing α with respect to ω . Setting $d\alpha/d\omega = 0$ yields a second algebraic equation. This equation and (2.46) were solved numerically for the unique buckling load α^* and the wave number ω^* by means of the standard Newton's root-finding technique.

2.5 Discussion of beam results. For Range I, the buckling load is computed for fixed values of β as a function of the mean square Δ^2 of the foundation stiffness for various values of γ . This buckling load is plotted for typical values of β and γ ($\beta = 0.5$, $\gamma = 0.2, 2$) in Fig. 3 along with the asymptotic solution given by (2.38). It may be noted that for this range the asymptotic formula gives an excellent result for variances of the same order as the mean (namely $\Delta^2 = O(1)$).

For Range II, the spectrum has two symmetric maxima at $\bar{p}$ and the asymptotic formula is valid for Δ^2 of order unity as shown in Fig. 4. For $\bar{p} = 4.0$ and $\bar{p} = 10$, the asymptotic formula gives the same result as obtained by solving (2.46) exactly for α^* . The Δ^2 range of validity of the asymptotic formula is reduced for Range II if the spectrum is strongly peaked at $\bar{p} < 2$. As already noted, the spectrum is strongly peaked if $\beta \ll \gamma$ and $\beta \ll 1$.

Figure 5 shows the variation of the buckling strength with β and γ . The buckling load is substantially reduced for small β and for values of γ near zero and two. These correspond to deviations of the foundation stiffness from the mean that are nearly sinusoidal of long wavelengths and wavelengths that are half the classical buckling wavelength. This result, which is predicted by the asymptotic formula, is expected: considering Eq. (2.7) we note that if $g = \cos 2\gamma$ then the classical buckling mode $\cos \gamma$ is one of the terms in the Fourier series solution of Eq. (2.7). For periodic deviations

FIG. 4. Infinite beam: buckling loads for Range II.

of long wavelength ($g = \cos \alpha y$ where $\alpha \ll 1$) the classical buckling mode approximately satisfies Eq. (2.7).

The rest of the discussion of the beam problem will be devoted to an assessment of the truncated hierarchy method. The infinite beam problem will be solved for the case of sinusoidal g and this will be compared to an independent perturbation solution.

FIG. 5. Infinite beam: dependence of buckling strength on autocorrelation parameters.

3. Assessment of the method: Beam on periodic foundation. The truncated hierarchy method will be used to solve the deterministic problem of an infinite beam for which g is sinusoidal. In particular we choose

$$g = \epsilon \cos 2y. \quad (3.1)$$

A second order hierarchy solution will also be derived for comparison with the first order hierarchy solution and an independently obtained perturbation solution. This will constitute a check of the truncation and closure approximations.

3.1 First order hierarchy solution. The spectrum of the sinusoidal foundation (3.1) is

$$S_o(\rho) = \frac{1}{4}\epsilon^2[\delta(\rho - 2) + \delta(\rho + 2)].$$

Substituting this into Eq. (2.27) and simplifying the results leads to

$$\omega^4 - 2\alpha\omega^2 + 1 - \frac{1}{4}\epsilon^2 \left\{ \frac{1}{(\omega - 2)^4 - 2\alpha(\omega - 2)^2 + 1} + \frac{1}{(\omega + 2)^4 - 2\alpha(\omega + 2)^2 + 1} \right\} = 0 \quad (3.2)$$

which corresponds to Eq. (2.28).

The buckling load α^* is obtained by minimizing α with respect to ω . Setting $d\alpha/d\omega = 0$ gives

$$4\omega^3 - 4\alpha\omega + \epsilon^2 \left\{ \frac{(\omega - 2)^3 - \alpha(\omega - 2)}{[(\omega - 2)^4 - 2\alpha(\omega - 2)^2 + 1]^2} + \frac{(\omega + 2)^3 - \alpha(\omega + 2)}{[(\omega + 2)^4 - 2\alpha(\omega + 2)^2 + 1]^2} \right\} = 0. \quad (3.3)$$

For small ϵ , we seek solutions to Eqs. (3.2) and (3.3) in the form

$$\begin{aligned} \alpha &= \alpha^* = 1 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \dots, \\ \omega &= \omega^* = 1 + \omega_1\epsilon + \dots \end{aligned} \quad (3.4)$$

Note that by Eqs. (2.10) and (2.11) $\alpha^* = \omega^* = 1$ for $\epsilon = 0$. Substituting these forms for α^* and ω^* into Eqs. (3.2) and (3.3) and equating coefficients of like powers of ϵ leads to

$$\alpha_1 = \pm \frac{1}{4}, \quad \alpha_2 = -\frac{1}{1024}, \quad \omega_1 = 0.$$

Of the two possible loads given by $\alpha_1 = \pm \frac{1}{4}$, $\alpha_1 = -\frac{1}{4}$ yields the lower load, thus

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{1}{1024}\epsilon^2 - \dots \quad (3.5)$$

3.2 Second order hierarchy solution. The second order hierarchy equations are obtained in exactly the same way as the first order hierarchy equation. The only modification is that the approximation (2.18) is not made. Two additional equations are obtained by multiplying (2.12) by $u(y)g(y + \mu)g(y + \tau)$ and $u(y)g(y + \mu)g(y + \tau)g(y + \eta)$. Closure is again achieved by an approximation in the last equation.

Define the following functions

$$\begin{aligned} R_2(\zeta, \tau, \mu) &\equiv \langle g(y + \mu)g(y + \tau)u(y + \zeta)u(y) \rangle, \\ R_3(\zeta, \eta, \mu, \tau) &\equiv \langle g(y + \eta)g(y + \mu)g(y + \tau)u(y + \zeta)u(y) \rangle. \end{aligned} \quad (3.6)$$

The correlation discard approximation gives

$$\begin{aligned} \langle g(y + \eta)g(y + \mu)g(y + \tau)g(y + \zeta)u(y + \zeta)u(y) \rangle \\ \cong \langle g(y + \eta)g(y + \mu)g(y + \tau)g(y + \zeta) \rangle \cdot \langle u(y + \zeta)u(y) \rangle. \end{aligned} \quad (3.7)$$

Multiplying Eq. (2.12) by the factors $u(y)$, $u(y)g(y + \mu)$, $u(y)g(y + \mu)g(y + \tau)$, and $u(y)g(y + \mu)g(y + \tau)g(y + \eta)$, and then averaging over y yields

$$\begin{aligned} R_u''''(\zeta) + 2\alpha R_u''(\zeta) + R_u(\zeta) &= -R_1(\zeta, \zeta), \\ R_1''''(\zeta, \mu) + 2\alpha R_1''(\zeta, \mu) + R_u(\zeta, \mu) &= -R_2(\zeta, \zeta, \mu), \\ R_2''''(\zeta, \tau, \mu) + 2\alpha R_2''(\zeta, \tau, \mu) + R_2(\zeta, \tau, \mu) &= -R_3(\zeta, \zeta, \mu, \tau), \\ R_3''''(\zeta, \eta, \mu, \tau) + 2\alpha R_3''(\zeta, \eta, \mu, \tau) + R_3(\zeta, \eta, \mu, \tau) \\ &= -\langle g(y + \eta)g(y + \mu)g(y + \tau)g(y + \zeta) \rangle R_u(\zeta) \end{aligned} \quad (3.8)$$

where as before prime denotes differentiation with respect to ζ , and R_1 is given by Eq. (2.15). Note that η , μ , and τ appear only as parameters in the equations (3.8).

