ON NONLINEAR STABILITY THEORY*

By A. J. PRITCHARD (*University of Warwick, Coventry, England*)

Introduction. As a first approach to the stability analysis of an equilibrium state of a dynamical system the perturbations are assumed to be infinitesimal, and the equations of the perturbed motion are linearized. If the system is autonomous, solutions which are exponential functions of time are sought and stability concluded if the perturbations decay to zero in time. On the other hand, if the perturbations grow in time the nonlinear terms must be considered and there exists the possibility that the perturbations are bounded for all time. The original equilibrium state is then distorted and may be replaced by some new state. This concept was first suggested by Landau [1] and an example is the Couette flow between two concentric, circular cylinders. If the outer cylinder is fixed and the speed of the inner cylinder is high enough the laminar Couette flow is replaced by a new circumferential flow with superimposed toroidal (or Taylor) vortices spaced periodically along the axis. Davey, Di Prima, and Stuart [2] have examined this problem by assuming a Fourier series representation of the disturbance, the coefficients of which are expanded in suitable powers and products of the "amplitudes" as functions of time. A similar approach for nonlinear stability problems has been suggested by Eckhaus [3] who assumes that the solution of the nonlinear problem may be expanded in a series of eigenfunctions of the linearized problem. The coefficients of the series are "amplitude" functions of time and are to be determined.

An alternative approach to stability problems is the direct method of Liapunov. The method has been applied successfully to stability problems involving linear and nonlinear ordinary differential equations. The stability and instability theorems have been extended to continuous systems by Movchan [4], [5] and Knops and Wilkes [6]. For continuous systems it is necessary to introduce a measure of the initial disturbance (metric ρ_0), and a measure (metric ρ) of the distance of the perturbed state from the equilibrium state. The stability of the system is then studied with respect to these metrics. Liapunov functionals are constructed, and the condition that the chosen, positive, bounded functional should be nonincreasing leads to criteria for stability.

In this paper a theorem is proved which extends the Liapunov stability theorem to obtain bounds for the solution of the nonlinear perturbed equations. It is then possible to consider the effects of the nonlinear terms and obtain estimates for the amplitude functions in problems which are unstable in the linear approximation. The approach enables the perturbed equations to be dealt with directly, and no assumptions have to be made about the relative orders of magnitude of the amplitude functions.

In Sec. 1 the Liapunov stability theorem and its extension are given, and in Sec. 2 these theorems are applied to some problems in hydrodynamics.

1. Basic Theorems. A specific system is considered and it is assumed that the variables z_1, z_2, $\cdots$ represent measurable values of the physical quantities at time t. If we associate with the set $(z_1, z_2, \cdots, t)$ a metric ρ the motion of the dynamical system may be represented as a path in a metric space. One such path will be taken as the un-

* Received July 10, 1968; revised version received March 4, 1969.

perturbed path and the metric ρ may then be taken as a suitable measure of the "distance" of a point in the perturbed path from a point in the unperturbed path. A further metric ρ_0 is also introduced which will serve as a generalized measure of the initial disturbance.

THEOREM 1: LIAPUNOV STABILITY THEOREM. *The unperturbed path will be uniformly, asymptotically stable with respect to the metrics ρ and ρ_0 if*

(i) *$\rho(z_1, \cdots, t)$ is a continuous function of t,*

(ii) *$\rho(z_1, \cdots, t)$ is continuous with respect to $\rho_0(z_1, \cdots, t)$, that is, given $\epsilon > 0$ there exists δ such that $\rho(z_1, \cdots, t) < \epsilon$ whenever $\rho_0(z_1, \cdots, t) < \delta$,*

(iii) *on the subspace consisting of the points which satisfy $\rho < R$ where R is a real positive number there exists a functional $V(z_1, \cdots, t)$ with the following properties:*

(a) *there exists a nondecreasing function $\gamma(\rho)$ such that*

$$dV/dt \leq -\gamma(\rho) < 0,$$

(b) *there exist nondecreasing functions $\alpha(\rho)$, $\beta(\rho)$ such that*

$$\beta(\rho) \geq V(z_1, \cdots, t) \geq \alpha(\rho) > 0,$$

(c) *the functional $V(z_1, \cdots, t)$ is continuous with respect to the metric ρ_0 on the set of initial instances T_0.*

If $\rho = \rho_0$ we say the unperturbed path is asymptotically stable with respect to a single metric ρ.

