MOST GENERAL SOLUTION OF A MULTIPLE LINEAR OPERATOR EQUATION*

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1. Introduction. In many problems of applied mathematics, it is necessary to solve an equation of the form

$$L_1 L_2 \cdots L_n \phi = 0, \tag{1}$$

where the L_i's are linear operators which commute and are not identical:

$$L_i(\phi + \psi) = L_i\phi + L_i\psi, \tag{2}$$

$$L_i L_j = L_j L_i , \tag{3}$$

$$L_i \neq L_j \quad \text{if} \quad i \neq j. \tag{4}$$

Equation (1) will in general have many solutions. The desired solution is the one which satisfies certain boundary conditions. In order to see what boundary conditions can be satisfied, and also in order to see if the solution is unique for a given set of boundary conditions, it is important to know the *most general solution* of (1). It is sometimes stated that this most general solution is

$$\phi = \phi_1 + \phi_2 + \cdots + \phi_n , \tag{5}$$

where ϕ_i is the most general solution of

$$L_i\phi_i = 0. \tag{6}$$

The purpose of the present paper is to find conditions for which (5) is indeed the most general solution of (1), and at the same time to show that under other conditions there may be additional solutions.

2. Most general solution of $L_1L_2\phi = 0$. We first consider the problem just formulated for the case of two operators. Let us denote the class formed by all solutions of (6) by Φ_i , with members ϕ_i . Then the equation

$$L_1L_2\phi = 0 \tag{7}$$

can be written

$$L_2\phi \in \Phi_1 \tag{8a}$$

or

$$L_2\phi = \phi_1 \tag{8b}$$

where ϕ_1 may be any of the members of Φ_1 . For given ϕ_1 , the most general solution of (8b) is

$$\phi = \phi_2 + p_1 \tag{9}$$

where p_1 is *any* particular solution of (8b); note that p_1 depends on ϕ_1 .

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In an attempt to find such a particular solution, we consider the function $L_2\phi_{1a}$, where ϕ_{1a} is any of the members of Φ_1. Since L_1 and L_2 commute, $L_1L_2\phi_{1a} = 0$ so that $L_2\phi_{1a} \in \Phi_1$, or

$$L_2\phi_{1a} = \phi_{1b} \tag{10}$$

where in general $\phi_{1b} \neq \phi_{1a}$. This allows us to find a particular solution for the case in which ϕ_1 in (8b) happens to be ϕ_{1b}: this particular solution is ϕ_{1a}. More generally, the solution of (7) is of the form $\phi = \phi_2 + \phi_{1a}$, provided ϕ_1 in (8b) belongs to the class Φ_{1b}, the members of which are formed by operating with L_2 on all members of Φ_1. We denote this symbolically by

$$L_2\Phi_1 = \Phi_{1b}. \tag{11}$$

It can be verified immediately that $\phi_1 + \phi_2$ always is a solution of (7). *We now conclude that*

$$\phi = \phi_1 + \phi_2 \tag{12}$$

is the most general solution of (7), *provided*

$$\Phi_{1b} = \Phi_1. \tag{13}$$

This is so because, if (13) is true, a particular solution ϕ_{1a} can be found for any ϕ_1 in (8b).[1] Condition (13) must be checked for each specific choice of L_1 and L_2. Note that (13) does not hold when $L_2\phi_1 = 0$ for all ϕ_1, i.e., when $L_1\psi = 0$ implies $L_2\psi = 0$. This condition is somewhat stronger than that given by (4).

As noted before, $\Phi_{1b} \leq \Phi_1$, i.e., any member of Φ_{1b} is a member of Φ_1. We now consider the case $\Phi_{1b} < \Phi_1$, so that there are members ϕ_1^1 of Φ_1 which are not members of Φ_{1b}. Denote the class consisting of all ϕ_1^1 by Φ_1^1, so that Φ_1^1 is the set-theoretic complement of Φ_{1b} in Φ_1. We must look for solutions of the equation

$$L_2\phi = \phi_1^1. \tag{14}$$

Assuming these solutions to exist, and denoting them by ϕ^1, we find that the most general solution of (7) is given by

$$\phi = \phi_1 + \phi_2 + \phi^1. \tag{15}$$

If no solutions ϕ^1 exist, the most general solution of (7) is still given by (12), even though $\Phi_{1b} < \Phi_1$. The question of the existence of solutions ϕ^1 cannot be attacked until L_1 and L_2 have been specified.