The equations (3.8) can easily be solved if the four-point correlation of g in the last of Eqs. (3.8) can be expressed as a function of lower order averages. However, as mentioned previously, the second order solution is sought only for sinusoidal deviation. For sinusoidal g given by Eq. (3.1) this average is

$$\begin{aligned} \langle g(y + \eta)g(y + \mu)g(y + \tau)g(y + \zeta) \rangle \\ = \frac{\epsilon^4}{4} \cos 2(\zeta - \mu) \cos 2(\tau - \eta) + \frac{\epsilon^4}{8} \cos 2[\zeta + \mu - \eta - \tau]. \end{aligned} \quad (3.9)$$

The differential operators in the equations (3.8) are identical so the solutions of (3.8) can be written in integral form using the same Green's function G as in Eq. (2.20). By eliminating R_1 , R_2 , and R_3 from these integral solutions it can be shown (details omitted) that the Fourier transform S_u of R_u satisfies the equation

$$\begin{aligned} \left[\frac{\epsilon^2}{16} Q(\rho) \{ Q(\rho) [Q(\rho - 2) + Q(\rho + 2)]^2 \right. \\ \left. + Q^2(\rho - 2)Q(\rho - 4) + Q^2(\rho + 2)Q(\rho + 4) \} - 1 \right] S_u(\rho) = 0 \end{aligned} \quad (3.10)$$

where $Q(k)$ is defined by Eq. (2.24) and the relation (3.9) has been used.

Thus, as in the first order hierarchy solution, the spectrum of the solution u consists of delta functions. The buckling shape again is sinusoidal. The sinusoidal solution is characteristic of this closure approximation except in cases where the basic equation is nonhomogeneous or nonlinear. An example of this is the beam on a nonlinear foundation with initial stress-free normal displacement (Fraser problem).

By setting the coefficient of $S_u(\rho)$ in (3.10) equal to zero for $\rho = \omega$, a functional relation is obtained between the load parameter α and the buckling mode parameter ω . As in Sec. 3.1 a solution is sought in the forms (3.4). In this case

$$\omega_1 = 0, \quad \alpha_1 = \pm \frac{1}{4}, \quad \alpha_2 = -3/2048;$$

thus the buckling load α^* is

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{3}{2048}\epsilon^2 - \dots \quad (3.11)$$

3.3 *Perturbation solution for sinusoidal foundation.* A perturbation solution is sought for the sinusoidal deviation (3.1) using an independent technique. Equation (2.7) becomes

$$\frac{d^4 u}{dy^4} + 2\alpha \frac{d^2 u}{dy^2} + u = -\epsilon \cos 2y \cdot u. \quad (3.12)$$

The amplitude ϵ is taken as the perturbation parameter and it is assumed that the normal displacement u and the load parameter α can be represented by the power series in ϵ :

$$u = \sum_{n=0}^{\infty} \epsilon^n u_n, \quad \alpha = 1 + \sum_{n=1}^{\infty} \epsilon^n \alpha_n. \quad (3.13)$$

Substituting the expressions (3.13) into Eq. (3.12) leads to

$$O(1): \mathcal{L}u_0 = 0 \quad (3.14a)$$

$$O(\epsilon): \mathcal{L}u_1 = -\cos 2y \cdot u_0 - 2\alpha_1 u_{0,yy} \quad (3.14b)$$

$$O(\epsilon^2): \mathcal{L}u_2 = -\cos 2y \cdot u_1 - 2\alpha_1 u_{1,yy} - 2\alpha_2 u_{0,yy}. \quad (3.14c)$$

More generally, the sequence of equations can be written in the form

$$O(\epsilon^n): \mathcal{L}u_n = -\cos 2y \cdot u_{n-1} - 2 \sum_{j=1}^n \alpha_j u_{n-j,yy} \quad (3.15)$$

where $(\cdot)_{,y} = d(\cdot)/dy$, and the operator $\mathcal{L}$ is defined by

$$\mathcal{L}(\cdot) \equiv \frac{d^4(\cdot)}{dy^4} + 2 \frac{d^2(\cdot)}{dy^2} + (\cdot). \quad (3.16)$$

Solving the above equations is elementary and hence only the solution will be noted. In determining the buckling load α^* the values of α_n which lead to the smallest value of α are taken. The α_n are determined by enforcing the boundedness condition (2.8), thus suppressing the solutions which lead to secular terms. To second order term in ϵ , the buckling load α^* is

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{1}{512}\epsilon^2. \quad (3.17)$$

3.4 *Comparison of hierarchy and perturbation solutions.* For easy reference the three solutions (first and second order hierarchy and perturbation) are repeated below. First order hierarchy:

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{1}{1024}\epsilon^2 - \dots$$

Second order hierarchy:

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{3}{2048}\epsilon^2 - \dots$$

Perturbation solution:

$$\alpha^* = 1 - \frac{1}{4}\epsilon - \frac{1}{512}\epsilon^2 - \dots$$

The hierarchy solutions are in excellent agreement with the perturbation solution. The second order hierarchy solution is an improvement on the first order solution, but the improvement is not worth the effort expended in its derivation. The most significant conclusion is that for amplitude ϵ of order unity, the error of the first order solution is less than 0.2%. Of course, in general, the error depends, in a nontrivial way, upon the problems considered. Note in passing that in the hierarchy method no explicit assumption is made about the size of the amplitude ϵ . Based on these results, the first order truncated hierarchy approximation will be used in the analysis of the cylinder. The rest of this report will be devoted to this analysis.

4. Infinite cylinder with axisymmetric imperfection. The buckling of circular cylindrical shells with initial random axisymmetric imperfection under axial compression is analyzed by means of the Kármán–Donnell equations. A cylindrical shell is characterized by its radial displacement $W(\bar{x}, \bar{y})$ (positive outward) and a stress function $F(\bar{x}, \bar{y})$ where $\bar{x}$ and $\bar{y}$ are the Cartesian coordinates in the axial and circumferential directions. The membrane stress resultants N_x , N_y , and N_{xy} are given as $N_x = F_{,\bar{y}\bar{y}}$, $N_y = F_{,\bar{x}\bar{x}}$, and $N_{xy} = -F_{,\bar{x}\bar{y}}$. The shell is assumed to have an initial stress-free radial axisymmetric displacement $\bar{W}(\bar{x})$ which is a homogeneous random function.

The coupled Donnell-type equations satisfied by the deflection W and the stress function F are

$$\frac{1}{Eh} \bar{\nabla}^4 F - \frac{1}{R} W_{,\bar{x}\bar{x}} + (\bar{W}_{,\bar{x}\bar{x}} + W_{,\bar{x}\bar{x}}) W_{,\bar{y}\bar{y}} - W_{,\bar{x}\bar{y}}^2 = 0 \quad (4.1)$$

and

$$D \bar{\nabla}^4 W + \frac{1}{R} F_{,\bar{x}\bar{x}} - (\bar{W}_{,\bar{x}\bar{x}} + W_{,\bar{x}\bar{x}}) F_{,\bar{y}\bar{y}} - W_{,\bar{y}\bar{y}} F_{,\bar{x}\bar{x}} + 2W_{,\bar{x}\bar{y}} F_{,\bar{x}\bar{y}} = 0 \quad (4.2)$$

where E is Young's modulus, ν is Poisson's ratio, h is shell thickness, R is shell radius, $D = Eh^3/12(1 - \nu^2)$ is the bending stiffness, and $\bar{\nabla}^4$ is the two-dimensional biharmonic operator. Equation (4.1) expresses the compatibility of the membrane strains while (4.2) is the equilibrium equation in the normal direction. The equilibrium equations in the axial and circumferential directions are satisfied automatically by the introduction of the stress function.

It is convenient to introduce the nondimensional quantities

$$x = q_0 \bar{x}/R, \quad y = q_0 \bar{y}/R, \quad w = W/h, \quad \bar{w} = \bar{W}/h, \quad f = q_0^2 F/Eh^2 R \quad (4.3)$$

where $q_0^4 = 12(1 - \nu^2)R^2/h^2$. The compatibility and equilibrium equations (4.1) and (4.2) then become

$$\nabla^4 f - w'' + 2c(\bar{w}'' + w'')\bar{w} - 2cw'^2 = 0 \quad (4.4)$$

and

$$\nabla^4 w + f'' - 2c(\bar{w}'' + w'')\bar{f} - 2c\bar{w}f'' + 4cw'f' = 0 \quad (4.5)$$

where $(\)' = \partial/\partial x$, $(\)\dot{ } = \partial/\partial y$, $c = (3(1 - \nu^2))^{1/2}$, and

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

4.1 Axisymmetric prebuckling state. For cylinders under axial compression, which is the only type of loading considered, the equations (4.4) and (4.5) admit an axisymmetric prebuckling solution. It is usually assumed that the cylinder is in a state of constant membrane stress and that the prebuckling radial displacement may be approximated by a constant. This is a good approximation except for a small boundary layer. However, since the analysis is restricted to only cylinders that are sufficiently long and with sufficiently "strong" end conditions to justify neglecting the effects of the boundary conditions, the above assumption is further justified.