THEOREM II. *Consider a functional $V(z_1, z_2 \cdots, t)$ and a metric $\rho(z_1, \cdots, t)$ such that:*

(i) *V and ρ are continuously differentiable in t,*

(ii) *$\alpha\rho^2 \leq V$, $\alpha > 0$,*

(iii) *$dV/dt \leq H(V)$ where $H(V)$ is a polynomial in V, the real positive roots of $H(V) = 0$ being assumed to be ordered by magnitude.*

Then if $H(V_0) \geq 0$ we have $\rho^2 \leq V_U/\alpha$ for all time and if $H(V_0) < 0$ we have $\rho^2 \leq V_0/\alpha$ for all time and as $t \to \infty$, $\rho^2 \to V_L/\alpha$ or $\rho^2 < V_L/\alpha$, where V_L, V_U are consecutive, real, positive roots of $H(V) = 0$ such that $V_L \leq V_0 \leq V_U$ and $V_0 = V(z_1, \cdots, t_0)$ is the initial value of V. If V_L, V_U do not exist we take $V_L = 0$ and $V_U = +\infty$.

Proof. Consider $H(V_0) < 0$ and assume $V_U > V > V_0$ at some time $\bar{t}$; then by the continuity of V there exists $\bar{\bar{t}} < \bar{t}$ such that $V_U > V > V_0$ for all $t \in (\bar{\bar{t}}, \bar{t})$ and $V(\bar{\bar{t}}) = V_0$. Therefore by the mean value theorem there exists $t_2 \in (\bar{\bar{t}}, \bar{t})$ and

$$\left.\frac{dV}{dt}\right|_{t=t_2} = \frac{V(\bar{t}) - V(\bar{\bar{t}})}{\bar{t} - \bar{\bar{t}}} > 0.$$

But $(dV/dt) \leq H(V) \leq 0$ for $V_U > V > V_0 > V_L$. We have a contradiction, so $V \leq V_0$ and $\rho^2 \leq V_0/\alpha$.

For $V_L \leq V \leq V_0$

$$dV/dt \leq K_0(V - V_L) \quad \text{where } K_0 \text{ is negative.}$$

Integrating we have

$$V - V_L \leq Ae^{K_0 t}$$

and as $t \to \infty$, $V \to V_L$ or $V < V_L$, so $\rho^2 \to V_L/\alpha$ or $\rho^2 < V_L/\alpha$.

Similar arguments may be invoked to prove the remainder of the theorem.

Note. (i) If $V_U = +\infty$ and $H(V_0) > 0$ the system is unstable.

(ii) If $V_L = 0$ and $H(V_0) < 0$ the system is asymptotically stable.

(iii) We may relate the bounds on ρ^2 to a measure of the initial disturbance ρ_0 if we further assume $V_0 \leq \beta\rho_0^2$.

2. In this section we apply the theorems to two stability problems associated with Burgers' models of turbulence [7]. Buis and Vogt [8] have used Theorem 1 to obtain sufficient conditions for asymptotic stability, and Eckhaus [3] has used asymptotic expansions with respect to suitably defined small parameters to obtain estimates for the amplitude functions. In Example III the theorems are applied to a system of ordinary differential equations governing the amplitude functions related to the transition from Couette flow to Taylor-vortex flow.

Example I. Burgers [7] has examined mathematical models which are similar to and simpler than the usual equations of hydrodynamics. He discusses these models, which in a sense form a mathematical model of turbulence, and indicates the bearing of his results on the hydrodynamical problem.

One such model is:

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + u - Ru\int_0^1 u^2\,dx - \frac{\partial(u^2)}{\partial x},$$

where $u(0) = u(1) = 0$ and is the analogue of flow in a channel. The variable u represents the velocity of the disturbed flow and there is turbulence in the system when u is different from zero. R is the Reynolds number.

Linear Problem.

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + u, \qquad u(0) = u(1) = 0.$$

Let

$$\rho_0^2 = \rho^2 = 2V = \int_0^1 u^2\,dx;$$

then

$$\frac{dV}{dt} = -\int_0^1 \left(\frac{1}{R}\left(\frac{\partial u}{\partial x}\right)^2 - u^2\right)dx.$$

Using the Rayleigh inequality

$$\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \geq \pi^2 \int_0^1 u^2\,dx$$

we have

$$\frac{dV}{dt} \leq -\left(\frac{\pi^2}{R} - 1\right)\int_0^1 u^2\,dx.$$

It is easily shown that for $R < \pi^2$ the conditions of Theorem 1 are satisfied, and so the system is asymptotically stable with respect to the metric ρ.

