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CONTINUOUS DEPENDENCE AND DIFFERENTABILITY PROPERTIES OF THE SOLUTION OF A FREE BOUNDARY PROBLEM FOR THE HEAT EQUATION*

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1. Introduction. The continuous dependence of the free boundary on the data in the following problem has been proved by Cannon and Douglas [1]:

$$\begin{gathered} u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x), \qquad u_x(0, t) = f(t), \\ u(s(t), t) = 0, \qquad -\lambda s'(t) + u_x(s(t), t) = 0, \qquad s(0) = a. \end{gathered} \tag{1.1}$$

Here $f(t) \geq 0$, $\phi(x) \leq 0$, $\phi(a) = 0$, $a \geq 0$. Thus the region $0 \leq x \leq a$ is initially solid with temperature distribution $\phi(x)$, and the region $a < x < \infty$ is liquid at the melting temperature 0. There is an outward flux of heat $f(t)$ at the fixed boundary $x = 0$ (we assume the thermal conductivity k, which should appear as a coefficient of $u_x(0, t)$, has been absorbed into $f(t)$). We have $\lambda = \rho l/k$, where ρ is the common density of liquid and solid and l is the latent heat. Let $s_k(t)$ be the free boundary corresponding to the data $f_k(t)$, $\phi_k(x)$, and a_k, $k = 1, 2$. We assume $a_2 \geq a_1$. Then Cannon and Douglas prove, under appropriate conditions on $f(t)$ and $\phi(x)$,

$$\begin{aligned} |s_1(t) - s_2(t)| \leq C\Big[a_2 - a_1 + \int_0^{a_1} |\phi_1(x) - \phi_2(x)|\,dx + \int_{a_1}^{a_2} \phi_2(x)\,dx \\ + \int_0^t |f_1(\tau) - f_2(\tau)|\,d\tau \Big], \qquad 0 \leq t \leq T. \end{aligned} \tag{1.2}$$

Here C depends on T and B, where the latter is the maximum of $||f_k||$ and $||\phi_k'||$, $k = 1, 2$, the norms being taken ove $0 \leq t \leq T$ and $0 \leq x \leq x \leq a_k$. If we replace the flux condition at the boundary $x = 0$ in (1.1) by $u(0, t) = f(t)$, where $f(t) \leq 0$, then Cannon and Hill [2] have proved the stability of the free boundary for that problem.

If we introduce a flux term $q(t) \geq 0$ at the free boundary directed into the solid we have the problem

$$\begin{gathered} u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x), \qquad u_x(0, t) = f(t), \\ u(s(t), t) = 0, \qquad -\lambda s'(t) + u_x(s(t), t) = q(t), \qquad s(0) = a. \end{gathered} \tag{1.3}$$

Here $a > 0$. While the free boundary in (1.1) is a nondecreasing function of t, this is not

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true of (1.3). In (1.3) $s(t)$ is, in general, neither nondecreasing nor nonincreasing. Furthermore the free boundary in (1.3) may reach $x = 0$. This occurs if there is a smallest time $t = \sigma$ such that $\lambda a + H(\sigma) = 0$, where

$$H(t) = \int_0^t [f(\tau) - q(\tau)] \, d\tau - \int_0^a \phi(x) \, dx. \tag{1.4}$$

If there is such a σ then $s(\sigma) = 0$ and $s(t) > 0$ for $t < \sigma$, otherwise $s(t) > 0$ for all t. An existence and uniqueness theorem for (1.3) has been proved in [6] under the hypothesis that $f(t)$, $q(t)$, and $\phi'(x)$ are continuous. In this paper we use the methods of [1], [2] to prove the continuous dependence of the solution of (1.3) on the data. The regularity hypotheses are the same as those for the existence and uniqueness theorem, but the restriction on the sign of $q(t)$ is unnecessary; i.e., the existence and uniqueness proof does not require that sign restriction nor does the continuous dependence proof. We get the inequality (2.2) below.

It has been proved by Jiang Li-Shang [5] that the free boundary of (1.1) is infinitely differentiable. We cannot expect this to be true of (1.3) since, by the second of the free boundary conditions, the regularity of the free boundary is tied to that of $q(t)$. Thus we know, from the existence theorem, that $q \in C^0$ implies $s \in C^1$, and more generally we expect that $q \in C^k$ implies $s \in C^{k+1}$. In this paper we prove the following theorem: if $f'(t)$, $q'(t)$, and $\phi''(x)$ are continuous then $s''(t)$ and $v'(t)(v(t) = u_x(s(t), t))$ are continuous on $0 < t < \sigma$; furthermore $t^{1/2}s''(t)$ and $t^{1/2}v'(t)$ are continuous on $0 \leq t < \sigma$ and $u_{xx}(x, t)$ has a finite limit at each boundary point except possibly $(0, 0)$ and $(a, 0)$. With these additional conditions of regularity on the data we can prove that $s'(t)$ and $u_x(x, t)$ depend continuously on the data. We do not give the proof here but refer the interested reader to the report [7].

In the proof of existence and uniqueness of the solution of (1.3) given in [6] use is made of continuous dependence in Lemma 1, which states that $u(x, t) \leq 0$. A proof that $u(x, t) \leq 0$ for any solution of (1.3), which does not use continuous dependence, is easily constructed. Suppose $u(x_1, t_1) > 0$. It is clear from the representation of u given by (3.2) that u is continuous on $0 \leq t < \sigma$, $0 \leq x \leq s(t)$. Since u is 0 on $x = s(t)$ and nonpositive on $t = 0$, by the maximum principle u achieves a positive maximum on $x = 0$, say at τ, where $\tau \leq t_1$. If $\tau < t_1$ then by a lemma of Friedman [4, p. 49] $u_x(0, \tau) < 0$ and this contradicts $u_x(0, \tau) = f(\tau) \geq 0$. If $\tau = t_1$ then the inside sphere property does not hold at the corner point $(0, t_1)$ but we can still conclude that $f(t_1) = u_x(0, t_1) \leq 0$, from which we conclude that $f(t_1) = 0$. If $f(t_1)$ is 0 we have no contradiction. But the argument above shows that $f(t) = 0$ wherever $u(x, t) > 0$. Hence if t_0 is the infimum of those values of $t \leq t_1$ for which there is some point (x, t) for which $u(x, t) > 0$ then $f(t) = 0$ for $t_1 \geq t \geq t_0$. Now $u(x, t_0) \leq 0$, $u(s(t), t) = 0$ and since $u_x(0, t) = f(t) = 0$ for $t_0 \leq t \leq t_1$ we may reflect the solution $u(x, t)$, $s(t)$ across the t axis. This extended solution is 0 on $x = \pm s(t)$ and nonpositive on $t = t_0$ and thus cannot be positive in the interior, a contradiction.