Thus the prebuckling state may be written as

$$f(x, y) = -\frac{1}{2c} \lambda y^2 + f_a(x), \quad w(x, y) = \frac{\sigma R}{E} + w_a(x) \quad (4.6)$$

where σ is the axial compressive stress and λ is a nondimensional load parameter given by

$$\lambda = (3(1 - \nu^2))^{1/2} \frac{\sigma R}{Eh}. \quad (4.7)$$

The classical buckling stress σ_c of a perfect cylindrical shell under axial compression is $Eh/R(3(1 - \nu^2))^{1/2}$ for which the load parameter λ has the value unity.

The substitution of the expressions (4.6) for w and f into the equations (4.4) and (4.5) leads to two linear ordinary differential equations for w_a and f_a :

$$f_a'''' - w_a'' = 0, \quad w_a'''' + f_a'' + 2\lambda(\bar{w}'' + w_a') = 0. \quad (4.8)$$

These equations may be solved explicitly for the axisymmetric prebuckling displacement w_a for any arbitrary imperfection. Eliminating f_a from (4.8) leads to the equation

$$w_a'''' + 2\lambda w_a'' + w_a = -2\lambda \bar{w}''. \quad (4.9)$$

Note that the polynomial portion of the solution has been accounted for by writing the prebuckling state in the form (4.6). The solution of (4.9) may be written in integral form as

$$w_a(x) = 2\lambda \int_{-\infty}^{\infty} G_0(x - y) \bar{w}(y) dy \quad (4.10)$$

where the Fourier transform Q_0 of the Green's function G_0 is given by

$$Q_0(\mu) = \mu^2/(\mu^4 - 2\lambda\mu^2 + 1). \quad (4.11)$$

The spectral density of the axisymmetric imperfection $\bar{w}$ is considered known. The bifurcation of the axisymmetric state into an asymmetric state depends on this spectral density. In anticipation of the subsequent buckling analysis, the dependence of the spectral densities of the normal prebuckling displacement w_a and the total displacement w on the spectrum of the initial imperfection will be derived.

The prebuckling displacement w_a is given by Eq. (4.10) which may be written as

$$w_a(x) = 2\lambda \int_{-\infty}^{\infty} G_0(y_1) \bar{w}(x - y_1) dy_1. \quad (4.12)$$

The imperfection $\bar{w}$ is assumed to have a zero mean; hence by (4.12)

$$\langle w_a(x) \rangle = 0.$$

We define the following autocorrelations:

$$\begin{aligned} R_{\bar{w}}(\zeta) &= \langle \bar{w}(x + \zeta) \bar{w}(x) \rangle \\ R_{\bar{w}_a}(\zeta) &= \langle w_a(x + \zeta) w_a(x) \rangle \\ R_{\bar{w}}(\zeta) &= \langle w(x + \zeta) w(x) \rangle \end{aligned} \quad (4.13)$$

and the cross-correlation

$$R_{\bar{w}w_a}(\zeta) = \langle w(x + \zeta) w_a(x) \rangle. \quad (4.14)$$

It follows from (4.12) that

$$w_a(x) w_a(x + \zeta) = 4\lambda^2 \iint_{-\infty}^{\infty} G_0(y_1) G_0(y_2) \bar{w}(x - y_1) \bar{w}(x + \zeta - y_2) dy_1 dy_2.$$

Taking the average over x and using (4.13) gives

$$R_{w_a}(\zeta) = 4\lambda^2 \iint_{-\infty}^{\infty} G_0(y_1) G_0(y_2) R_{\bar{w}}(\zeta + y_1 - y_2) dy_1 dy_2. \quad (4.15)$$

The spectral density of w_a is the inverse Fourier transform of its autocorrelation R_{w_a} . Thus by Eq. (4.15)

$$S_{w_a}(\mu) = 4\lambda^2 Q_0^2(\mu) S_{\bar{w}}(\mu).$$

Similarly, multiplying (4.12) by $w(x + \zeta)$, noting that the total displacement w is given by

$$w(x) = \bar{w}(x) + w_a(x) \quad (4.16)$$

and averaging over x gives the cross-correlation of w_a and w :

$$\begin{aligned} R_{\bar{w}w_a}(\zeta) &= 4\lambda^2 \iint_{-\infty}^{\infty} G_0(y_1) G_0(y_2) R_{\bar{w}}(\zeta + y_1 - y_2) dy_1 dy_2 \\ &\quad + 2\lambda \int_{-\infty}^{\infty} G_0(y_1) R_{\bar{w}}(\zeta + y_1) dy_1. \end{aligned}$$

Taking the inverse Fourier transform leads to

$$S_{\bar{w}w_a}(\mu) = 2\lambda Q_0(\mu) [2\lambda Q_0(\mu) + 1] S_{\bar{w}}(\mu)$$

which simplifies to

$$S_{\bar{w}w_a}(\mu) = -2\lambda\mu^2(\mu^4 + 1)Q_4(\mu)S_{\bar{w}}(\mu) \quad (4.17)$$

where

$$Q_4(\mu) = -1/(\mu^4 - 2\lambda\mu^2 + 1)^2. \quad (4.18)$$

Similarly, it can be shown that

$$S_{\bar{w}}(\mu) = -(\mu^4 + 1)^2 Q_4(\mu) S_{\bar{w}}(\mu). \quad (4.19)$$

4.2 *Buckling of the axisymmetric state.* Figure 6 represents qualitatively the nature of the buckling of axially compressed imperfect cylindrical shells. The curve OAC represents the prebuckling equilibrium state discussed above. At some point A the nonlinear equations may admit an asymmetric solution which is infinitesimally adjacent to the prebuckling axisymmetric state. It is generally known that there is a loss in the load-carrying capacity at bifurcation; hence the bifurcation path AB falls as shown.

The bifurcation equations are derived from the nonlinear equations (4.4) and (4.5) by seeking solutions in the form

$$\begin{aligned} f(x, y) &= -(1/2c)\lambda y^2 + f_a(x) + f_1(x, y), \\ w(x, y) &= \sigma\nu R/E + w_a(x) + w_1(x, y) \end{aligned} \quad (4.20)$$

where $f_1(x, y)$ and $w_1(x, y)$ are infinitesimal asymmetric nondimensional quantities. Substitution of (4.20) into the nonlinear equations (4.4) and (4.5) and the use of (4.8) leads upon linearization with respect to f_1 and w_1 to

$$\begin{aligned} \nabla^4 f_1 - w_1'' + 2c(\bar{w}'' + w_a'')\bar{w}_1 &= 0, \\ \nabla^4 w_1 + 2\lambda w_1'' + f_1'' - 2cw_a\bar{w}_1 - 2c(\bar{w}'' + w_a'')f_1 &= 0. \end{aligned} \quad (4.21)$$

The y dependence is eliminated by letting

$$w_1(x, y) = \phi(x) \cos \bar{n}y, \quad f_1(x, y) = \psi(x) \cos \bar{n}y \quad (4.22)$$

to obtain

$$\mathcal{L}_1\psi = \phi'' + 2\bar{n}^2 c(\bar{w}'' + w_a'')\phi \quad (4.23)$$

FIG. 6. Infinite cylinder: hypothetical load-deflection curve.

