ON THE USE OF FOURIER TRANSFORMS FOR THE SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF ELASTOSTATICS*

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In this note we derive in a heuristic manner a condition which should be satisfied in order that Fourier transforms may be used for the solution of two-dimensional boundary value problems in the mathematical theory of elasticity.

It is well known [1, p. 405] that the components of the displacement vector given by the relations

$$U_x(x, y) = -\frac{1}{2(1-\eta)} \int_0^\infty \xi^{-1} e^{-\xi|y|}(1 - 2\eta - \xi\,|y|)\psi(\xi) \sin \xi x \, d\xi, \tag{1}$$

η being Poisson's ratio, and

$$U_y(x, y) = \frac{1}{2(1-\eta)} \int_0^\infty \xi^{-1} e^{-\xi|y|}(2 - 2\eta + \xi\,|y|)\psi(\xi) \cos \xi x \, d\xi, \tag{2}$$

where $\psi(\xi)$ is an arbitrary integrable function, are suitable for constructing solutions of the two-dimensional problems of elastostatics, where the elastic field is symmetric about the x-axis. These solutions have the property that the shear component of the stress tensor vanishes for $y = 0$ and that all the components of the stress tensor and the displacement vector approach zero for large distances from the origin. The normal component of the stress-tensor for $y = 0$ is given by the relation

$$\sigma_{yy}(x, 0) = -\frac{E}{2(1-\eta^2)} \int_0^\infty \psi(\xi) \cos \xi x \, d\xi. \tag{3}$$

Inherent in the use of Fourier transforms is the assumption of integrability for the components of the stress tensor and displacement vector.

Formally, since

$$\frac{1}{\pi}\int_0^\infty \cos \xi x \, d\xi = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\xi x} \, d\xi = \delta(x), \tag{4}$$

we obtain the relation

$$\int_0^\infty \sigma_{yy}(x, 0)\, dx = -\frac{E}{2\pi(1-\eta^2)} \psi(0). \tag{5}$$

Since $U_y(x, 0)$ is finite and is seen from (2) to be equal to $\int_0^\infty \xi^{-1}\psi(\xi) \cos \xi x \, d\xi$, it is necessary that $\psi(\xi)$ must vanish for $\xi = 0$, or else the integral will be divergent. It follows that

$$\int_0^\infty \sigma_{yy}(x, 0)\, dx = 0 \tag{6}$$

and also that Fourier transform methods are applicable only if the above condition is satisfied.

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Examples. (i) Let us consider the problem of a Griffith crack occupying the segment $-1 \leq x \leq 1$ in an infinite isotropic elastic sheet opened by a uniform internal pressure p_0. The normal component of stress is given by $-p_0$ for the crack surface and for $x > 1$ (see [1]), $\sigma_{yy}(x, 0) = -p_0[1 - x/(x^2 - 1)^{1/2}]$ and it is easily verified that $\int_0^\infty \sigma_{yy}(x, 0)\, dx = 0$. Physically, this means that the algebraic sum of the normal load transmitted to the medium across the plane $y = 0$ is zero.

(ii) As a second example, we consider the indentation problem of the semispace by frictionless punch which produces a specified even displacement for $-1 \leq x \leq 1$, it being assumed that the region on the boundary not immediately under the punch (i.e. $|x| > 1$) is stress-free. In this case, the auxiliary function $\psi(\xi)$ is to be determined by the pair of equations

$$\int_0^\infty \xi^{-1}\psi(\xi) \cos \xi x \, d\xi = f(x), \qquad 0 \leq x \leq 1 \tag{7}$$

$$\int_0^\infty \psi(\xi) \cos \xi x \, d\xi = 0, \qquad x > 1. \tag{8}$$

Recall that

$$\sigma_{yy}(x, 0) = \int_0^\infty \psi(\xi) \cos \xi x \, d\xi$$

and in view of (8), we find on integrating the above equation from t to ∞, that

$$\int_t^1 \sigma_{yy}(x, 0)\, dx = \int_0^\infty \frac{\psi(\xi)}{\xi} \sin \xi t \, d\xi, \qquad 0 \leq t < 1. \tag{9}$$

Since $\psi(\xi)$ is integrable and vanishes for $\xi = 0$, it is obvious that

$$\int_0^1 \sigma_{yy}(x, 0)\, dx = \int_0^\infty \sigma_{yy}(x, 0)\, dx = 0.$$

It is also implied that the normal stress is both compressive and tensile under the punch, a fact which is corroborated by the relation

$$\int_0^1 \frac{f(x)\, dx}{\sqrt{(1 - x^2)}} = 0 \tag{10}$$

obtained by Sneddon [2, p. 101] and Lowengrub [3, p. 71] for the existence of the solution of (7) and (8).

While for the sake of convenience we have considered the transfer of load across the plane $y = 0$, it can be concluded that across any plane $y = k$ the sum of the load transferred to the medium $y > k$ is equal to zero.

References

[1] I. N. Sneddon, *Fourier transforms*, McGraw Hill, New York, 1951
[2] I. N. Sneddon, *Mixed boundary value problems in potential theory*, North Holland, Amsterdam, 1966
[3] M. Lowengrub, *Some dual trigonometric equations with an application to elasticity*, Int. J. Engng. Sci. **4**, pp. 69–79 (1966)