2. Continuous dependence of the solution of (1.3) **on the data.** Let $T < \sigma$ and let $||g||$ and $||g||_\epsilon$ be the norms of any continuous function $g(t)$ on $0 \leq t \leq T$ and $\epsilon \leq t \leq T$. Let $||\phi'||$ and $u_1(t)$ be norms of $\phi'(x)$ and $u_x(x, t)$ on $0 \leq x \leq a$ and $0 \leq x \leq s(t)$, and let $||u_x||$ and $||u_x||_\epsilon$ be the norms of $u_x(x, t)$ on $0 \leq t \leq T$, $0 \leq x \leq s(t)$ and $\epsilon \leq t \leq T$, $0 \leq x \leq s(t)$. Then we have the following lemma.

LEMMA. *Define n, $\gamma > 0$, and B by*

$$n = [||q||^2 T/\pi\lambda^2] + 1, \qquad \gamma = [1 - \lambda^{-1} ||q|| (T/n\pi)^{1/2}]^{-1},$$
$$B = ||f|| \gamma[(4\gamma)^n - 1]/(4\gamma - 1) + (4\gamma)^n \max (||\phi'||, f(0)),$$

where, in the definition of n, $[x]$ is the largest integer $\leq x$. Then $||v|| \leq B$, $||u_x|| \leq B$, and $0 \geq u(x, t) \geq B(x - s(t))$.

This lemma corresponds to (1.4) and (1.5) of [1] and to Lemma 2 of [2]. The regularity conditions on the data are the same as for the existence theorem. We divide the interval $\epsilon \leq t \leq T$ into n equal parts, $h = (T - \epsilon)/n$, and define $||v||_k = \max v(t)$ on $\epsilon + (k - 1)h \leq t \leq \epsilon + kh$ (we note that $v(t) \geq 0$). From the inequality (33) of [6]

$$||v||_1 < \zeta[||f|| + 4u_1(\epsilon)], \qquad \zeta = [1 - \lambda^{-1} ||q|| (h/\pi)^{1/2}]^{-1}.$$

Since $u_x(x, t)$ is continuous on $\epsilon \leq t \leq \epsilon + h$, $0 \leq x \leq s(t)$ the maximum principle implies (it is clear from (3.2) below that $u_x(x, t)$ also satisfies the heat equation on $0 < x < s(t)$, $0 < t \leq T$)

$$u_1(\epsilon + h) \leq \max (||f||, u_1(\epsilon), ||v||_1) \leq \zeta[||f|| + 4u_1(\epsilon)].$$

Again from (33) of [6]

$$||v||_2 \leq \zeta[||f|| + 4u_1(\epsilon + h)] \leq ||f||(\zeta + 4\zeta^2) + (4\zeta)^2 u_1(\epsilon).$$

It follows by induction that

$$||v||_k \leq ||f||\zeta[(4\zeta)^k - 1]/(4\zeta - 1) + (4\zeta)^k u_1(\epsilon).$$

Thus

$$||v||_\epsilon = \max (||v||_1 , ||v||_2 , \cdots , ||v||_n) \leq ||f||\zeta[(4\zeta)^n - 1]/(4\zeta - 1) + (4\zeta)^n u_1(\epsilon). \tag{2.1}$$

By the maximum principle

$$||u_x||_\epsilon = \max (||f||_\epsilon , u_1(\epsilon), ||v||_\epsilon).$$

Then $||u_x||_\epsilon$ is less than or equal to the right side of (2.1). As $\epsilon \to 0$ $u_x(x, \epsilon) \to \phi'(x)$ for $x \neq 0$, $u_x(0, \epsilon) = f(\epsilon) \to f(0)$, $\zeta \to \gamma$, $||v||_\epsilon \to ||v||$, $||f||_\epsilon \to ||f||$, $||u_x||_\epsilon \to ||u_x||$, and the right side of (2.1) tends to B. Thus $||v|| \leq B$ and $||u_x|| \leq B$. Finally

$$0 \geq u(x, t) = -\int_x^{s(t)} u_\xi(\xi, t)\, d\xi \geq -\int_x^{s(t)} B\, d\xi = B(x - s(t)).$$

We can now prove the following continuous dependence theorem.

THEOREM 1. *Let $u^k(x, t)$, $s_k(t)$ be the solution of* (1.3) *corresponding to the data $f_k(t)$, $q_k(t)$, $\phi_k(x)$, and a_k, $k = 1, 2$. Let $a_2 \geq a_1$. Then, on $0 \leq t \leq T$,*

$$||s_1 - s_2||_t \leq C\left[\lambda(a_2 - a_1) + \int_0^{a_1} |\phi_1(x) - \phi_2(x)|\, dx + \int_{a_1}^{a_2} \phi_2(x)\, dx \right.$$
$$\left. + \int_0^t (|f_1(\tau) - f_2(\tau)| + |q_1(\tau) - q_2(\tau)|)\, d\tau\right], \tag{2.2}$$

where $||s_1 - s_2||_t = \max |s_1(\tau) - s_2(\tau)|$ on $0 \leq \tau \leq t$ and C is given by

$$C = 2\lambda^{-1}[1 + 4T^{1/2}C_1] \exp (4\pi C_1^2 T), \qquad C_1 = \pi^{-1/2}B[1 + B(T/\pi)^{1/2}] \exp (B^2T/4).$$

On the common domain of definition of $u^1(x, t)$ *and* $u^2(x, t)$ *we have*

$$|u^1(x, t) - u^2(x, t)| \leq ||\phi_1 - \phi_2|| + 4t^{1/2} ||f_1 - f_2||_t + B ||s_1 - s_2||_t . \tag{2.3}$$

To prove the theorem we proceed as in [1]. We obtain the following equation by integrating $u_{xx} = u_t$ over the domain $0 < x < s(\tau)$, $0 < \tau < t$:

$$s(t) = a + \lambda^{-1} \int_0^t [f(\tau) - q(\tau)]\, d\tau - \lambda^{-1} \int_0^a \phi(x)\, dx + \lambda^{-1} \int_0^{s(t)} u(x, t)\, dx. \tag{2.4}$$