and

$$\mathfrak{L}_2\phi = -\psi'' - 2\bar{n}^2cw_a\phi - 2\bar{n}^2c(\bar{w}'' + w_a'')\psi \quad (4.24)$$

where

$$\begin{aligned} \mathfrak{L}_1(\cdot) &= (\cdot)'''' - 2\bar{n}^2(\cdot)'' + \bar{n}^4(\cdot), \\ \mathfrak{L}_2(\cdot) &= (\cdot)'''' + 2(\lambda - \bar{n}^2)(\cdot)'' + \bar{n}^4(\cdot). \end{aligned} \quad (4.25)$$

The solution of Eq. (4.23) can be written in the integral form

$$\psi(x) = \int_{-\infty}^{\infty} G_1(x-y)\phi''(y) dy + 2\bar{n}^2c \int_{-\infty}^{\infty} G_1(x-y)[\bar{w}''(y) + w_a''(y)]\phi(y) dy$$

where G_1 is the Green's function which, as will become apparent, need not be determined. However, its Fourier transform Q_1 is given by

$$Q_1(\mu) = 1/(\mu^2 + \bar{n}^2)^2. \quad (4.26)$$

Substituting for ψ in Eq. (4.24) leads to

$$\begin{aligned} \mathfrak{L}_2\phi(x) + \int_{-\infty}^{\infty} G_1''(x-y)\phi''(y) dy \\ = -2\bar{n}^2cw_a(x)\phi(x) - 2\bar{n}^2c \int_{-\infty}^{\infty} G_1''(x-y)[w_a''(y) + \bar{w}''(y)]\phi(y) dy \\ - 2\bar{n}^2c[\bar{w}''(x) + w_a''(x)] \int_{-\infty}^{\infty} G_1''(x-y)\phi(y) dy \\ - 4\bar{n}^4c^2[\bar{w}''(x) + w_a''(x)] \int_{-\infty}^{\infty} G_1(x-y)[\bar{w}''(y) + w_a''(y)]\phi(y) dy. \end{aligned} \quad (4.27)$$

Here and in the rest of the report prime denotes differentiation with respect to the argument. For convenience the linear operator on the left-hand side of Eq. (4.27) is defined as $\mathfrak{L}_3$; thus

$$\mathfrak{L}_3\phi(x) \equiv \mathfrak{L}_2\phi(x) + \int_{-\infty}^{\infty} G_1''(x-y)\phi''(y) dy. \quad (4.28)$$

If G_3 is the Green's function for the operator $\mathfrak{L}_3$ over the infinite interval $-\infty < x < \infty$ satisfying the equation

$$\mathfrak{L}_2G_3(x) + \int_{-\infty}^{\infty} G_1''(x-y)G_3''(y) dy = -\delta(x)$$

and the conditions

$$G_3(x) = \frac{dG_3}{dx} = 0 \quad \text{for } |x| = \infty,$$

then its Fourier transform Q_3 is given by

$$Q_3(\mu) = \frac{-(\mu^2 + \bar{n}^2)^2}{(\mu^2 + \bar{n}^2)^4 - 2\lambda\mu^2(\mu^2 + \bar{n}^2)^2 + \mu^4}. \quad (4.29)$$

Equation (4.27) represents an eigenvalue problem for the load parameter λ . This

equation will be solved for the lowest eigenvalue λ^* which is the buckling load. The buckling load will be sought for random imperfections using the hierarchy method.

5. Buckling of the infinite cylinder. The hierarchy equations for the infinite cylinder will be derived using Eq. (4.27). Here the averages are taken over the length of the cylinder since the imperfection is axisymmetric and the circumferential dependence of the equations has been eliminated by the choice (4.22). Thus as before the angular bracket is defined by

$$\langle \cdots \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cdots dx.$$

5.1 Hierarchy equations for the infinite cylinder. Equation (4.27) may be written as

$$\begin{aligned} \mathfrak{L}_3 \phi(x + \zeta) = & -2\bar{n}^2 c w_a(x + \zeta) \phi(x + \zeta) \\ & - 2\bar{n}^2 c \int_{-\infty}^{\infty} G_1''(y) w''(x + \zeta - y) \phi(x + \zeta - y) dy \\ & - 2\bar{n}^2 c w''(x + \zeta) \int_{-\infty}^{\infty} G_1''(y) \phi(x + \zeta - y) dy \\ & - 4\bar{n}^4 c^2 w''(x + \zeta) \int_{-\infty}^{\infty} G_1(y) w''(x + \zeta - y) \phi(x + \zeta - y) dy \end{aligned} \quad (5.1)$$

where the relation (4.16) has been used, and $\mathfrak{L}_3$ is defined by Eq. (4.28). It is necessary to define the following averages over x :

$$\begin{aligned} R_\phi(\zeta) &= \langle \phi(x + \zeta) \phi(x) \rangle, \\ R_{11}(\zeta, \mu) &= \langle \phi(x + \zeta) \phi(x) w_a(x + \mu) \rangle, \\ B_{11}(\zeta, \mu) &= \langle \phi(x + \zeta) \phi(x) w(x + \mu) \rangle, \\ R_a(\zeta, \mu, \rho, \tau) &= \langle \phi(x + \zeta) \phi(x) w(x + \mu) w(x + \rho) w(x + \tau) \rangle. \end{aligned} \quad (5.2)$$

Multiplying Eq. (5.1) by $\phi(x)$ and averaging over x gives

$$\begin{aligned} \mathfrak{L}_3 R_\phi(\zeta) = & -2\bar{n}^2 c R_{11}(\zeta, \zeta) - 2\bar{n}^2 c \int_{-\infty}^{\infty} G_1''(y) B_{11, \mu\mu}(\zeta - y, \mu) \Big|_{\mu=\zeta-y} dy \\ & - 2\bar{n}^2 c \int_{-\infty}^{\infty} G_1''(y) B_{11, \mu\mu}(\zeta - y, \mu) \Big|_{\mu=\zeta} dy - 4\bar{n}^4 c^2 \int_{-\infty}^{\infty} G_1(y) R_w''''(y) R_\phi(\zeta - y) dy \end{aligned} \quad (5.3)$$

where definitions (5.2) and the closure approximation

$$\langle \phi(x + \zeta) \phi(x) w(x + \mu) w(x) \rangle \cong \langle \phi(x + \zeta) \phi(x) \rangle \langle w(x + \mu) w(x) \rangle \quad (5.4)$$

have been used. The order of differentiation and averaging have been interchanged where necessary. The solution of Eq. (5.3) can be written in integral form using the Green's functions G_3 :

$$\begin{aligned} R_\phi(\zeta) = & 2\bar{n}^2 c \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) R_{11}(\xi_1, \xi_1) d\xi_1 \\ & + 2\bar{n}^2 c \iint_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1''(\xi_1 - \eta) B_{11, \mu\mu}(\eta, \mu) \Big|_{\mu=\eta} d\eta d\xi_1 \end{aligned}$$

$$\begin{aligned}
& + 2\bar{n}^2 c \int \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1'(\eta) R_{11, \mu\mu}(\xi_1 - \eta, \mu) \Big|_{\mu=\xi_1} d\eta d\xi_1 \\
& + 4\bar{n}^4 c^2 \int \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1(\eta) R_{\psi}''''(\eta) R_{\phi}(\xi_1 - \eta) d\eta d\xi_1 . \quad (5.5)
\end{aligned}$$

In order to determine R_{11} Eq. (5.1) is multiplied by $\phi(x)\psi(x + \mu)$ and averaged over x . The result is a differential equation for R_{11} whose solution may be written in integral form as

$$\begin{aligned}
R_{11}(\zeta, \mu) &= 2\bar{n}^2 c \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) R_{\phi}(\xi_1) R_{\psi\psi}(\mu - \xi_1) d\xi_1 \\
&+ 2\bar{n}^2 c \int \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1''(y) R_{\psi}''(\mu + y - \xi_1) R_{\phi}(\xi_1 - y) dy d\xi_1 \\
&+ 2\bar{n}^2 c \int \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1''(y) R_{\psi}''(\mu - \xi_1) R_{\phi}(\xi_1 - y) dy d\xi_1 \\
&+ 4\bar{n}^4 c^2 \int \int_{-\infty}^{\infty} G_3(\zeta - \xi_1) G_1(y) R_{a, \rho\rho\tau\tau}(\zeta, \mu, \rho, \tau) \Big|_{\tau=\zeta-\xi_1-\nu, \rho=\xi_1} dy d\xi_1 . \quad (5.6)
\end{aligned}$$

An expression for R_{11} need not be determined since its Fourier transform is easily expressed as a Fourier transform of R_a . The hierarchy may be truncated here by approximating $R_a(\zeta, \mu, \rho, \tau)$ by a product of lower order averages. The result of making this approximation will be noted later after a differential equation has been derived for R_a .