Nonlinear Problem. Let

$$\rho_0^2 = \rho^2 = V = \int_0^1 u^2\,dx;$$

then

$$\frac{dV}{dt} = 2\int_0^1 \left[u^2 - \frac{1}{R}\left(\frac{\partial u}{\partial x}\right)^2\right] dx - 2R\left[\int_0^1 u^2\,dx\right]^2,$$

and so

$$\frac{dV}{dt} \leq 2RV\left[\frac{1}{R}\left(1 - \pi^2/R\right) - V\right].$$

Again for $R < \pi^2$ we have asymptotic stability.

$$\left.\begin{array}{l}\text{If } R > \pi^2\text{, recalling Theorem 2, for } 0 \leq V_0 < (1/R)(1 - \pi^2/R) \text{ or } \rho_0^2 < (1/R)\cdot \\ (1 - \pi^2/R) \text{ we have } \rho^2 \leq 1/R(1 - \pi^2/R) \text{ for all time.} \\ \quad\text{For } \rho_0^2 \geq (1/R)(1 - \pi^2/R) \text{ we have } \rho^2 < \rho_0^2 \text{ and } \rho^2 \to (1/R)(1 - \pi^2/R) \text{ as } t \to \infty \\ \text{or } \rho^2 < (1/R)(1 - \pi^2/R).\end{array}\right\}\text{(A)}$$

The above results give bounds for the turbulence in the system, and it is instructive to compare the level with that obtained by Eckhaus [3]. By taking $u = \sum_0^\infty A_n \sin n\pi x$ and using suitably defined small parameters Eckhaus obtains, for $R = 2\pi^2$, the following estimates for the amplitude functions:

$$A_0 = \pm.55/\pi, \quad A_1 = .16/\pi, \quad A_2 = \mp.075/\pi, \quad A_3 = .031/\pi,$$

and so $\sum A_n^2 \sim 1/3\pi^2$. For $R = 2\pi^2$ and $u = \sum A_n \sin n\pi x$ we have $\rho^2 = \frac{1}{2}\sum A_n^2$ and the results (A) give

$$\sum A_n^2 \to \frac{1}{2\pi^2} \quad \text{as} \quad t \to \infty \qquad \text{or} \qquad \sum A_n^2 < \frac{1}{2\pi^2}.$$

Example II. Consider

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + \frac{2}{\sqrt{R}}\frac{\partial}{\partial x}(xu) + x^2 u - u\frac{\partial u}{\partial x} - R^2 u\int_0^1 u^2\,dx$$

and

$$u(0) = u(1) = 0.$$

Linear Problem. Let

$$\rho^2 = \rho_0^2 = V = R\int_0^1 u^2 e^{\sqrt{R}x^2}\,dx;$$

then

$$\frac{dV}{dt} = -2\int_0^1 \left[e^{\sqrt{R}x^2}\left(\frac{\partial u}{\partial x}\right)^2 - Re^{\sqrt{R}x^2}\left(x^2 + \frac{2}{\sqrt{R}}\right)u^2\right] dx.$$

Using the inequality

$$\int_0^1 \left[\frac{d}{dx}\left(e^{1/2\sqrt{R}x^2}u\right)\right]^2 dx \geq \pi^2\int_0^1 e^{\sqrt{R}x^2}u^2\,dx,$$

we have

$$\frac{dV}{dt} \leq -2\left(\frac{\pi^2}{R} - \frac{1}{\sqrt{R}}\right)\int_0^1 Ru^2 e^{\sqrt{R}x^2}\,dx.$$

It is easily shown that for $R < \pi^4$ the conditions of Theorem 1 are satisfied and the system is asymptotically stable with respect to the metric ρ.

Nonlinear Problem. Let

$$\rho^2 = \rho_0^2 = V = \int u^2 \, dx;$$

then

$$\frac{dv}{dt} = 2 \int_0^1 \left[-\frac{1}{R} \left(\frac{\partial u}{dx}\right)^2 + \left(\frac{1}{\sqrt{R}} + x^2\right) u^2 \right] dx - 2R^2 \left[\int_0^1 u^2 \, dx \right]^2.$$

Making use of the Rayleigh inequality, and the inequality

$$\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \geq (5.562)^2 \int_0^1 x^2 u^2 \, dx,$$

which can easily be established by variational methods, we have

$$\frac{1}{2} \frac{dV}{dt} \leq -\gamma V - R^2 V^2, \qquad \text{where} \quad \gamma = \pi^2 \left(\frac{1}{R} - \frac{1}{30.94}\right) - \frac{1}{\sqrt{R}}.$$

From which, using Theorem 1, we have asymptotic stability with respect to the metric ρ for $R < 17.7$.

$$\left.\begin{array}{l} \text{For } R > 17.7 \text{ we have, by Theorem II if } \rho_0^2 < -\gamma/R^2 \text{ then } \rho^2 \leq -\gamma/R^2 \text{ for all time;} \\ \text{if } \rho_0^2 \geq -\gamma/R^2 \text{ then } \rho^2 \leq \rho_0^2 \text{ and as } t \to \infty,\ \rho^2 \to -\gamma/R^2 \text{ or } \rho^2 < -\gamma/R^2 \end{array}\right\} \text{(B)}$$

