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BOOK REVIEWS

VOLUME XXVII *1969*

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2b}}{\cos \frac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq (25)" is acceptable, but not "the preceding Eq " Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

BOOK REVIEW SECTION

Theory of max-min and its application to weapons allocation problems. By John M. Danskin. Springer-Verlag New York, Inc., New York, 1967. viii + 126 pp. $8.00.

Books consisting entirely of original work and presenting an entirely new theory are rare. They are to be acclaimed, their authors crowned with laurel. Let Danskin wear his for this theory of two-player conflicts which are generally distinct from games. They are profound in theory and of basic military value in practice.

Let $F(x, y)$ be a smooth, numerically valued function; the two arguments are usually each vectors over a simplex in a Euclidean space, but might be taken more generally. The idea here is to find and study

$$\max_x \min_y F(x, y). \tag{1}$$

This reverts, of course, to classical game theory, should the order of max and min not affect the result. The theory here is of the cases where such is not true; our concern is with (1) and not with the min-max.

The idea can be extended; in later chapters Danskin copes with

$$\min_z \max_x \min_y F(x, y, z)$$

and even goes as far as four tiers.

Even though the given $F(x, y)$ may be nicely smooth (usually some simple analytic expression) this need not be true of

$$\phi(x) = \min_y F(x, y)$$

and so we cannot maximize ϕ by the classical methods employing its derivatives. Thus new concepts are needed and they comprise the theoretical sections of the book.

The practical problems that motivated Danskin are those of large-scale preparation for war. Suppose nation A seeks an optimal allocation of its budget among several major weapons systems (Minuteman, Polaris, etc.). The rival nation B will learn of A's massive decisions, certainly when they are implemented, and so, to counter them, will devise an optimal allocation of his defensive means. Thus A's original maximization must take account B's subsequent minimization and hence A faces a max-min problem. Similarly, if A allocates his attack strength over various targets of B, he must reckon on B's distributing his defenses over these same targets so as to minimize the effectiveness of A's allocation. These are typical of the military planning applications of the theory; perhaps, as Danskin hints, there are others in economic or social domains.

After an introductory chapter, *Theory of max-min* builds its pure theory. Essentially it is a calculus of the not necessarily differentiable $\phi(x)$. It is shown, nevertheless, that ϕ has a directional derivative in all "admissible" directions and this is obtained by minimizing an expression in the partials of F; the formula reduces to the classic case if ϕ is smooth. Similarly Danskin extends other basic concepts to his new class of functions: the chain rule, law of the mean, the Lagrange multiplier theorem.

So far all is fine mathematics and pleasant reading, but when the concepts are applied to military problems the going gets rougher. This is not due to the presentation, for Danskin's style is singularly careful and lucid, but to the problems themselves. They are tough. They cannot help but be so, for they embody not only all the classical difficulties of maximizing, but must anticipate the decisions of a minimizing opponent as well. It is to the author's credit that he solves many. Toward even the incomplete solutions, Danskin has thrown the lance further than his predecessors, who had often struggled with these bafflers before him.

We think it a pity that the basics for the underlying military models are not systematically and adequately discussed. One may write a book on mechanics and presuppose the reader's knowledge of Newton's laws. But may one so take for granted laws of firing tactics and simply state, as Danskin does often, that such and such is the "damage function"? Actually the probabilistic outcome of many weapons fired at many targets (or one large one) is a hazy matter; various analyses, based on different aiming schemes and intermediate degrees of knowledge, lead to various formulas. Danskin always takes the damage arising from n weapons as proportional to a negative exponential of n. This is often satisfactory but not always. One wonders what would have occurred if Danskin had used other formulas and why he never even discusses this point.

But some inklings of the basics are scattered throughout the book. The reader who is interested in conning them first may look at pages 16 (bottom), 41 (middle), 52, 85 (bottom), and 87 (middle).

RUFUS ISAACS (*Baltimore, Md.*)

Numerical methods for two-point boundary-value problems. By Herbert B. Keller. Blaisdell Publishing Company, Waltham, Mass., 1968. viii + 184 pp. $7.50.

This monograph presents an excellent brief introduction to the theory and practice of numerical methods for the solution of two-point boundary-value problems for ordinary differential equations. Instead of aiming at an all-inclusive presentation in as general a setting as possible, the author (wisely, in the reviewer's opinion) restricts himself to an elementary yet thoroughly rigorous treatment of essentially only two methods: the initial-value or shooting methods and the finite difference methods. Their discussion for second order linear and nonlinear equations and first order systems in Chapters 2 and 3 forms the core of the book. Chapter 1 is concerned with some of the basic mathematical prerequisites, such as existence theorems for initial-value and boundary-value problems. In Chapter 4 the author briefly takes up integral equation methods based on the use of Green's functions, and in Chapter 5 he discusses eigenvalue problems and the general setting of continuity methods. Chapter 6 is devoted entirely to practical examples and computational exercises, which are of independent interest; they illustrate cases to which the theory as presented does not apply (though more general theoretical results would cover them), yet for which the numerical methods described are well-behaved.