It is noteworthy that in the model problem we did not encounter a function corresponding to R_a . The reason for this becomes apparent when we compare Eqs. (2.12) and (5.1). Although both equations are linear in their respective displacement functions u and ϕ , the coefficient of ϕ in the last term of Eq. (5.1) is quadratic in the known random function ψ .

In obtaining a truncated equation for R_a the following approximation will be made:

$$\begin{aligned}
&\langle \psi(x + \zeta) \psi(x + \mu) \psi(x + \rho) \psi(x + \tau) \phi(x + \zeta) \phi(x) \rangle \\
&\cong \langle \phi(x + \zeta) \phi(x) \rangle \langle \psi(x + \zeta) \psi(x + \mu) \psi(x + \rho) \psi(x + \tau) \rangle. \quad (5.7)
\end{aligned}$$

For certain special cases the four-point correlation may be written as a sum of the product of autocorrelation functions of ψ when $\langle \psi \rangle = 0$. For a one mode sinusoidal imperfection, ψ is also sinusoidal and

$$\begin{aligned}
&\langle \psi(x + \zeta) \psi(x + \mu) \psi(x + \rho) \psi(x + \tau) \rangle \\
&= \kappa [\langle \psi(x + \zeta) \psi(x + \mu) \rangle \langle \psi(x + \rho) \psi(x + \tau) \rangle \\
&\quad + \langle \psi(x + \zeta) \psi(x + \rho) \rangle \langle \psi(x + \mu) \psi(x + \tau) \rangle \\
&\quad + \langle \psi(x + \zeta) \psi(x + \tau) \rangle \langle \psi(x + \mu) \psi(x + \rho) \rangle] \quad (5.8)
\end{aligned}$$

where $\kappa = 0.5$. If the imperfection $\bar{w}$ is a Gaussian random function then ψ is also Gaussian³ since by Eqs. (4.12) and (4.16) ψ is a linear transformation of $\bar{w}$. For ψ Gaussian it can be shown⁴ that Eq. (5.8) holds with $\kappa = 1$.

³Theorem; a useful reference is Middleton [8].

⁴Ibid., p. 343.

A differential equation for R_a is obtained if (5.1) is multiplied by $\phi(x)\psi(x + \mu) \cdot \psi(x + \rho)\psi(x + \tau)$ and averaged over x . The equation is omitted; however, the solution may be written by means of (4.13), (4.14), (5.7) and (5.8) as

$$R_a(\zeta, \mu, \rho, \tau) = \kappa \int_{-\infty}^{\infty} G_3(\zeta - \xi_2) F(\xi_2, \mu, \rho, \tau) d\xi_2 \quad (5.9)$$

where

$$\begin{aligned} F(\zeta, \mu, \rho, \tau) = 2\bar{n}^2 c \bigg\{ & [R_{\psi\psi a}(\zeta - \mu)R_{\psi}(\rho - \tau) \\ & + R_{\psi\psi a}(\zeta - \rho)R_{\psi}(\mu - \tau) + R_{\psi\psi a}(\zeta - \tau)R_{\psi}(\mu - \rho)]R_{\phi}(\zeta) \\ & + R_{\psi}(\rho - \tau) \int_{-\infty}^{\infty} G_1''(y)R_{\psi}''(\zeta - y - \mu)R_{\phi}(\zeta - y) dy \\ & + R_{\psi}(\mu - \tau) \int_{-\infty}^{\infty} G_1''(y)R_{\psi}''(\zeta - y - \rho)R_{\phi}(\zeta - y) dy \\ & + R_{\psi}(\mu - \rho) \int_{-\infty}^{\infty} G_1''(y)R_{\psi}''(\zeta - y - \tau)R_{\phi}(\zeta - y) dy \\ & + [R_{\psi}''(\zeta - \mu)R_{\psi}(\rho - \tau) + R_{\psi}''(\zeta - \rho)R_{\psi}(\mu - \tau) \\ & + R_{\psi}''(\zeta - \tau)R_{\psi}(\mu - \rho)] \int_{-\infty}^{\infty} G_1''(y)R_{\phi}(\zeta - y) dy \bigg\}. \end{aligned} \quad (5.10)$$

As mentioned above, R_a may be approximated by a product of lower order averages which for either sinusoidal or Gaussian imperfection leads to $R_a = 0$. Since by (5.9) $\kappa = 0$ implies $R_a = 0$, the approximation $R_a = 0$ can be obtained in the final results by setting $\kappa = 0$.

Taking the inverse Fourier transform of Eq. (5.5) and eliminating R_{11} and R_a by means of (5.6) and (5.9), the following homogeneous equation is obtained for the Fourier transform S_{ϕ} :

$$S_{\phi}(\omega)[1 - 4\bar{n}^4 c^2 Q_3(\omega)(I_1 - 4\bar{n}^4 c^2 \kappa I_2)] = 0 \quad (5.11)$$

where

$$\begin{aligned} I_1 = & \int_{-\infty}^{\infty} [2\lambda + \omega^2(\mu^4 + 1)Q_1(\omega)]^2 \mu^4 Q_3(\mu - \omega)Q_4(\mu)S_{\psi}(\mu) d\mu \\ & + \int_{-\infty}^{\infty} [4\lambda + 2\omega^2(\mu^4 + 1)Q_1(\omega)](\mu^4 + 1)\mu^4(\mu - \omega)^2 Q_1(\mu - \omega)Q_3(\mu - \omega)Q_4(\mu)S_{\psi}(\mu) d\mu \\ & + \int_{-\infty}^{\infty} \mu^4(\mu^4 + 1)^2 Q_1(\mu - \omega)Q_4(\mu)S_{\psi}(\mu) d\mu \\ & + \int_{-\infty}^{\infty} \mu^4(\mu - \omega)^4(\mu^4 + 1)^2 Q_1^2(\mu - \omega)Q_3(\mu - \omega)Q_4(\mu)S_{\psi}(\mu) d\mu, \end{aligned} \quad (5.12)$$

$$\begin{aligned} I_2 = & \iint_{-\infty}^{\infty} A_1(\mu, \rho)Q_3(\rho)Q_4(\mu)Q_4(\rho + \omega)S_{\psi}(\mu)S_{\psi}(\rho + \omega) d\rho d\mu \\ & + \iint_{-\infty}^{\infty} A_2(\mu, \rho)Q_3(\rho)Q_3(\mu)Q_4(\rho + \omega)Q_4(\mu + \omega)S_{\psi}(\rho + \omega)S_{\psi}(\mu + \omega) d\rho d\mu \end{aligned} \quad (5.13)$$

and

$$\begin{aligned}
 A_1(\mu, \rho) &= \mu^4(\rho + \omega)^4(\mu^4 + 1)^2\{2\lambda + [1 + \rho^2 Q_1(\rho)][(\rho + \omega)^4 + 1]\} \\
 &\quad \cdot \left\{ 2\lambda Q_1(\mu + \rho) Q_3(\rho) + \rho^2 [(\rho + \omega)^4 + 1] Q_1(\rho) Q_1(\mu + \rho) Q_3(\rho) + \omega^2 Q_1(\omega) \right. \\
 &\quad \cdot [(\rho + \omega)^4 + 1] Q_1(\rho + \mu + \omega) [Q_3(\mu) + Q_3(\rho + \omega)] + \frac{\omega^2}{\bar{n}^4} Q_1(\omega) Q_3(\mu) \Big\} \\
 A_2(\mu, \rho) &= (\rho + \omega)^4(\mu + \omega)^4[(\rho + \omega)^4 + 1][(\mu + \omega)^4 + 1] \\
 &\quad \cdot \{2\lambda + [1 + \rho^2 Q_1(\rho)][(\rho + \omega)^4 + 1]\} \\
 &\quad \cdot \{2\lambda + \mu^2[(\mu + \omega)^4 + 1] Q_1(\mu)\} [Q_1(\mu + \rho + \omega) + Q_1(\omega)].
 \end{aligned}$$

Q_1 is defined by (4.26), Q_3 by (4.29) and Q_4 by (4.18).