Let $\alpha(t)$ and $\beta(t)$ be, respectively, the minimum and maximum of $s_1(t)$ and $s_2(t)$ and let $\delta(t) = \beta(t) - \alpha(t)$. Then from (2.4) we get

$$\begin{aligned} \delta(t) \leq a_2 - a_1 &+ \lambda^{-1} \left| \int_0^t [f_1(\tau) - f_2(\tau) - q_1(\tau) + q_2(\tau)]\, d\tau \right| \\ &+ \lambda^{-1} \left| \int_0^{a_1} [\phi_1(x) - \phi_2(x)]\, dx \right| - \lambda^{-1} \int_{a_1}^{a_2} \phi_2(x)\, dx \\ &+ \lambda^{-1} \left| \int_0^{\alpha(t)} [u^1(x, t) - u^2(x, t)]\, dx \right| + \lambda^{-1} \int_{\alpha(t)}^{\beta(t)} u^j(x, t)\, dx, \end{aligned} \tag{2.5}$$

where $j = 2$ if $\alpha = s_1$ and $\beta = s_2$ and $j = 1$ otherwise. From the lemma we get

$$-u^j(\alpha(t), t) = |u^1(\alpha(t), t) - u^2(\alpha(t), t)| \leq B\, \delta(t). \tag{2.6}$$

On $0 < x < \alpha(t)$, $0 < t \leq T$ we write

$$u^1(x, t) - u^2(x, t) = v^1(x, t) + v^2(x, t) + v^3(x, t) \tag{2.7}$$

where each v^k satisfies the heat equation and the following boundary and initial conditions:

$$\begin{aligned} &v_x^1(0, t) = 0, \quad v_1(\alpha(t), t) = 0, \quad v^1(x, 0) = \phi_1(x) - \phi_2(x), \\ &v_x^2(0, t) = f_1(t) - f_2(t), \quad v^2(\alpha(t), t) = 0, \quad v^2(x, 0) = 0, \\ &v_x^3(0, t) = 0, \quad v^3(\alpha(t), t) = u^1(\alpha(t), t) - u^2(\alpha(t), t), \quad v^3(x, 0) = 0. \end{aligned}$$

The argument now proceeds precisely as in [1] to yield the inequality (2.2). Since we may reflect $v^1(x, t)$ across the t axis, an application of the maximum principle shows that $|v^1| \leq ||\phi_1 - \phi_2||$, the norm referring to $0 \leq x \leq a_1$. If $v^\pm(x, t)$ are solutions of the heat equation in the domain $0 < t < T$, $x > 0$ corresponding to the conditions

$$v_x^\pm(0, t) = \mp |f_1(t) - f_2(t)|, \qquad v^\pm(x, 0) = 0,$$

then we have $v^+(x, t) \leq v^2(x, t) \leq v^-(x, t)$ in the domain $0 < t < T$, $0 < x < \alpha(t)$. Since

$$v^\pm(x, t) = 2 \int_0^t \mp |f_1(\tau) - f_2(\tau)|\, K(x, t; 0, \tau)\, d\tau,$$

where K is the fundamental solution of the heat equation (see Sec. 3), we have $|v^2| \leq 4t^{1/2} ||f_1 - f_2||_t$, the norm referring to $0 \leq \tau \leq t$. We may reflect $v^3(x, t)$across the t axis and derive, from the maximum principle and (2.6), $|v^3| \leq B\, ||\delta||_t$. Thus (2.3) is proved. It is easily seen that Theorem 1 also implies continuous dependence on λ.

3. Differentiability properties of the solution of (1.3). The fundamental solution of the heat equation and the Green's and Neumann's function for the first quadrant are

$$K(x, t; \xi, \tau) = [4\pi(t - \tau)]^{-1/2} \exp[-(x - \xi)^2/4(t - \tau)],$$
$$G(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(-x, t; \xi, \tau),$$
$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau).$$

It is shown in [6] that (1.3) is equivalent to

$$v(t) = 2\int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau - 2\int_0^t f(\tau)N_x(s(t), t; 0, \tau)\, d\tau + 2\int_0^a \phi'(\xi)G(s(t), t; \xi, 0)\, d\xi, \tag{3.1a}$$

$$s(t) = a - \int_0^t q(\tau)\, d\tau + \int_0^t v(\tau)\, d\tau. \tag{3.1b}$$

More precisely, if $u(x, t)$, $s(t)$ is a solution of (1.3) on $0 \le t \le T$, $0 \le x \le s(t)$ then $v(t)$, $s(t)$ is a solution of (3.1a, b) on $0 \le t \le T$, and conversely, if $v(t)$, $s(t)$ is a solution of (3.1a, b) on $0 \le t \le T$ then $s(t)$ together with $u(x, t)$, defined by

$$u(x, t) = \int_0^t v(\tau)N(x, t; s(\tau), \tau)\, d\tau - \int_0^t f(\tau)N(x, t; 0, \tau)\, d\tau + \int_0^a \phi(\xi)N(x, t; \xi, 0)\, d\xi, \tag{3.2}$$

is a solution of (1.3) on $0 \le t \le T$, $0 \le x \le s(t)$. The existence theorem proved in [6] shows that if $f(t)$, $q(t)$, and $\phi'(x)$ are continuous then $v(t)$ and $s'(t)$ are continuous. The following theorem shows the implications, for (1.3), of one additional degree of regularity on the data.

THEOREM 2. *Let $f(t)$ and $q(t)$ have continuous first derivatives on $t \ge 0$ and let $\phi(x)$ have a continuous second derivative on $0 \le x \le a$. Then, for any $T < \sigma$, $v'(t)$ and $s''(t)$ exist and are continuous on $0 < t \le T$, $t^{1/2}v'(t)$ and $t^{1/2}s''(t)$ are continuous on $0 \le t \le T$, and $u_{xx}(x, t)$ has a finite limit at each boundary point except possibly* (0, 0) *and* (a, 0).