Each section within a chapter is provided with a set of problems, some fairly difficult, and each chapter closes with supplementary notes and references to the ample bibliography. A five-page appendix on function space approximation methods points the reader towards more advanced topics such as the methods of Ritz and Galerkin.

H. A. ANTOSIEWICZ (*Los Angeles, California*)

Darstellungen von Gruppen. By H. Boerner. Springer-Verlag, Inc., Berlin, Heidelberg, New York, 1967. xiii + 317 pp. $14.50.

This is a second edition of a well-known book on group theory. It is slightly enlarged as compared to the first edition of 1955. The style has not changed; i.e., the book gives the elements of group theory and representation theory in a way that does not require any previous knowledge of the field. The author makes comments and side remarks on the theorems and points out applications in physics. The fact that the author does not try to present the material in its most general mathematical form does not limit the value of the book for applications. The underlying field is, for instance, assumed to be of characteristic zero and algebraically closed. A physicist can therefore always think of the field of complex numbers (which seems to be sufficient for him).

After an introductory chapter on linear spaces (finite dimensional) and matrices there follows a chapter on groups. Besides the abstract group the author pays special attention to the symmetric and alternating groups and to those continuous groups which are isomorphic to the linear (matrix) groups. The representation theory is throughout the book confined to the finite dimensional case. The author makes frequent use of the group algebra. The representations of the symmetric group are treated in detail. The Young diagram technique is then used to classify the representations of the full linear, the unimodular and the unitary matrix groups. The physically important rotation and Lorentz groups are treated in more detail.

The parts which are new in this second edition concern mainly ray representations and induced representations. Ray representations are particularly interesting for physics because the vectors in the representation space are physically observable only up to a phase. The technique of inducing representations of a group from those of a subgroup has found important applications in the theory of infinite dimensional representations. Here only finite dimensional representations are discussed, which, however, have applications in the theory of crystals.

From the point of view of physics much interest has during the last years been directed towards the infinite dimensional representations of certain continuous groups. This branch of group theory is not represented in the book and one understands the author when he says that an addition of it would have damaged the framework of the book. In spite of this, the book represents an important source for those who use group theory for applications. At the same time it could be used as a solid and yet not heavy textbook on group theory.

A. KIHLBERG (*Göteborg*)

Theory of orbits: the restricted problem of three bodies. By Victor Szebehely. Academic Press, Inc., New York, London, 1967. xvi + 668 pp. $25.00.

The restricted problem of three bodies is the problem of determining the motion of a particle of negligible mass which moves in the gravitational field of two point-masses revolving about each other in circles and in their plane. First studied by Euler and Lagrange in 1772, it has been the object of attention of most of the prominent workers in celestial mechanics ever since, as being the most tractable non-trivial case of the general problem of three bodies moving under their mutual influence.

The author's treatment of the problem is clear and comprehensive. I think no contribution of importance has been overlooked up to April 1967, when the manuscript was finished. Special attention is paid to the Soviet literature, where many important contributions have appeared during the last three decades. A chapter is devoted to numerical explorations, which have been numerous since the proliferation of modern calculating machines, although important contributions were made in the late nineteenth century. The subject of regularization, which is a method of removing the mathematical singularities associated with collisions between two of the three bodies, is comprehensively treated. A final chapter is devoted to the extension of the restricted problem to cases where the motions are not co-planar, to cases where the two massive bodies move in ellipses instead of circles, and to other modifications of the problem.

It is clear that the subject, which was regarded as purely theoretical until a decade ago, may now be applied to a number of practical problems in space technology, the earth being one massive body, the moon, the sun, or a planet the other, and a space probe the one of negligible mass. Indeed, practical applications are already being made.

G. M. CLEMENCE (*New Haven*)

Optimization in control theory and practice. By I. Gumowski and C. Mira. Cambridge University Press, Cambridge, 1968. ix + 242 pp. $10.50.

This book has as "its prime objective the study of such problems which are likely to stimulate an exchange of ideas between the designers and the theoreticians." To this end the first chapter is devoted to a discussion of the "gap between theory and practice." Chapter 2 is a general discussion of extremal problems and Chapter 3 presents Carathéodory's equivalent formulation. In Chapter 4 the relation between this and dynamic programming and the maximum principle is considered. The book concludes with Chapter 5 which outlines some methods for approximating solutions of optimization problems.