Equation (5.11) corresponds to Eq. (2.23) for the beam. Thus we conclude, as in the case of the beam, that, based on the first order hierarchy approximation, the cylinder buckles into one sinusoidal mode shape. The buckling load, which is determined by setting the bracketed quantities in Eq. (5.11) equal to zero, depends on the imperfection spectrum but not on the particular realization.

For nontrivial solution of (5.11),

$$4\bar{n}^4 c^2 Q_3(\omega) (I_1 - 4\bar{n}^4 c^2 \kappa I_2) = 1. \quad (5.14)$$

This gives the load parameter λ in terms of ω , $\bar{n}$ and the imperfection spectrum S_ω . For any given imperfection the buckling load λ^* may be obtained by minimizing λ with respect to the two reduced wave numbers ω and $\bar{n}$. The reduced circumferential wave number $\bar{n}$ is subject to the condition that the actual wave number $\bar{n}q_0$ be an integer. However, this requirement is hardly restrictive for large wave numbers for which Donnell's equations are valid. It is therefore assumed that $\bar{n}$ varies continuously.

It is convenient at this stage to apply the result (5.14) to a deterministic problem in order to assess its range of validity. Thus as a check buckling loads will be determined in the next section for imperfections in the shape of the classical axisymmetric buckling mode of the perfect cylinder for comparison with independently obtained results.

5.2 Cylinder with sinusoidal imperfection. Several studies have been made to determine the effect of sinusoidal axisymmetric imperfections on the buckling strength of cylindrical shells under axial compression. In particular, the results of Koiter [9] and a more recent investigation by Almroth [10] are compared with the results of this analysis for the special case of imperfection in the shape of the axisymmetric buckling mode

$$\bar{W} = h\bar{\xi} \cos x. \quad (5.15)$$

The imperfection (5.15) has the delta function spectrum

$$S_{\bar{w}}(\omega) = \frac{1}{4}\bar{\xi}^2 [\delta(\omega - 1) + \delta(\omega + 1)]. \quad (5.16)$$

This spectrum is substituted into Eq. (5.14) and the integrals are easily evaluated to yield an algebraic relation between λ , ω , $\bar{n}$ and $\bar{\xi}$. Because of the implicit nature of the λ , ω , and $\bar{n}$ dependence in the relation, it was found convenient to solve for $\bar{\xi}$. Thus for any given λ the quadratic equation for $\bar{\xi}^2$ was solved for various values of the reduced wave number ω and $\bar{n}$ to determine the smallest corresponding value of $\bar{\xi}$.

FIG. 7. Buckling of long cylinders with periodic axisymmetric imperfection of amplitude $\bar{\xi}h$.

As already noted in the last section, $\kappa = 0.5$ for this case. The dependence of λ^* on $\bar{\xi}$ is shown in Fig. 7 along with results of Koiter and Almroth. Almroth's result was obtained by varying the axial and circumferential wavelength of the buckled pattern whereas Koiter varied the circumferential wavelength but fixed the axial wavelength at twice the wavelength of the classical axisymmetric buckling mode. It should prove interesting to obtain an asymptotic expression for λ^* for small amplitude sinusoidal imperfection. This will be done in the following subsection.

5.2.1 Asymptotic expression for λ^* for sinusoidal imperfections. For small values of the imperfection amplitude $\bar{\xi}$, I_2 may be neglected and Eq. (5.14) becomes

$$4\bar{n}^4 c^2 Q_3(\omega) I_1 = 1. \quad (5.17)$$

Substituting (5.16) into the expression (5.12) for I_1 leads to

$$\begin{aligned} I_1 = & \bar{\xi}^2 [\lambda + \omega^2 Q_1(\omega)]^2 Q_4(1) \{ Q_3(1 - \omega) + Q_3(1 + \omega) \} \\ & + 2\bar{\xi}^2 [\lambda + \omega^2 Q_1(\omega)] Q_4(1) \{ (1 - \omega)^2 Q_1(1 - \omega) Q_3(1 - \omega) \\ & + (1 + \omega)^2 Q_1(1 + \omega) Q_3(1 + \omega) \} + \bar{\xi}^2 Q_4(1) \{ Q_1(1 - \omega) + Q_1(1 + \omega) \} \\ & + \bar{\xi}^2 Q_4(1) \{ (1 - \omega)^4 Q_1^2(1 - \omega) Q_3(1 - \omega) + (1 + \omega)^4 Q_1^2(1 + \omega) Q_3(1 + \omega) \}. \end{aligned} \quad (5.18)$$

An asymptotic series is sought in the form

$$\begin{aligned} \omega &= \omega_0 + \omega_1 \bar{\xi}^p + \dots, \\ \bar{n} &= n_0 + n_1 \bar{\xi}^p + \dots, \\ \lambda &= 1 + \lambda_1 \bar{\xi}^p + \dots. \end{aligned} \quad (5.19)$$

The classical asymmetric wave numbers ω_0 , n_0 are related by the equation

$$\omega_0^2 - \omega_0 + n_0^2 = 0. \quad (5.20)$$

Substitution of (5.19) into (5.17) using (5.18) and equating lowest order terms in $\bar{\xi}$ leads to

$$p = \frac{1}{2}, \quad \lambda_1 = -(3c/2)^{1/2}. \quad (5.21)$$

It is interesting to note that λ_1 does not depend explicitly upon n_0 and ω_0 . This is because the imperfection is chosen in the shape of the classical axisymmetric buckling mode.

The buckling load is given by

$$\lambda^* = 1 - (3c/2)^{1/2} \bar{\xi}^{1/2}. \quad (5.22)$$

This is exactly the same result obtained by Koiter in his general theory [11] as well as in the special theory [9]. The asymptotic expression which is plotted in Fig. 7 is a good approximation for $\bar{\xi} < 0.2$. In practice imperfections are not periodic but random. The buckling load α^* will be determined for small mean square imperfection and for exponential-cosine imperfection.