We know that (1.3) and (3.1*a, b*) are equivalent. If we assume $v'(t)$ exists then

$$\frac{d}{dt}\int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau = v(0)N_x(s(t), t; a, 0) + \int_0^t \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial t}\right)v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau. \tag{3.3}$$

We obtain (3.3) by adding the two equations (3.4) below, letting $\epsilon \to 0$, and differentiating with respect to t.

$$\begin{aligned}\int_\epsilon^t \int_0^{\eta-\epsilon} \frac{\partial}{\partial\tau} v(\tau)N_x(s(\eta), \eta; s(\tau), \tau)\, d\tau\, d\eta &= \int_\epsilon^t \{v(\eta - \epsilon)N_x(s(\eta), \eta; s(\eta - \epsilon), \eta - \epsilon) - v(0)N_x(s(\eta), \eta; a, 0)\}\, d\eta, \\ \int_\epsilon^t \int_0^{\eta-\epsilon} \frac{\partial}{\partial\eta} v(\tau)N_x(s(\eta), \eta; s(\tau), \tau)\, d\tau\, d\eta &= \int_0^{t-\epsilon} \{v(\tau)N_x(s(t), t; s(\tau), \tau) - v(\tau)N_x(s(\tau + \epsilon), \tau + \epsilon; s(\tau), \tau)\}\, d\tau.\end{aligned} \tag{3.4}$$

Next we get (3.5) below by differentiating under the integral on the left, using $G_{xx} = G_{\xi\xi}$, $G_x = -N_\xi$, and integrating partially:

$$\frac{d}{dt}\int_0^a \phi'(\xi)G(s(t), t; \xi, 0)\,d\xi$$

$$= \phi'(a)G_\xi(s(t), t; a, 0) - \phi'(0)G_\xi(s(t), t; 0, 0) - \int_0^a \phi''(\xi)G_\xi(s(t), t; \xi, 0)\,d\xi$$

$$- s'(t)[\phi'(a)N(s(t), t; a, 0) - \phi'(0)N(s(t), t; 0, 0] + s'(t)\int_0^a \phi''(\xi)N(s(t), t; \xi, 0)\,d\xi. \tag{3.5}$$

Finally we have

$$\frac{d}{dt}\int_0^t f(\tau)N_x(s(t), t; 0, \tau)\,d\tau = f(0)N_x(s(t), t; 0, 0) + \int_0^t \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial t}\right)f(\tau)N_x(s(t), t; 0, \tau)\,d\tau. \tag{3.6}$$

Equation (3.6) is proved in the same way as (3.3). On differentiating (3.1a, b) and using $v(0) = \phi'(a)$ we get

$$\begin{aligned} v'(t) = {} & -2s'(t)[\phi'(a)N(s(t), t; a, 0) - \phi'(0)N(s(t), t; 0, 0)] \\ & + 2[\phi'(0) - f(0)]N_x(s(t), t; 0, 0) \\ & + 2\int_0^t \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial t}\right)v(\tau)N_x(s(t), t; s(\tau), \tau)\,d\tau \\ & + 2\int_0^t \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial t}\right)f(\tau)N_x(s(t), t; 0, \tau)\,d\tau \\ & - 2\int_0^a \phi''(\xi)[G_\xi(s(t), t; \xi, 0) - s'(t)N(s(t), t; \xi, 0)]\,d\xi, \end{aligned} \tag{3.7a}$$

$$s'(t) = \lambda^{-1}(v(t) - q(t)), \qquad s(0) = a. \tag{3.7b}$$

Conversely on integrating (3.7a, b), using $v(0) = \phi'(\mathrm{a})$, we get (3.1a, b). Thus, if (3.7a, b) has a continuous solution $v'(t)$, $s(t)$ (then $s'(t)$, $s''(t)$ exist and are continuous) then (3.1*a*, *b*) has a solution $v(t)$, $s(t)$ such that $v'(t)$, $s'(t)$, $s''(t)$ exist and are continuous.

It is now necessary to prove that the system (3.7a, b), together with $v(0) = \phi'(a)$, has a unique solution. The uniqueness of the solution of (3.7a, b) follows from the uniqueness of the solution of (3.1a, b) since a solution of (3.7a, b) is also a solution of (3.1a, b). The uniqueness of the solution of (3.1a, b) has been proved in [6]. Turning to the question of existence we follow the procedure in [3, II] and eliminate the singularity $t^{-1/2}$ in (3.7a) by introducing $V(t) = t^{1/2}v'(t)$. The system assumes the following form:

$$\begin{aligned} V(t) = {} & -2s'(t)t^{1/2}[\phi'(a)N(s(t), t; a, 0) - \phi'(0)N(s(t), t; 0, 0)] \\ & + 2t^{1/2}[\phi'(0) - f(0)]N_x(s(t), t; 0, 0) \\ & - t^{1/2}\int_0^t \left\{\left[\tau^{-1/2}V(\tau)\,\frac{s(t) - s(\tau)}{t - \tau} + v(\tau)\,\frac{s'(t) - s'(\tau)}{t - \tau}\right.\right. \\ & \left. - v(\tau)\,\frac{(s(t) - s(\tau))^2(s'(t) - s'(\tau))}{2(t - \tau)^2}\right]K(s(t), t; s(\tau), \tau) \end{aligned}$$

$$+ \left[\tau^{-1/2} V(\tau) \frac{s(t) + s(\tau)}{t - \tau} + v(\tau) \frac{s'(t) + s'(\tau)}{t - \tau}\right.$$

$$\left. - v(\tau) \frac{(s(t) + s(\tau))^2 (s'(t) + s'(\tau))}{2(t - \tau)^2}\right] K(-s(t), t; s(\tau), \tau)\Big\} \, d\tau$$

$$- t^{1/2} \int_0^t \left\{\frac{f'(\tau)s(t)}{t - \tau} + \frac{f(\tau)s'(t)}{t - \tau} - \frac{f(\tau)s^2(t)s'(t)}{2(t - \tau)^2}\right\} K(s(t), t; 0, \tau) \, d\tau$$

$$- 2t^{1/2} \int_0^a \phi''(\xi)[G_\xi(s(t), t; \xi, 0) - s'(t)N(s(t), t; \xi, 0)] \, d\xi, \tag{3.8a}$$

$$v(t) = \phi'(a) + \int_0^t \tau^{-1/2} V(\tau) \, d\tau, \tag{3.8b}$$

$$s'(t) = \lambda^{-1}(v(t) - q(t)), \qquad s(0) = a. \tag{3.8c}$$

We prove now three lemmas which together prove the theorem. The first of these lemmas states that there is a solution of (3.8a, b, c) for sufficiently small t.