In Chapter 1 the authors contend that "the separation into theoretical and practical factions" began with the development of modern abstract linear system theory. This is a gross but unimportant over-simplification. The important point they have to make is that the major obstacle to communication between the designer and the theoretician is the failure to distinguish between inductive and deductive models on the one hand and *a priori* and *a posteriori* models on the other. The designer tends to believe that every deductive model is an *a posteriori* model (one that can be applied only after a system has been successfully designed). Working separately, the theoretician may not care or may not seem to care whether or not his deductive model is an *a priori* (realistic) model. The authors then go on to discuss the general properties required of an *a priori* model based on Hadamard's conditions for a "well posed" problem in partial differential equations. As the history of differential equations teaches us, identifying *a priori* models is not always easy, and the identification comes about from both experience and theory. Optimal control theory is about at the point today where the theory of partial differential equations was 50 years ago. Experience shows that some optimal control theory is useful, but we know very little theoretically about the insensitivity (stability) under perturbations of optimal syntheses of feedback control. The authors fail to point out in this regard and with reference to Chapter 4 that L. D. Berkovitz [J. Math. Anal. Appl. 3(1961), 146–169] made an important, but incomplete, contribution in this direction with his introduction of the concept of a "normal" synthesis. In 1961 V. G. Boltyanskii [Siam J. Control 4(1966), 326–361, English translation] introduced a similar concept which he called a "regular" synthesis. This type of synthesis is, however, not completely stable under perturbations, as can be seen from Boltyanskii's examples of regular syntheses. From the point of view of Section 14.6 of this book these are examples of pathological optimization and, as pointed out in this section, parametric embedding is a useful technique here. There are a number of concrete examples of this within recent optimal control theory to which the authors might have referred. It might be helpful to note here that work done today on what in this book are called "inert" systems is to be found in English under the name of "structural stability", a topic which is, as the authors point out, certainly relevant to *a priori* models.

The book itself has almost nothing to do with control theory and certainly does nothing to show that optimal control theory provides an *a priori* model for feedback control. Many things which are said are wrong and others are quite misleading. Much of the writing is pretentious and most of the references are of the "*a posteriori*" (after the fact) variety. In the opinion of the reviewer this book is not likely to stimulate a useful exchange of ideas between designers and theoreticians.

J. P. LaSalle (*Providence, R. I.*)

Discontinuous and optimal control. By I. Flügge-Lotz. McGraw-Hill Book Co., New York, 1968. x + 286 pp. $16.50.

This book is a sequel to a very noteworthy predecessor (*Discontinuous Automatic Control*, Princeton University Press, 1953), which was one of the first "modern" treatments of the dynamics of control systems. Then as now Professor Flügge-Lotz's work had centered around the very diverse dynamical phenomena which result from the use of discontinuous control functions. This problem area has a double significance in control technology and has played an important part in the theoretical developments of the last 15 years. First, the problem arises in trying to use relays as control amplifiers (simplicity?); second, the problem arises in trying to construct control laws which make optimal use of the limited power available for control (perfection?). The first book covered only the first problem, the present book treats both.

Since the literature of optimal control is currently measured in bookshelf-feet, one should not expect of course an encyclopedic treatment in *any* book. Here the following topics are emphasized: specific results on second-order linear systems with bounded control variable and (usually) linear control laws (Chapter 2); similar results for third and higher order systems, with some examples from engineering practice (Chapter 3), a quick review of optimal control theory along the lines of Halkin's geometric formalism, together with many interesting examples involving discontinuous control (Chapter 4), plus numerous other problems which are outside the conventional theoretical setup (Chapter 5). The book integrates the contributions of Professor Flügge-Lotz's many doctoral students and collaborators; it is a well-written research monograph rather than an exhaustive textbook. It can be (and has been) used as a text in graduate courses.

In view of the circumstances of this review and the series in which the book appears, the reviewer has to say how things relate to applied mathematics. This is not an easy matter. The book abounds in contributions, but they are not in the form of theorems. There is no heavy mathematical apparatus; the techniques are linear algebra and elementary analysis, with a decidedly geometric flavor. In other words, in the conventional sense of the word the mathematical contents are meager. This is largely due to the fact that there is little mathematics today without continuity and *discontinuity* is in the center of attention in this book. To put it differently, the mature mathematician interested in applications will find here a wealth of problems as well as carefully worked out special cases. To bring these results within the deductive machinery of modern mathematics is a real challenge.