5.3 *Asymptotic expression for λ^* for arbitrary axisymmetric imperfection spectrum.* For imperfection with small mean square, I_2 is neglected in Eq. (5.14) which may then be written as

$$4\bar{n}^4 c^2 Q_3(\omega) \left\{ \int_{-\infty}^{\infty} H(\mu) Q_3(\mu - \omega) Q_4(\mu) S_{\bar{w}}(\mu) d\mu + \int_{-\infty}^{\infty} \mu^4 (\mu^4 + 1)^2 Q_1(\mu - \omega) Q_4(\mu) S_{\bar{w}}(\mu) d\mu \right\} = 1 \quad (5.23)$$

where

$$H(\mu) = [2\lambda + \omega^2(\mu^4 + 1)Q_1(\omega)]^2 \mu^4 + \mu^4(\mu - \omega)^4(\mu^4 + 1)^2 Q_1^2(\mu - \omega) + [4\lambda + 2\omega^2(\mu^4 + 1)Q_1(\omega)](\mu^4 + 1)\mu^4(\mu - \omega)^2 Q_1(\mu - \omega). \quad (5.24)$$

Note that $H(0) = 0$. An asymptotic expression for the buckling load λ^* is obtained by expanding ω , $\bar{n}$ and λ in the series

$$\begin{aligned} \lambda^* &= \lambda = 1 + \lambda_1 \delta^q + \dots, \\ \omega^* &= \omega = \omega_0 + \omega_1 \delta^q + \dots, \\ \bar{n}^* &= \bar{n} = n_0 + n_1 \delta^q + \dots \end{aligned} \quad (5.25)$$

where ω_0 and n_0 are related by Eq. (5.20) and $\delta^2 \equiv R_{\bar{w}}(0)$ is the root mean square of the imperfection. The weighted spectrum $\tilde{S}_{\bar{w}}$ is defined by

$$S_{\bar{w}}(\rho) = \delta^2 \tilde{S}_{\bar{w}}(\rho). \quad (5.26)$$

Using the expansions (5.25) it is easily shown that

$$Q_3(\omega) = 1/2 \omega_0^2 \lambda_1 \delta^q. \quad (5.27)$$

The first integral in Eq. (5.23) is evaluated asymptotically in Appendix B for the cases $\omega_0 = 0.5$ and $\omega_0 \neq 0.5$. Unlike the case of imperfection in the shape of the axisymmetric buckling mode, the buckling load depends on the values of ω_0 and n_0 . The second integral does not contribute to the lowest order term.

Case A: $\omega_0 \neq 0.5$. Equating the zeroth order terms in Eq. (5.23) using (B.6), (5.24), and (5.27) and simplifying leads to $q = 4/3$ and

$$\lambda_1 = - \left\{ \frac{128\sqrt{2}\pi c^2 \omega_0^3 (1 - \omega_0)^2 (1 + 8\omega_0^4)^2 \bar{S}_w(2\omega_0)}{(1 + 2\omega_0)^4 (1 - 2\omega_0)^5} \right\}^{2/3}$$

Case B: $\omega_0 = 0.5$. In this case we use the asymptotic expression (B.9) and Eqs. (5.24) and (5.27) in (5.23) retaining only the lowest order terms to obtain $q = 4/7$ and

$$\lambda_1 = - \left\{ \frac{9\pi c^2 \bar{S}_w(1)}{2\sqrt{2}} \right\}^{2/7}.$$

The load parameter for this case depends more strongly on the imperfection than for Case A, and hence in general the buckling load is given by Case B. The desired asymptotic formula for the buckling load of long imperfect cylinders with axisymmetric imperfection spectrum $S_w(\rho) = \delta^2 \bar{S}_w(\rho)$ is

$$\lambda^* = 1 - \left[\frac{9\pi c^2}{2\sqrt{2}} \bar{S}_w(1) \right]^{2/7} \delta^{4/7}. \quad (5.28)$$

We note for the purpose of comparison Koiter's asymptotic result (also obtained in this analysis as (5.22)) that

$$\lambda^* = 1 - (3c/2)^{1/2} \bar{\xi}^{1/2} \quad (5.29)$$

for imperfection in the shape of the classical axisymmetric buckling mode $\bar{w} = \bar{\xi} \cos x$. By comparing Eqs. (5.28) and (5.29) we conclude that the degree of dependence on imperfection size of the degradation of the buckling load due to axisymmetric imperfection is nearly the same for both sinusoidal and random imperfection ($\frac{1}{2}$ as compared to $\frac{4}{7}$).

5.4 Numerical evaluation of λ^ for exponential-cosine autocorrelation.* The nature of the dependence of λ^* upon the parameters of the exponential-cosine autocorrelation

$$R_w(\eta) = \delta^2 e^{-\beta|\eta|} \cos \gamma \eta \quad (5.30)$$

will be investigated for a small value of the mean square imperfection δ^2 . The corresponding spectral density is

$$S_w(\rho) = \frac{\delta^2 \beta(\rho^2 + \beta^2 + \gamma^2)}{\pi[\rho^4 + 2(\beta^2 - \gamma^2)\rho^2 + (\beta^2 + \gamma^2)^2]}.$$

The buckling loads are determined numerically for $\delta^2 = 0.005$ as functions of the correlation parameters β and γ . For $\beta = 0$, $\gamma = 1$ the imperfection is in the shape of the classical axisymmetric buckling mode with amplitude 0.1. From Fig. 7 we observe that for this amplitude the present result is in good agreement with Almroth's. From Fig. 8 it is noted that neglecting I_2 in (5.14) (that is, setting $\kappa = 0$) does not significantly affect the buckling load given by $\kappa = 0.5$.

In calculating the buckling loads it was found convenient to combine complex variable theory with numerical method in a novel way. The integrals in Eq. (5.23) are extended to the complex plane and evaluated using the calculus of residues. However, the expressions for the residues are so involved that they are best evaluated numerically. A good check on the algebra and programming involved is that the imaginary parts of the residues should be negligible. (Computational round-off errors may lead to small but negligible imaginary parts.) The result of this calculation is shown in Fig. 9.

FIG. 8. Influence of κ on λ^* for cylinders with periodic axisymmetric imperfection of amplitude ξh .

5.5 *Discussion of infinite cylinder results.* A comparison of the present result with previously obtained results (Fig. 7) shows that both the asymptotic formula and the first order hierarchy solution are in good agreement with Koiter's and Almroth's for sinusoidal imperfections of amplitudes that are small compared to the shell thickness. The first order hierarchy approximation gives the same asymptotic formula as was

FIG. 9. Infinite cylinder: dependence of buckling strength on autocorrelation parameters.

obtained by Koiter. The approximation is not good for imperfections of the order of the shell thickness. This inadequacy may be due to the correlation discard method which yields sinusoidal buckling mode when the equilibrium equation is linear in the buckling displacement, or it may be due to the consideration of only the first order approximation.

However, the approximation yields excellent results for small mean square imperfection. Fig. 9 shows, as already noted by Koiter and as predicted by the asymptotic formula (5.28), that for sinusoidal or nearly sinusoidal axisymmetric imperfections the most degrading imperfection is one in the shape of the classical axisymmetric buckling mode. In addition, for moderate values of β the buckling load is nearly independent of γ for a large range of values of γ .

6. Concluding remarks. A modified hierarchy method has been used to analyze two stochastic buckling problems. In both the model problem and cylinder problem it is found that the buckling loads of the structures depend only on the spectral density of the random properties and not on the particular realization. Thus each member of an ensemble of axially compressed infinite cylinders with random axisymmetric imperfection has the same buckling strength which depends on the spectral density of the imperfection.

As already noted, the technique yields simple but accurate asymptotic expressions for the buckling strengths for small variances of the stochastic function. For the cylinder with axisymmetric imperfection the degree of dependence on imperfection size of the degradation of the buckling strength is nearly the same for both random and one mode sinusoidal imperfections. Since small variances are expected when structures are carefully fabricated, the asymptotic results may prove useful in structural design.

Furthermore, for small variances the buckling loads depend only on the values of the spectral densities at certain wave numbers and not on the shape of the spectrum. It should prove worthwhile to develop experimental techniques for determining spectral densities for elastic structures. In statistical communication theory such techniques have been developed and possibly some of these may be modified for application to statistical structural mechanics.

The method used here can be utilized for determining the buckling loads of structures whose buckling equations are nonlinear. An analysis which is omitted here shows that this method gives the same results obtained by Fraser [2] for infinitely long imperfect beams on nonlinear foundations. A next step would be to use this method for analyzing the buckling of cylinders with imperfections that are not necessarily axisymmetric.