LEMMA 1. *There is a t_0 such that there is a continuous solution $V(t)$, $v(t)$, $s'(t)$ of* (3.8a, b, c) *on $0 \le t \le t_0$ (it follows from* (3.8b, c) *that $t^{1/2}s''(t)$ is continuous on $0 \le t \le t_0$).*

To prove the lemma we define the Banach space $C(t_0)$ of continuous functions $V(t)$ on $0 \le t \le t_0$ with norm $\|V(t)\| = \max |V(t)|$. Here t_0 is to be determined. Let $C(t_0, M)$ be the closed sphere of functions satisfying $\|V\| \le M$. Then (3.8a), together with (3.8b, c), defines a mapping $W = S(V)$ of $C(t_0, M)$ into $C(t_0)$. From (3.8b, c) we get

$$\|v\| \le \phi'(a) + 2t_0^{1/2}M = m_1, \tag{3.9}$$

$$\|s'\| \le \lambda^{-1}(m_1 + \|q\|) = m_2, \tag{3.10}$$

$$|s(t) - s(\tau)| \le m_2(t - \tau). \tag{3.11}$$

Here all norms of functions of t, and all appearing later in the proof of this lemma, are taken over the interval $0 \le t \le t_0$. We select M and t_0 subject to

$$2m_2 t_0 = 2\lambda^{-1}(\phi'(a) + 2t_0^{1/2}M + \|q\|)t_0 \le a. \tag{3.12}$$

Then from (3.11), taking $\tau = 0$,

$$a/2 \le s(t) \le 3a/2. \tag{3.13}$$

We note that since $s''(t) = \lambda^{-1}(V(t)/t^{1/2} - q(t))$

$$\begin{aligned} \frac{s'(t) - s'(\tau)}{t - \tau} &\le \lambda^{-1}(M/\tau^{1/2} + \|q'\|) \\ &\le \lambda^{-1}\tau^{-1/2}(M + t_0^{1/2}\|q'\|) = m_3\tau^{-1/2}, \end{aligned} \tag{3.14}$$

for $\tau \le t$. We can now derive upper bounds for the absolute value of each of the terms appearing on the right of (3.8a); in the derivation of these bounds we use

$$x \exp(-ax) \le (ae)^{-1}, \qquad \int_0^t [\tau(t - \tau)]^{-1/2} \, d\tau = \pi.$$

We write $\delta = |\phi'(0) - f(0)|$ and write the bounds, in (3.15) below, in the same order as the terms appear in (3.8a).

$$|W| \le 2m_2\,||\phi'|| + 6a^{-1}\,\delta + t_0^{1/2}(Mm_2 + m_1m_3 + m_1m_2^3t_0^{1/2})$$
$$+ t_0^{1/2}\int_0^t \left[\frac{3Ma}{\tau^{1/2}(t-\tau)} + \frac{2m_1m_2}{t-\tau} + \frac{9m_1m_2a^2}{(t-\tau)^2}\right]K(a, t; 0, \tau)\,d\tau$$
$$+ t_0^{1/2}\int_0^t \left[\frac{3a\,||f'||}{2(t-\tau)} + \frac{m_2\,||f||}{t-\tau} + \frac{9a^2m_2\,||f||}{8(t-\tau)^2}\right]K(a/2, t; 0, \tau)\,d\tau$$
$$+ 2\,||\phi''|| + 4t_0^{1/2}\,||\phi''||\,m_2\,. \tag{3.15}$$

In the first term in the first integral we introduce the change of variable $\tau = t\zeta$. Then that term is equal to

$$\left(\frac{t_0}{4\pi}\right)^{1/2}\int_0^1 \frac{3Ma}{\zeta^{1/2}(1-\zeta)^{3/2}t}\exp\left(-\frac{a^2}{4t(1-\zeta)}\right)d\zeta \le t_0^{1/2}\int_0^1 \frac{Ma}{\zeta^{1/2}(1-\zeta)^{3/2}}\,\frac{4(1-\zeta)}{a^2e}\,d\zeta$$
$$\le 5Ma^{-1}t_0^{1/2}.$$

In the other five terms in the two integrals we introduce the change of variable $x = (t-\tau)^{-1}$. Then t no longer appears explicitly in the integrands. The limits of integration are t^{-1} and ∞, and if we replace t^{-1} by t_0^{-1} we increase each of the five terms. With this replacement let $B(t_0)$ be the sum of these five terms; $B(t_0)$ is a decreasing function of a. Then we have from (3.15) $||W|| \le D + t_0^{1/2}R$, where

$$D = 2\lambda^{-1}(\phi'(a) + ||q||)\,||\phi'|| + 6a^{-1}\,\delta + 2\,||\phi''||,$$
$$R = B(t_0) + 5Ma^{-1} + 4\,||\phi''||\,m_2 + m_1m_2^3t_0^{1/2} + m_1m_3 + Mm_2 + 4\lambda^{-1}M\,||\phi'||.$$

We choose $M = D + 1$; M depends on t_0 through $||q||$. We choose t_0 so that $t_0^{1/2}R < 1$. We note also that (3.12) has to be satisfied. Then $||W|| \le M$ and therefore $W = s(V)$ maps $C(t_0, M)$ into itself.

We prove now that $W = s(V)$ is a contraction for appropriately chosen t_0. Let $W_k = s(V_k)$, $k = 1, 2$, $\epsilon = ||V_1 - V_2||$. Then we prove that $||W_1 - W_2|| \le t_0^{1/2}F\epsilon$, where F is a function of the quantities

$$a,\ t_0,\ \delta,\ \phi'(a),\ ||\phi'||,\ ||\phi''||,\ ||f||,\ ||f'||,\ \lambda^{-1},\ M,\ m_1,\ m_2,\ m_3,$$

which is continuous in t_0 for $t_0 \ge 0$. Thus by choosing t_0 subject to $t_0^{1/2}F < 1$ as well as $t_0^{1/2}R < 1$ and (3.12) the mapping $W = S(V)$ of $C(t_0, M)$ into itself is a contraction. From (3.8b, c) we derive the following inequalities

$$|v_1(t) - v_2(t)| \le 2t^{1/2}\epsilon \le 2t_0^{1/2}\epsilon, \qquad ||v_1 - v_2|| \le 2t_0^{1/2}\epsilon,$$
$$|s_1'(t) - s_2'(t)| \le 2\lambda^{-1}t^{1/2}\epsilon \le 2\lambda^{-1}t_0^{1/2}\epsilon, \qquad ||s_1' - s_2'|| \le 2\lambda^{-1}t_0^{1/2}\epsilon,$$
$$|s_1(t) - s_2(t)| \le 2\lambda^{-1}t^{3/2}\epsilon \le 2\lambda^{-1}t_0^{3/2}\epsilon, \qquad ||s_1 - s_2|| \le 2\lambda^{-1}t_0^{3/2}\epsilon.$$