Among the printing errors, some (like the unusual definition of the squared Euclidean norm on the top of page 158) are probably attributable to the production process rather than to the author. McGraw-Hill's editing and typography are passable. The figures are nicely organized and very well drawn.

R. E. KALMAN (*Stanford*)

Introduction to measure and probability. By J. F. C. Kingman and S. J. Taylor. Cambridge University Press, London and New York, 1966. x + 401 pp. $12.50.

This book gives an excellent introduction to measure theory and integration and provides a good background for the mathematical foundation of modern probability theory. Unlike many of the books written for probability theory which start out with an introduction of probabilistic notions and then develop measure theoretic tools as they become necessary, this book treats measure and integration as primary objects of study, and then takes up probability theory as an extension (motivated by practical and intuitive notions of probability) of some aspects of measure theory. In the first part of the book, the authors develop the concepts of measure and integral in a general setting and discuss their properties quite extensively. Fundamental ideas in measure and integration theory are explained carefully with particular emphasis on Lebesgue measure and Lebesgue integral on the real line as important examples, and complete proofs are given of all the basic theorems regardless of their relevance to probability theory. Furthermore, the discussions of these concepts lead in a natural way to a treatment of related topics in mathematical analysis which, together with measure and integration, form a core of a typical first-year graduate real variables course. Thus, a reader can find in the first part of this book an introduction to functional analysis (including among other things discussions of the Hahn-Banach theorem, Riesz representation theorem and identification of continuous linear functionals on L^p spaces), the Daniel theory of integration, differentiation theorems on the real line and a construction of the Haar measure for locally compact groups. The treatment of these topics is not very extensive, but the authors succeeded quite well in explaining the essentials in a neat and clear manner.

The reviewer feels that the first part of this book can serve as an excellent textbook for a first-year graduate real variables course.

In Chapter 10, where the second part of the book begins, the authors explain how the intuitive notion of probability can be fitted into the scheme of measure theory and then identify the traditional probability nomenclature with corresponding measure theoretic terminologies. After the notions of independence, random variables and distribution functions are described, characteristic functions are defined and important theorems concerning them are proved in detail. With the aid of this important tool, classical limit theorems for sums of independent random variables are derived in Chapter 13. Throughout these discussions, machinery developed in the first half of the book is used in an essential way. The book also discusses the notion of joint distributions of several random variables and that of conditioning, and in the last chapter introduces stochastic processes through the discussion of renewal processes and describes some important examples. Measure theoretic difficulties connected with the treatment of continuous parameter stochastic processes are explained, and a way to get around these difficulties is shown in the case of the Wiener process. The second part of the book is a bit thin in its coverage in comparison to the extensive first half, and the reviewer feels that this part by itself will be inadequate for use as a textbook for graduate level introduction to mathematical probability theory. While it is true that modern probability theory has become quite diversified and it is therefore impossible to include an introduction to every aspect of the theory in a book of this kind, the reviewer nevertheless feels that some discussion of martingales and Markov chains, for example, could have been included in this book. Finally, since probability theory is treated in this book more or less as an extension of measure theory, a student will not be able to cultivate a habit of "thinking probabilistically" by reading this book; however, it is certain that he will be able to gain an excellent mathematical background for rigorous treatment of probability theory and its diverse applications.

YUJI ITO (*Providence, R. I.*)

Varieties of groups. By Hanna Neumann. Springer-Verlag New York, Inc., New York, 1967. xi + 192 pp. $11.50.

This book is a milestone in the literature of group theory. The contributions to the study of varieties of groups by Hanna Neumann, B. H. Neumann, and Peter Neumann have been so major that it is fitting that this book should have been written by a member of this distinguished family.

A variety of groups is a class of equationally defined groups. The study of varieties began in the 1930's with Garrett Birkhoff's paper on "The structure of abstract algebras" and owes much to Philip Hall's lectures on universal algebras. But the greatest flowering of the study of varieties of algebras has been in group theory, and this book assembles this information for the first time.

The first chapter of this book gives the basic facts. These include the necessary information on free groups and relatively free groups and a proof that a class of groups closed with respect to taking subgroups, factor groups, and cartesian products is a variety and conversely that a variety has these closure properties.

The second chapter defines an associative product for varieties and gives a number of relations between different varieties. The important class of nilpotent varieties is treated in chapter three and also some significant implications with respect to the Burnside problem.

The fourth chapter "Miscellaneous properties of relatively free groups" covers a great many topics including Hopfian, Schrerer, and splitting properties, and some relations to homological algebra.