APPENDIX A

Asymptotic evaluation of an integral arising in the beam problem. The first nonzero term in the asymptotic expansion of the integral

$$I = \int_0^\infty \frac{f(\mu) d\mu}{\mu^4 - 2\alpha\mu^2 + 1} \quad (\text{A.1})$$

will be derived. The integral occurs in a modified form in Sec. 2.3. The function $f(\mu)$ is assumed well-behaved and bounded for $0 \leq \mu < \infty$. The load parameter α is expanded in the series

$$\alpha = 1 + \alpha_1 \Delta^k + \dots \quad (\text{A.2})$$

where Δ is the small standard deviation of the foundation. Since by (A.2)

$$\mu^4 - 2\alpha\mu^2 + 1 = (\mu^2 - 1)^2 - 2\alpha_1\mu^2\Delta^k + O(\Delta^{2k}),$$

the main contribution to the integral is from the neighborhood of $\mu = 1$, say $|\mu - 1| \leq \epsilon$. Define a new independent variable η by the relation

$$\mu^2 - 1 = \Delta^* \eta.$$

The integral I is then approximated by

$$I \sim \frac{1}{2} \Delta^* f(1) \int_{-2\epsilon/\Delta^*}^{2\epsilon/\Delta^*} \frac{d\eta}{\eta^2 \Delta^{2s} - 2\alpha_1 \Delta^k}.$$

Equating the orders of the terms of the integrand and performing the integration with the limits replaced by $\pm \infty$ gives

$$I \sim \frac{\pi f(1)}{2(-2\alpha_1)^{1/2}} \Delta^{-k/2}. \quad (\text{A.3})$$

The integral in Eq. (2.31) can be written as

$$\int_{-\infty}^{\infty} \frac{\bar{S}_s(\mu - \omega) d\mu}{\mu^4 - 2\alpha\mu^2 + 1} = \int_0^{\infty} \frac{\bar{S}_s(\mu - \omega) + \bar{S}_s(\mu + \omega)}{\mu^4 - 2\alpha\mu^2 + 1} d\mu$$

by using the fact that the spectrum is even. It is easily deduced by comparison with (A.1) and the result (A.3) that

$$\int_{-\infty}^{\infty} \frac{\bar{S}_s(\mu - \omega) d\mu}{\mu^4 - 2\alpha\mu^2 + 1} \sim \frac{\pi}{2(-2\alpha_1)^{1/2}} [\bar{S}_s(0) + \bar{S}_s(2)] \Delta^{-k/2}; \quad (\text{A.4})$$

similarly,

$$\int_{-\infty}^{\infty} \frac{\bar{S}'_s(\mu - \omega) d\mu}{\mu^4 - 2\alpha\mu^2 + 1} \sim \frac{\pi}{2(-2\alpha_1)^{1/2}} [\bar{S}'_s(0) + \bar{S}'_s(2)] \Delta^{-k/2}. \quad (\text{A.5})$$

APPENDIX B

Asymptotic evaluation of an integral arising in the cylinder problem. An asymptotic formula is sought for the integral

$$J = \int_{-\infty}^{\infty} H(\mu) Q_3(\mu - \omega) Q_4(\mu) S_w(\mu) d\mu$$

occurring in Eq. (5.23), where $H(\mu)$ is well behaved and bounded for $-\infty < \mu < \infty$. By Eq. (5.26)

$$J = \delta^2 \int_{-\infty}^{\infty} H(\mu) Q_3(\mu - \omega) Q_4(\mu) \bar{S}_w(\mu) d\mu. \quad (\text{B.1})$$

The asymptotic formula is sought using the series expansions (5.25), namely

$$\begin{aligned} \omega &= \omega_0 + \omega_1 \delta^a + \dots \\ \bar{n} &= n_0 + n_1 \delta^a + \dots \\ \lambda &= 1 + \lambda_1 \delta^a + \dots \end{aligned} \quad (\text{B.2})$$

where by (5.20)

$$\omega_0^2 - \omega_0 + n_0^2 = 0. \quad (\text{B.3})$$

It is anticipated that the main contribution to the integral is from the neighborhood of $\mu = 2\omega_0$ and $\mu = 0$ but since $H(0) = 0$ only the neighborhood of $\mu = 2\omega_0$ is significant. A new variable η defined by the relation

$$\mu = 2\omega_0 + \eta\delta^r \quad (\text{B.4})$$

is introduced. Thus if the infinite limits on the integral are replaced by $2\omega_0 \pm \epsilon$, then the limits on η become $\pm \epsilon/\delta^r$. It is easily shown that

$$\begin{aligned} \mu^4 - 2\lambda\mu^2 + 1 &= (1 - 4\omega_0^2)^2 - 8\omega_0(1 - 4\omega_0^2)\eta\delta^r \\ &\quad + 2(12\omega_0^2 - 1)\eta^2\delta^{2r} + O(\delta^{3r}) + 8\omega_0^2\lambda_1\delta^q + O(\delta^{2q}) \end{aligned} \quad (\text{B.5})$$

and

$$\begin{aligned} [(\mu - \omega)^2 + n^2]^4 - 2\lambda(\mu - \omega)^2[(\mu - \omega)^2 + n^2]^2 + (\mu - \omega)^4 \\ = -2\omega_0^4\lambda_1\delta^q + 4\omega_0^2(2\omega_0 - 1)^2\eta^2\delta^{2r} + 4\omega_0(2\omega_0 - 1)(4\omega_0^2 + 2\omega_0 - 1)\eta^3\delta^{3r} \\ + (16\omega_0^4 + 48\omega_0^3 - 20\omega_0^2 - 4\omega_0 + 1)\eta^4\delta^{4r} + O(\delta^{2q}) + O(\delta^{5r}). \end{aligned}$$

Since by Eq. (B.5) the lowest order term in the expansion of $Q_4(\mu)$ will be different depending upon the value of ω_0 , there are two distinct cases, namely $\omega_0 = 0.5$ and $\omega_0 \neq 0.5$.

Case A: $\omega_0 \neq 0.5$. If $\omega_0 \neq 0.5$, then J may be approximated by

$$J \sim -\frac{\delta^{2+r}\omega_0^2 H(2\omega_0)\bar{S}_w(2\omega_0)}{(1 - 4\omega_0^2)^4} \int_{-\epsilon/\delta^r}^{\epsilon/\delta^r} \frac{d\eta}{-2\omega_0^4\lambda_1\delta^q + 4\omega_0^2(2\omega_0 - 1)^2\eta^2\delta^{2r} + O(\delta^{2q}) + O(\delta^{3r})}.$$

We demand that the first two terms in the denominator of the integrand be of the same order. Thus

$$2r = q$$

and by extending the limits of integration to $\pm \infty$ we have

$$J \sim \frac{-\pi}{4\omega_0(1 + 2\omega_0)^4(1 - 2\omega_0)^5} \left(-\frac{2}{\lambda_1}\right)^{1/2} H(2\omega_0)\bar{S}_w(2\omega_0)\delta^{2-q/2}. \quad (\text{B.6})$$

Case B: $\omega_0 = 0.5$. In this case

$$\mu^4 - 2\lambda\mu^2 + 1 = -2\lambda_1\delta^q + 4\eta^2\delta^{2r} + \dots \quad (\text{B.7})$$

and

$$[(\mu - \omega)^2 + n^2]^4 - 2\lambda(\mu - \omega)^2[(\mu - \omega)^2 + n^2]^2 + (\mu - \omega)^4 = -\frac{1}{8}\lambda_1\delta^q + \eta^4\delta^{4r} + \dots \quad (\text{B.8})$$

Only the lowest order terms in these expansions have been retained. Demanding, as in case A, that the lowest order terms in Eq. (B.7) be of the same order gives $2r = q$. The second term on the right-hand side of Eq. (B.8) becomes of a higher order than the first and the integral J is approximated by

$$J \sim -\delta^{2-5q/2} \frac{H(1)\bar{S}_w(1)}{8\lambda_1} \int_{-\epsilon/\delta^r}^{\epsilon/\delta^r} \frac{d\eta}{\left[\eta^2 + \left(-\frac{\lambda_1}{2}\right)\right]^2}.$$

Extending the limits of integration to $\pm \infty$ and integrating gives

$$J \sim -\frac{H(1)\tilde{S}_w(1)\pi}{(-2\lambda_1)^{5/2}} \delta^{2-5q/2}. \quad (\text{B.9})$$

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