Considering now the difference of the first terms of $W_1 = S(V_1)$, $W_2 = S(V_2)$ we have

$$2t^{1/2}(s_1'(t) - s_2'(t))[\phi'(a)N(s_1(t), t; a, 0) - \phi'(0)N(s_1(t), t; 0, 0)]$$
$$+ 2t^{1/2}s_2'(t)[\phi'(a)N(s_1(t), t; a, 0) - \phi'(0)N(s_1(t), t; 0, 0)$$
$$- \{\phi'(a)N(s_2(t), t; a, 0) - \phi'(0)N(s_2(t), t; 0, 0\}]. \tag{3.16}$$

The absolute value of the first term of (3.16) is $\leq 4\lambda^{-1} \|\phi'\| t_0^{1/2}\epsilon$. Using the mean value theorem the second term of (3.16) is

$$2t^{1/2}s_2'(t)(s_1(t) - s_2(t))\{\phi'(a)N_x(\sigma(t), t; a, 0) - \phi'(0)N_x(\sigma(t), t; 0, 0)\} \tag{3.17}$$

where $\sigma(t)$ lies between $s_1(t)$ and $s_2(t)$. Thus the absolute value of (3.17) is $\leq 12a \|\phi'\| m_2\lambda^{-1}t_0^{1/2}\epsilon$ and therefore the absolute value of (3.16) is $\leq (1 + 3am_2)4\lambda^{-1} \|\phi'\| t_0^{1/2}\epsilon$. The difference of the second terms is

$$\begin{aligned} &2t^{1/2}(\phi'(0) - f(0))[N_x(s_1(t), t; 0, 0) - N_x(s_2(t), t; 0, 0)] \\ &\qquad = 2t^{1/2}(\phi'(0) - f(0))(s_1(t) - s_2(t))N_{xx}(\sigma(t), t; 0, 0). \end{aligned} \tag{3.18}$$

Using (3.13), which is true for $\sigma(t)$ also, and $x \exp(-ax) \leq (ae)^{-1}$, the absolute value of (3.18) is $\leq 8\lambda^{-1} \delta t_0^{1/2}\epsilon$. The difference of the third terms we write as follows:

$$\begin{aligned} -t^{1/2}\Bigg\{&\int_0^t \tau^{-1/2}(V_1(\tau) - V_2(\tau))\frac{s_1(t) - s_1(\tau)}{t - \tau} K(s_1(t), t; s_1(\tau), \tau)\, d\tau \\ &+ \int_0^t \tau^{-1/2}V_2(\tau)\left[\frac{s_1(t) - s_1(\tau)}{t - \tau} - \frac{s_2(t) - s_2(\tau)}{t - \tau}\right]K(s_1(t), t; s_1(\tau), \tau)\, d\tau \\ &+ \int_0^t \tau^{-1/2}V_2(\tau)\frac{s_2(t) - s_2(\tau)}{t - \tau}[K(s_1(t), t; s_1(\tau), \tau) - K(s_2(t), t; s_2(\tau), \tau]\, d\tau\Bigg\}. \end{aligned} \tag{3.19}$$

The absolute value of the first term of (3.19) if $\leq t_0^{1/2}m_2\epsilon$. The bracket in the second term can be written $s_1'(t_1) - s_2'(t_1)$, where $\tau \leq t_1 \leq t$, so that the absolute value of the second term of (3.19) is $\leq 2\lambda^{-1}Mt_0\epsilon$. The bracket in the third term can be written

$$\begin{aligned} &\{s_1(t) - s_1(\tau) - [s_2(t) - s_2(\tau)]\}K_x(x_1, t; 0, \tau) \\ &\qquad = \left\{\frac{s_1(t) - s_1(\tau)}{t - \tau} - \frac{s_2(t) - s_2(\tau)}{t - \tau}\right\}\frac{x_1}{4[\pi(t - \tau)]^{1/2}}\exp\left(-\frac{x_1^2}{4(t - \tau)}\right), \end{aligned}$$

where x_1 lies between $s_1(t) - s_1(\tau)$ and $s_2(t) - s_2(\tau)$, so that the absolute value of the third term of (3.19) is $\leq 3a\lambda^{-1}Mm_2t_0\epsilon$. Thus the absolute value of (3.19) is less than or equal to

$$\epsilon t_0^{1/2}(m_2 + 2\lambda^{-1}t_0^{1/2}M + 3a\lambda^{-1}Mm_2t_0^{1/2}).$$

The difference of the fourth terms can be written

$$\begin{aligned} -t^{1/2}\Bigg\{&\int_0^t (v_1(\tau) - v_2(\tau))\frac{s_1'(t) - s_1'(\tau)}{t - \tau} K(s_1(t), t; s_1(\tau), \tau)\, d\tau \\ &+ \int_0^t v_2(\tau)\left[\frac{s_1'(t) - s_1'(\tau)}{t - \tau} - \frac{s_2'(t) - s_2'(\tau)}{t - \tau}\right]K(s_1(t), t; s_1(\tau), \tau)\, d\tau \\ &+ \int_0^t v_2(\tau)\frac{s_2'(t) - s_2'(\tau)}{t - \tau}[K(s_1(t), t; s_1(\tau), \tau) - K(s_2(t), t; s_2(\tau), \tau)]\, d\tau\Bigg\}. \end{aligned} \tag{3.20}$$

The absolute value of the first term of (3.20) is $\leq m_3t_0^{1/2}\epsilon$. The bracket in the second term can be written

$$s_1''(t_1) - s_2''(t_1) = \lambda^{-1}(v_1'(t_1) - v_2'(t_1)) = \lambda^{-1}t_1^{-1/2}(V_1(t_1) - V_2(t_1)),$$

where $\tau \leq t_1 \leq t$. Therefore the absolute value of that bracket is $\leq \lambda^{-1}\tau^{-1/2}\epsilon$ and the absolute value of the second term is $\leq \lambda^{-1}m_1t_0^{1/2}\epsilon$. The absolute value of the third term is $6a\lambda^{-1}m_1m_3t_0\epsilon$. Thus the absolute value of (3.20) is less than or equal to

$$\epsilon t_0^{1/2}(m_3 + \lambda^{-1}m_1 + 6a\lambda^{-1}m_1m_3t_0).$$

The general form of the argument is now clear and we may regard Lemma 1 as proved.