The last chapter "On the laws of finite groups" includes a proof of the finite basis theorem for the laws of finite groups. This asserts that for a finite group G, that there exists a finite set of laws such that every law satisfied by G is a law of the variety defined by this finite set. The concepts of a critical group and a Cross variety are most effective in this proof.

Scattered throughout the book is a series of unsolved problems. These are an index of the vitality of the subject and a challenge to the reader.

MARSHALL HALL, JR. (*Pasadena*)

Lectures on functional equations and their applications. By Janos Aczél. Academic Press, New York and London, 1966. xix + 510 pp. $19.50.

This book is written by one of the leading experts in the field of functional equations and, as might be expected, it is extremely scholarly, informative and interesting. In the interests of brevity Aczél interprets his title in a somewhat narrow sense, so that difference equations (for example) are not included. Differential equations of the sort that Truesdell takes as a starting point for his theory of special functions are also excluded. The great variety of topics which is nevertheless considered shows that this decision to restrict the scope was wise. Complete proofs are given for nearly all the main theorems, and the book contains applications of great diversity. (For example, functional equations are used to characterize the dot and cross product, and to characterize those force fields that admit a suitable concept of "center of gravity." The functional equations associated with transmission-line theory are presented, but the author does not alienate his mathematical readers by giving technical details.) The book is, on the whole, elementary, and is written in a pleasant, informal style that enables one to dip into it on almost any page. The bibliography of over 100 pages is enhanced by references to it scattered in footnotes throughout the text. Inevitably there are some omissions in a work of this scope; e.g.,the reviewer's discussion of compound Poisson processes by functional equations (1953) is not referenced, though relevant. Aczél also modestly omits his own name from the index of authors. In summary, this book is warmly recommended as a most welcome addition to mathematical literature.

R. M. REDHEFFER (*Los Angeles*)

Théorie des groupes. By Henry Bacry. Gordon & Breach, New York, 1967. xvi + 450 pp. $17 paper, $22 hardbound.

This lucid book is a compendium of practically all aspects of group theory and representation theory of interest to the elementary particle physicist. The level is about that of Hamermesh's book

(M. Hamermesh, *Group Theory and its Applications to Physical Problems*, Addison-Wesley, 1962), though its coverage is wider. The clean and economical treatment of the pure mathematical parts, in the French style following Bourbaki, is very welcome—we signal in particular the treatment of vector spaces and that of the topology of group spaces, covering groups, etc. In its character as a course at the intermediate level rather than a treatise, many theorems are given without proof; by the same token, a few deep subjects have been left out, e.g., the unitary representation theory of noncompact groups (except for a survey of that for the Poincaré group). On the other hand the finite-dimensional representations of semi-simple Lie groups are found both by the classical method: irreducible tensors, Young diagrams, etc., and *via* the weight spaces of Lie algebras. The useful techniques employing Young diagrams to reduce product representations, to reduce representations with respect to various subgroups, etc., are exhaustively illustrated.

The final chapters turn to a general discussion of symmetries of elementary particles, probably sufficient to orient the mathematician with no previous experience in physics, and to a reasonably complete survey of the successes and failures of "unitary" (SU(3) and SU(6)) symmetry. There are some useful appendices and tables on Lie algebras at the end.

R. L. INGRAHAM (*University Park, N. M.*)

Nuclear forces: introduction to theoretical nuclear physics. By Gernot Eder. M. I. T. Press, Cambridge, Mass., 1968. ix + 349 pp. $17. 50.

This is one of the best books on nuclear theory I have ever seen. The style is refreshingly clear yet economical; the author drives through this big subject at an even, rapid pace. The mode, popular in the literature, of jargon coupled with a cunning absence of definitions is avoided. The definitions are clear and concise, the notation consistent, stripping false mystery from many topics usually accessible only to "insiders." The emphasis is on theory (and it is remarkable to see how far consistent theories can go, in this subject where the basic dynamics is still unknown), with little attention to examples or specific nuclei.

The author starts with an exposition of the standard phase shift analysis of nuclear scattering and the classical theory of the two nucleon potential. Single-particle problems are next treated: shell model and variations thereon with different potentials, calculations of dipole, quadrupole moments, etc. Many-body theory and the theory of nuclear reactions, resonance formalism, optical model, etc., follow next. The book closes with a brief excursion into quantum field theory, including an adequate treatment of β-decay and related weak interactions, nuclear electromagnetic interactions, and a taste of the field theory of strong interactions.