3. Symmetry with respect to interchange on indices. In view of the initial formulation of the problem, it is clear that the result must be symmetric with respect to interchange of the indices 1 and 2. However, this symmetry is not apparent in (13) and (14). We now consider this point in more detail.

Suppose no solutions ϕ^1 exist, either because $\Phi_{1b} = \Phi_1$ or because (14) has no solution.

[1] In finding the most general solution of $L_1\phi = 0$, some care must be taken in deciding what are allowed solutions. For example, both ϕ_1 and ϕ_2 can be allowed to have singularities, provided these cancel in the sum $\phi_1 + \phi_2$. Also, restrictions on ϕ_1 and ϕ_2 being multivalued must be regarded with respect to $\phi_1 + \phi_2$.

Then the most general solution of (7) is $\phi = \phi_1 + \phi_2$, and this leads us to conclude that either (a) $\phi_{2b} = \Phi_2$, or (b) there is no solution to the equation

$$L_1\phi = \phi_2^1 ; \tag{16}$$

the definition of ϕ_2^1 is analogous to that of ϕ_1^1 . Without saying more about L_1 and L_2 , neither (a) nor (b) can be ruled out.

On the other hand, if solutions ϕ^1 do exist, then the most general solution of (7) is $\phi = \phi_1 + \phi_2 + \phi^1$, and this is possible only if there exist functions ϕ_2^1 for which (16) has a solution.

4. Extension to n operators. The extension to more than two operators is almost immediate, since all operators except 1 can be grouped together into a new operator so that we again must consider the equation for two operators:

$$L_1L_2 \cdots L_n\phi = L_IL_n\phi = 0. \tag{17}$$

The question to be considered now is if $L_n\Phi_I$ is equal to or smaller than Φ_I . Since Φ_I will be of the form $\Phi_1 + \Phi_2 \cdots + \Phi_{n-1} + \Phi^1$, where Φ^1 stands for any extra solutions obtained in previous steps, the question becomes a multiple one: is $L_n\Phi_i = \Phi_i$ for $i = 1, 2 \cdots n - 1$, and is $L_n\Phi^1 = \Phi^1$? Clearly, if

$$L_j\Phi_i = \Phi_i \quad \text{for all} \quad j > i \tag{18}$$

the most general solution of (17) is

$$\phi = \sum_{i=1}^{n} \phi_i . \tag{19}$$

Since by assumption all of the operators commute, the labeling is arbitrary. If (18) is not satisfied for one or more i, j combinations, extra solutions may have to be included in (19); the argument then is so close to that given in Sec. 2 as not to warrant further discussion.

5. Final remarks. Since the existence of solutions ϕ^1 requires rather special conditions, it is useful to include an example hsowing such a solution. The following example is due to L. Gross. Let

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix},$$

where a, b, c, d and e are arbitrary constants. Then $L_1\phi = 0$ and $L_2\phi = 0$ have the solutions

$$\phi_1 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ e \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} a \\ 0 \\ 0 \\ d \\ 0 \end{pmatrix},$$

respectively. L_1 and L_2 satisfy conditions (2)–(4). Furthermore, $L_1\psi = 0$ does not imply $L_2\psi = 0$, nor does $L_2\psi = 0$ imply $L_1\psi = 0$. We have

$$L_1L_2 = L_2L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so that $L_1L_2\phi = 0$ has the solution

$$\phi = \begin{pmatrix} a \\ b \\ 0 \\ d \\ e \end{pmatrix}.$$

We see that $\phi = \phi_1 + \phi_2 + \phi^1$, where

$$\phi^1 = \begin{pmatrix} 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We note that in order to find the most general solution for any problem (1), it is first of all necessary that Φ_1 and Φ_2 be known. If this is the case, there is no difficulty in checking (13), or its alternative condition $\Phi_{2b} = \Phi_2$. Either of these conditions guarantees (12) to be the most general solution. If neither of the two conditions is satisfied, it is necessary to look into the existence of solutions ϕ^1 of (14), or alternatively of $L_1\phi = \phi_2^1$.

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