Lemma 2. *Suppose* (3.8a, b, c) *has a solution on* $0 \leq t < T$, *where* $T < \sigma$ *so that* $s(t) > 0$ *on* $0 \leq t \leq T$. *Then* $u_{xx}(x, t)$ *has a finite limit at each boundary point except possibly* (0, 0) *and* (a, 0).

The existence of a solution of (3.8a, b, c) implies that $v'(t)$ exists and is continuous on $0 < t < T$. Starting from (3.2) we form $u_{xx}(x, t)$. Using the following equations

$$N_{xx}(x, t; s(\tau), \tau) = -(d/d\tau)N(x, t; s(\tau), \tau) + s'(\tau)G_\xi(x, t; s(\tau), \tau),$$

$$N_{xx}(x, t; 0, \tau) = -N_\tau(x, t; 0, \tau), \qquad N_{xx}(x, t; \xi, 0) = N_{\xi\xi}(x, t; \xi, 0),$$

and performing several partial integrations, we arrive at

$$\begin{aligned} u_{xx}(x, t) = {} & [\phi'(0) - f(0)]N(x, t; 0, 0) \\ & + \int_0^t v'(\tau)N(x, t; s(\tau), \tau)\, d\tau + \int_0^t v(\tau)s'(\tau)G_\xi(x, t; s(\tau), \tau)\, d\tau \\ & - \int_0^t f'(\tau)N(x, t; 0, \tau)\, d\tau + \int_0^a \phi''(\xi)N(x, t; \xi, 0)\, d\xi. \end{aligned} \tag{3.21}$$

Lemma 2 follows from (3.21) since each term on the right of (3.21) has a finite limit at each boundary point except (0, 0) and $(a, 0)$.

We need upper and lower bounds for $s(t)$. Let $J(t) = \lambda a + H(t)$, where $H(t)$ is defined by (1.4). Then $J(t)$ is positive for $0 \leq t < \sigma$. Let $m(t) = 2\, ||f||_t\, (t/\pi)^{1/2} + 2\, ||\phi||$, where the norms are taken on $0 \leq \tau \leq t$ and $0 \leq x \leq a$. Then $s(t)$ is subject to the bounds

$$J(t)/(m(t) + \lambda) \leq s(t) \leq \lambda^{-1}J(t). \tag{3.22}$$

To prove (3.22) we note that since $u(x, t) \leq 0$ (2.4) implies the right half of (3.22). Since $v(t) \geq 0$ (3.2) implies

$$0 \geq u(x, t) \geq -||f||_t \int_0^t [\pi(t - \tau)]^{-1/2}\, d\tau - 2\, ||\phi|| = -m(t).$$

Thus, from (2.4), $\lambda s(t) \geq J(t) - m(t)s(t)$ and the left side of (3.22) follows.

We will need a bound on $u_2(t) = \max |u_{xx}(x, t)|$ on $0 \leq x \leq s(t)$ in the discussion below; we derive it now from (3.21). Norms of functions of t will be taken with respect to the interval $0 \leq t \leq T$. The absolute value of the first term on the right of (3.21) is $\leq \delta t^{-1/2}$. The absolute value of the second term is $\leq \pi^{1/2}\, ||V||$. The absolute value of the fourth term is $\leq 2T^{1/2}\, ||f'||$. The absolute value of the fifth term is $\leq 2\, ||\phi''||$. We need to estimate the third term. Let α and β be respectively the minimum of the left side and the maximum of the right side of (3.22) on $0 \leq t \leq T$. Then $\alpha \leq s(t) \leq \beta$, and $\alpha > 0$ since $T < \sigma$. The absolute value of the third term on the right of (3.21) is less than or equal to

$$(16\pi)^{-1/2}\,||v||\,||s'||\int_0^t\left\{\frac{x-s(t)+s(t)-s(\tau)}{(t-\tau)^{3/2}}\exp\left(-\frac{(x-s(t)+s(t)-s(\tau))^2}{4(t-\tau)}\right)\right.$$
$$\left.+\left|\frac{x+s(\tau)}{(t-\tau)^{3/2}}\exp\left(-\frac{(x+s(\tau))^2}{4(t-\tau)}\right)\right|\right\}d\tau \tag{3.23}$$

and the integral in (3.23) is less than or equal to

$$\int_0^t\frac{|x-s(t)|}{(t-\tau)^{3/2}}\exp\left(-\frac{(x-s(t))^2}{4(t-\tau)}\right)\exp\frac{(s(t)-x)(s(t)-s(\tau))}{2(t-\tau)}\,d\tau$$
$$+\int_0^t\frac{||s'||}{(t-\tau)^{1/2}}\,d\tau+\int_0^t\frac{2\beta}{(t-\tau)^{3/2}}\exp\left(-\frac{\alpha^2}{4(t-\tau)}\right)d\tau$$
$$\leq\int_0^t\frac{|x-s(t)|}{(t-\tau)^{3/2}}\exp\left(-\frac{(x-s(t))^2}{4(t-\tau)}\right)\exp(\beta\,||s'||/2)\,d\tau+2T^{1/2}\,||s'||+4\beta\pi^{1/2}/\alpha$$
$$\leq 2\pi^{1/2}\exp(\beta\,||s'||/2)+2T^{1/2}\,||s'||+4\beta\pi^{1/2}/\alpha.$$

We have $||v||\leq B$ and $||s'||\leq\lambda^{-1}(||q||+B)$. Thus if we write

$$L=(16\lambda^2\pi)^{-1/2}B(||q||+B)\cdot\{2\pi^{1/2}\exp(\beta\lambda^{-1}(||q||+B)/2)+2T^{1/2}\lambda^{-1}(||q||+B)+4\beta\pi^{1/2}/\alpha\} \tag{3.24}$$

we have

$$u_2(t)\leq\delta t^{-1/2}+\pi^{1/2}\,||V||+2T^{1/2}\,||f'||+2\,||\phi''||+L. \tag{3.25}$$

Actually, the inequality (3.25) has no force until we prove that $||V||$ is finite, i.e., that V is bounded on $0\leq t<T$. This is asserted in the following lemma. We note, however, that (3.25) remains valid if we replace $||V||$ on the right by max $|V(\tau)|$ on $0\leq\tau\leq t$. This remark will be used later in this section.