The chapter on many-body problems is especially good. Here it is shown clearly how quantum field theory analogues in nonrelativistic form—annihilation and creation operators acting on a "vacuum," S-operator expansion, Green's functions, and diagrams of various sorts can be used fruitfully. Applications are made to surface oscillations and fission. The BCS method of quasi-particles, energy gap and so on, is shown to apply to deformed nuclei.

The translation is excellent, and the incidence of typographical errors seems to be minimal.

R. L. INGRAHAM (*University Park, N. M.*)

The elements of probability and sampling. By Frank A. Friday. Barnes and Noble, New York, 1968. xvi + 130 pp. $6.00.

This book constitutes an attempt to explain in layman's language the basics of probability and sampling. The exposition is entirely non-mathematical. The author, an economist, primarily visualizes business men and politicians as presumptive readers. The book may be elucidating, and even entertaining, for a reader lacking a statistical background.

Most of the material contained in this book would ordinarily be covered in an elementary college course on probability. The absence of mathematical tools has to be counterbalanced, as usual, by a very detailed exposition, which would seem tedious to a reader with only a slight mathematical background. The author does, however, provide a good deal of historical background from the times of

Fermat, Pascal and Bernoulli, as well as some philosophical discussion on the meaning of probability and randomness, for example, in connection with the von Mises notion of probability as the limit of relative frequency. Such aspects would probably not be covered in an introductory probability course.

The topic of gambling takes the author into explaining about permutations, combinations, the buildup of Pascal's triangle, binomial coefficients, and binomial probabilities. An explanation is given of de Moivre's approximation of the binomial distribution by normal curve probabilities, and the setting of confidence limits on the observed relative frequency r/n. There is a short discussion on the hypergeometric distribution. The final chapter is concerned with the elements of sampling, explaining why average measurement obtained from a sample can be reliable as an estimate of a population characteristic.

Carl-Erik Särndal (*Washington, D. C.*)

Functions of a complex variable: constructive theory. By V. I. Smirnov and N. A. Lebedev. M. I. T. Press, Cambridge, Mass., 1968. ix + 488 pp. $12.00.

This book, a translation from the Russian, is based upon a series of lectures on "The Constructive Theory of Functions of a Complex Variable" given at Leningrad University by Smirnov and later by Lebedev. The problems here have to do principally with the approximation of functions of one complex variable by means of polynomials or by rational functions. Questions of existence of approximations, representation of approximations, degree of approximations, best approximations are all considered. This is what the Russians mean by "the constructive theory of functions". Some of the devices used for approximations include interpolation, least-square and weighted least-square approximations, and expansions in terms of various domain polynomials. While this book goes into many topics in depth and the Russian literature is presumably well covered, the same cannot be asserted for the post-1930 non-Russian literature. The techniques are mainly those of classical analysis. The translation, prepared by Scripta Technica, Ltd., is satisfactory. The book should appeal principally to the specialist in approximation theory who will be grateful for the Russian material.

Table of Contents. Chapter 1. Uniform approximation of functions by polynomials and rational functions. Chapter 2. Faber polynomials and problems of representing a regular function by a polynomial series. Chapter 3. Quadratic approximation. Functions orthogonal with respect to a domain. Chapter 4. Functions orthogonal with respect to a contour. Chapter 5. Questions related to the best uniform approximation.

Philip J. Davis (*Providence, R. I.*)

Praktische Funktionenlehre. By Friedrich Tölke. Band 5. Allgemeine Weierstrassche Funktionen und Ableitungen nach dem Parameter. Integrale der Theta Funktionen und Bilinear-Entwicklungen. Springer-Verlag, Berlin/Heidelberg/New York, 1968. viii + 158 pp. $18.25.

Volumes 2 and 3 of this work were reviewed in Quarterly of Applied Mathematics, vol. 26, no. 3. In the first chapter of the present volume 5, "general" (i.e., the traditional) Weierstrass functions are presented and the formulae concerning them are developed from the double series representations. The "special" Weierstrass functions are also expressed in double series and related to Weierstrass' $\wp$. There follows a chapter containing formulae for the derivatives of elliptic functions with respect to the parameter κ and the modulus k. This is followed by a chapter on repeated integrals of theta functions (with respect to ξ), the so-called D-functions, which are shown to arise in connection with certain inhomogeneous partial differential equations.

"Many-dimensional" theta and D-functions are products of theta functions and of their integrals, and are shown to arise in connection with the diffusion equation in several dimensions. Theta and D-functions with an imaginary parameter are solutions of partial differential equations of the character of the Schrödinger equation. One chapter is devoted to each of these two subjects, and in a further chapter, the last of the volume, Green's functions are constructed for some partial differential equations.