The functional V used here is different from that used in the linearized problem. Buis and Vogt [8] have shown, using

$$\rho_0^2 = \rho^2 = V = R \int_0^1 e^{\sqrt{R}x^2} u^2 \, dx,$$

that the nonlinear problem is asymptotically stable for $R < \pi^4$ and disturbances bounded by max $|u| < \frac{3}{2}[1/R(\pi^2/\sqrt{R} - 1)]$. Eckhaus [3] has considered the problem by assuming $u = \sum_0^\infty A_n u_n$, where u_n are the eigenfunctions of the linearized problem

$$u_n = e^{-1/2\sqrt{R}x^2} \sin (n + 1)\pi x.$$

In many problems it will not be possible to use Theorem 2 directly. However we may use the theorem after first obtaining the equations satisfied by the amplitude functions. This approach will be illustrated in the next example.

Example III. Davey, Di Prima, and Stuart [2] have shown that the amplitude equations to the third order which provide a model for the transition (with increasing Taylor number, T) from Couette flow to Taylor-vortex flow, are

$$\frac{dx}{dt} = \epsilon X - X^3 - XY^2 - 6X\,|Z|^2 - 2X\,|V|^2 - 2Y(Z\tilde{V} + \tilde{Z}V)$$

$$\frac{dy}{dt} = \epsilon Y - Y^3 - YX^2 - 6Y\,|V|^2 - 2Y\,|Z|^2 - 2X(Z\tilde{V} + \tilde{Z}V)$$

$$\frac{dz}{dt} = \sigma Z - 3Z\,|Z|^2 - 2Z\,|V|^2 - 3ZX^2 - \gamma ZY^2 - (3 - \gamma)VXY - \tilde{Z}V^2$$

$$\frac{dv}{dt} = \sigma V - 3V\,|V|^2 - 2V\,|Z|^2 - 3VY^2 - \gamma VX^2 - (3 - \gamma)ZXY - \tilde{V}Z^2$$

where~ denotes complex conjugate, γ is a complex number whose real part γ_r is in the range $0 < \gamma_r \leq 3$. The functions X, Y are real and Z and V are complex. The parameter ϵ is real, but σ is complex, and $\epsilon > \sigma_r$.

The laminar Couette flow corresponds to $X \equiv Y \equiv Z \equiv V = 0$ and $\epsilon \lessgtr 0$ for $T \lessgtr T_c$ where T_c is the critical Taylor number. Stuart [9] has constructed a Liapunov functional L for the system. He considers

$$\rho_0^2 = \rho^2 = L = X^2 + Y^2 + |Z|^2 + |V|^2$$

and so

$$\frac{dL}{dt} = \epsilon L - L^2 - (\epsilon - \sigma_r)(|Z|^2 + |V|^2) - \gamma_r(X^2 |V|^2 + Y^2 |Z|^2) - 2(|Z|^4 + |V|^4)$$
$$- (\tilde{Z}V + Z\tilde{V})^2 - 7(X^2 |Z|^2 + Y^2 |V|^2) + (7 - \gamma_r)XY(Z\tilde{V} + \tilde{Z}V).$$

Therefore

$$dL/dt \leq \epsilon L - L^2.$$

By Theorem 1 we have for $\epsilon < 0$(i.e. $T < T_c$) the laminar Couette flow is asymptotically stable. For $T > T_c$, i.e. $\epsilon > 0$, we may use Theorem 2; then for $0 < \rho_0^2 < \epsilon$ we have $\rho^2 \leq \epsilon$ for all time; for $\rho_0^2 \geq \epsilon$ we have $\rho^2 \leq \rho_0^2$ and $\rho^2 \to \epsilon$ as $t \to \infty$ or $\rho^2 < \epsilon$.

We have therefore obtained estimates for the bounds of the amplitude functions. These estimates have been obtained by assuming that the amplitude equations may be truncated at the third order. Stuart [9] makes use of these estimates in justifying this truncation.

Conclusions. It has been shown that Theorem 2 may be useful in estimating bounds for the solution of nonlinear problems, when the linearized problem is unstable. Much depends on the ability to construct a functional V satisfying certain conditions. In many problems it will not be possible to construct a functional by dealing directly with the equations of motion. However, by assuming the solution may be expanded in a Fourier series, or a series of eigenfunctions of the linearized problem, it is possible to obtain the amplitude equations and to construct a functional from these equations.

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