LEMMA 3. *Suppose* (3.8a, b, c) *has a solution on* $0\leq t<T$, *where* $T<\sigma$ *so that* $s(t)>0$ *on* $0\leq t\leq T$. *Then* $V(t)$ *is bounded on* $0\leq t<T$.

Since a solution of (3.8a, b, c) implies a solution of (3.1a, b), Lemma 2 of [6] implies that $v(t)$, $s(t)$, and $s'(t)$ are continuous on $0\leq t\leq T$. We want to show that $v'(t)$ is bounded in the vicinity of $t=T$. Taking the origin of the time axis at $T-\mu$, μ to be determined, we may write a system analogous to (3.8a, b, c) with $\phi''(\xi)$ replaced by $u_{xx}(\xi, T-\mu)$, a replaced by $s(T-\mu)$, $V(t)$ replaced by $V^*(t')=t'^{1/2}v^{*\prime}(t')$, where $t'=t-(T-\mu)$ and $v^*(t')=v(t)$. We refer to this system as (3.8*a, b, c). We note that since $u_x(0, T-\mu)=f(T-\mu)$ if $\mu\neq T$ the second term of (3.8a) does not have a corresponding term in (3.8*a). Let $||V^*||=\max|V^*(t')|$ on $0\leq t'\leq t_1$ where $t_1<\mu$. Then from (3.8*a) we may write an inequality, analogous to $||W||\leq D+t^{1/2}R$, which followed from (3.15). The argument leading to (3.15) needs the following modifications. In place of (3.13) we write $\alpha\leq s(t')\leq\beta$. We note that α and β do not depend on μ. In place of (3.14) we use

$$\left|\frac{s'(t)-s'(\tau)}{t-\tau}\right|\leq\lambda^{-1}\tau^{-1/2}(||V^*||+\mu\,||q'||),$$

where the norm $||q'||$ refers to the interval $0 \leq t \leq T$. This will also be true of the norms of v, q, f, and f' appearing in the remainder of this argument. We replace m_1 by $||v||$, m_2 by $\lambda^{-1}(||v|| + ||q||)$, m_3 by $\lambda^{-1}(||V^*|| + \mu\, ||q'||)$, M by $||V^*||$, and t_0 by μ. We get

$$||V^*|| \leq \mu^{1/2}F\, ||V^*|| + G, \tag{3.26}$$

where

$$F = \lambda^{-1}(2\, ||v|| + ||q||) + \beta\alpha^{-2}, \tag{3.27a}$$

$$G = 2\lambda^{-1}(||v|| + ||q||)u_1(T - \mu) + \cdots + [2 + 4\mu^{1/2}\lambda^{-1}(||v|| + ||q||)]u_2(T - \mu). \tag{3.27b}$$

The dots on the right of (3.27b) stand for terms involving $||v||$, $||q||$, $||q'||$, $||f||$, $||f'||$, α, β, and μ. The right side of (3.27b) is finite for any choice of μ, the last term by Lemma 2. We may choose μ so that $1 - \mu^{1/2}F > 0$. With this choice of μ (3.26) implies the boundedness $V^*(T')$ on $0 \leq t' < \mu$ and therefore the boundedness of $V(t)$ on $0 \leq t < T$.

We may now complete the proof of the theorem. Let T^* be the supremum of those T such that (3.8a, b, c) has a solution on $0 \leq t \leq T$. By Lemma 1 $T^* > 0$. We wish to prove that $T^* = \sigma$. Suppose $T^* < \sigma$; then $s(T^*) > 0$. We write (3.8*a, b, c) with time origin at $T^* - \mu$, μ to be determined. The t_0 for which we can establish a solution for (3.8*a, b, c) by the contracting mapping principle depends on inequalities involving $u_1(T^* - \mu)$, $u_2(T^* - \mu)$, $s(T^* - \mu)$, and the norms of f, f', q, q' on $T^* - \mu \leq t \leq T^* - \mu + t_0$. In these inequalities we may replace $u_1(T^* - \mu)$ and $u_2(T^* - \mu)$ by quantities depending on $||\phi||$ and on the norms of f, f', q, q', and V on $0 \leq t \leq T^*$; this follows from $u_1(T^* - \mu) \leq B$ (we replace T by T^* in B) and from the inequality (3.25) (but in the use of (3.25) we suppose μ so small that $\delta(T^* - \mu)^{1/2} \leq 2\delta(T^*)^{-1/2}$). We may also replace the norms of f, f', q, q' on $T^* - \mu \leq t \leq T^* - \mu + t_0$ by the norms of these same functions on $0 \leq t \leq T^* + t_0$. We may replace $s(T^* - \mu)$ by α or β, whichever is appropriate. Here, as before, α and β are, respectively, the minimum of the left side and the maximum of the right side of (3.22) on $0 \leq t \leq T^*$. As an example of the steps we have indicated above, consider inequality (3.12), which reads

$$2\lambda^{-1}t_0[u_x(s(T^* - \mu), T^* - \mu) + ||q|| + 2t_0^{1/2}M] \leq s(T^* - \mu),$$

$$M = 2\lambda^{-1}[u_x(s(T^* - \mu), T^* - \mu) + ||q||]u_1(T^* - \mu) + 2u_2(T^* - \mu) + 1. \tag{3.28}$$

If we let C be the right side of (3.25) with $t^{-1/2}$ replaced by $2(T^*)^{-1/2}$ then

$$2\lambda^{-1}t_0\{B + ||q|| + 2t_0^{1/2}[2\lambda^{-1}(B + ||q||)B + 2C + 1]\} \leq \alpha$$

implies (3.28). The effect of the changes discussed above is to decrease t_0 and to make it independent of μ. Hence by choosing μ less than t_0 we can extend the solution past T^*. We conclude then that $T^* = \sigma$. The remaining part of the theorem follows from Lemma 2.

In regard to the conclusion, in Theorem 2, that $s(t) \in C^2$, it is very likely that the hypotheses on $f(t)$ and $\phi(x)$ are too stringent. Indeed it is a reasonable conjecture that $s(t) \in C^{k+1}$ if $q(t) \in C^k$ and $f(t)$ and $\phi(x)$ are merely continuous with a finite number of jump discontinuities.

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