As in volumes 2 and 3, there is an abundance of formulae and diagrams directed towards the practical use of these functions. There is a brief list of references (7 items) but there is no index.

A. ERDÉLYI (*Edinburgh, Scotland*)

Hilbert spaces of entire functions. By Louis de Branges. Prentice-Hall, Englewood Cliffs, N. J., 1968. ix + 326 pp. $11.00.

The simplest Hilbert space of entire functions has as its elements the entire functions of exponential type that are of integrable square on the real axis; the famous Paley-Wiener theorem characterizes this space as the class of Fourier transforms of functions that are of integrable square and vanish outside a finite interval. This theorem has numerous applications—including, as the author shows, the whole L^2 theory of Fourier transforms. After a first chapter which collects the results that are needed from the classical and modern theory of entire functions, the author obtains the generalizations of Fourier analysis (generalized eigenfunction expansions) appropriate for a wide class of other Hilbert spaces of entire functions, and then gives applications which indicate that the theory is not only mathematically elegant but likely eventually to be useful. In the third chapter the theory is tied in with the theory of special functions via the determination of the structure of spaces of entire functions with special properties—for (elementary) example, that $F(-z)$ belongs to the space when $F(z)$ does, or that $a^{1+\nu} F(az)$ belongs to the space when $F(z)$ does. Each example leads eventually to the introduction in a natural way of a particular class of special functions (Bessel functions, Jacobi polynomials, hypergeometric functions, etc.). Thus the special functions arise from expansions rather than as solutions of differential equations. Other applications are to the theory of local operators (which transform a function vanishing in a neighborhood of x_0 into a function that vanishes at x_0), and hence to the problem of when a Fourier transform can vanish in an interval without vanishing identically. As the author points out in the preface, his Hilbert spaces of entire functions are the invariant subspaces appropriate for a class of transformations, and these transformations are represented by integrals that display the invariant subspaces; however, the reader must supply the details of this interpretation for himself, since the phrase "invariant subspace" does not appear between the preface at the beginning and the notes at the end. Considering that everything that the average applied mathematician needs to know about the classical Fourier transform is condensed into half a dozen pages of this book, it may well be a considerable time before the users of mathematics will be able to bring the elegant mathematics of the rest of the book to bear on their problems.

R. P. BOAS (*Evanston, Ill.*)

Dynamic programming: sequential scientific management. By A. Kaufmann and R. Cruon. Academic Press, New York and London, 1967. xv + 278 pp. $12.00.

This is a remarkable book. It contains material advanced enough to interest the expert, yet is carefully written enough so that beginners (with advanced calculus) can use it profitably. The authors treat discrete-time multistage decision processes in which the duration is either finite or infinite. Random effects may be present or not. Many examples are worked in great detail. Students in a class at UCLA responded enthusiastically and especially liked the emphasis on the theory of graphs. The treatment of convergence problems when the planning horizon is infinite is thorough and provocative. We look forward to a promised second volume on adaptive control.

R. KALABA (*Santa Monica, California*)

The theory of rotating fluids. By H. P. Greenspan. Cambridge University Press, New York, 1968. xii + 327 pp. $15.00.

Laboratory experiments and experience with large-scale flow in the oceans and atmosphere indicate that slow hydrodynamical motions in a rapidly rotating fluid are such that Coriolis forces approximately balance pressure forces nearly everywhere; when the balance is perfect the flow is said to be "geostrophic" (a term coined by Napier Shaw in 1916; $\gamma\tilde{\eta}$ = Earth, $\sigma\tau\rho o\phi\acute{\eta}$ = turning). "Geostrophic motion" satisfies the meteorologist's "thermal wind" equation when the fluid is baroclinic, which reduces to the celebrated Proudman-Taylor theorem when the fluid is barotropic (i.e. when the density is uniform for a liquid, or dependent only on pressure for a gas). Unfortunately, the equations of geostrophic motion are mathematically degenerate and ageostrophic effects must, therefore, be taken into account in any acceptable theoretical procedure.

Four types of ageostrophic effect occur in general, associated respectively with time-variations in flow pattern, with the acceleration of fluid particles relative to the rotating frame, with viscous friction, and with magneto-hydrodynamic processes if we include electrically-conducting fluids, such as the Earth's core. The first step in any theoretical study is usually the ranking of ageostrophic effects in order of importance. If either friction or time-variations in the flow pattern predominate then the mathematical problem can usually be linearised and treated analytically. More commonly, however, particle accelerations or magnetohydrodynamic effects predominate and the resulting mathematical equations are non-linear and generally intractable, except by numerical methods.

Several years ago, largely through pioneering studies by meteorologists and oceanographers, the number and variety of available solutions to "linear problems" in the hydrodynamics of rotating fluids reached the level at which it became possible to offer a course in the subject that would satisfy graduate students in applied mathematics, and the most successful part of Professor Greenspan's book is the outcome of such a course. The book attempts to give a comprehensive account of the theory of the hydrodynamics of rotating fluids (excluding magnetohydrodynamic effects).

In the first chapter (27 pages), a few simple laboratory demonstrations are described, the equations of motion are presented and the rudiments of vorticity theory and boundary-layer theory are outlined. In chapter 2, the theory of the Ekman boundary layer is treated at some length. As meteorologists were probably the first to discover, Ekman-layer suction is the dominant viscous process in the vorticity-balance equation, a process brilliantly analysed by Professor Greenspan under the title of "spin-up". According to the theory, the typical interval of time required for friction to bring about substantial changes in vorticity in a barotropic fluid is the geometric mean of the viscous diffusion time (typically several months at least for the Earth's atmosphere) and the period of rotation. "Spin-up" is a recurring theme throughout the book.

A rotating barotropic fluid is capable of supporting inertial oscillations with frequencies less than twice the angular speed of basic rotation. "Rossby-Haurwitz" waves (Laplace's second-class tidal oscillations) comprise that class of inertial oscillation with which the meteorologist and oceanographer are particularly familiar. These waves are but one phenomenon due essentially to the spatial variation of the axial distance between the upper and lower surfaces of the fluid, western boundary currents (e.g. the Gulf Stream) and Taylor columns being others. Linear theories of these phenomena are treated in detail in chapter 2.

Chapters 3 (52 pages), 4 (40 pages) and 5 (46 pages) include further discussions of some of the phenomena introduced in chapter 2, taking into account, where possible, non-linear effects and the absence of bounding surfaces. In chapter 6 (29 pages), experimental and theoretical investigations of the various barotropic and baroclinic flows are reviewed, with the greatest emphasis on the onset of instabilities in Ekman layers. Baroclinc instability, the process responsible for the conversion of potential energy due to solar heating of the atmosphere into the kinetic energy of large-scale motions, is treated in outline. Bénard convection in a rotating fluid is not included, presumably because it has been thoroughly treated elsewhere (by Chandrasekhar) fairly recently.

The book carries a detailed notation guide, a bibliography and author index and a subject index, and is fairly lavishly illustrated, which may account for its high price. This reviewer noticed only one or two minor errors. The book will constitute a useful addition to the libraries of applied mathematicians and fluid dynamicists and of mathematically inclined astronomers, geophysicists, meteorologists, oceanographers and engineers.

R. Hide (*Bracknell, England*)

Algebraic theory of machines, languages and semigroups. Edited by M. A. Arbib. Academic Press, New York, 1968. xvi + 359 pp. $16.00.

In their joint thesis work Kenneth Krohn and John Rhodes made a profound contribution to automata theory by relating the semigroup properties of a finite automaton to its decomposition or realization from simpler component automata. With every finite automaton they associated the semigroup of transformations induced by the automaton on its set of states and showed that:

1. every finite automaton can be realized by a loop-free (or even cascade) connection of two-state automata and automata whose semigroups are simple groups;

2. every simple group "contained" in the semigroup of an automaton must be "contained" in the semigroup of one of the component automata into which it is decomposed. This work created considerable interest and initiated further research in the relations between automata and semigroups, enriching both areas.

The book under review is an attempt "to carry the reader from elementary theory all the way to hitherto unpublished results" in the semigroup-theoretic approach to machines (automata). The book contains contributions by eight authors who presented papers at the conference on the Algebraic Theory of Machines, Languages, and Semigroups held at Asilomar, California in 1966, and also material from seminars conducted by Rhodes at Berkeley. In general, the book is a success. The material is well selected, nicely organized and edited and some of it is very well written. As claimed, the book does provide a good introduction to the semigroup-theoretic approach to machines in its initial four chapters and then proceeds to a detailed discussion of the central theorems of the decomposition theory. The following chapter defines the complexity of machines in terms of the minimal number of group machines required for its realization. After one other expository chapter and a chapter on homomorphisms and semilocal theory of finite semigroups, these results are applied to gain deeper insight into the complexity theory. This is followed by a chapter on topological semigroups, "in hope of hastening the day when many of the results developed for finite semigroups and machines in previous chapters will be extended to topological semigroups and machines." The last three chapters deal with formal languages and are less directly related to the spirit of the semigroup-theoretic approach developed so far in the book. In particular, the chapter on context-free languages does not even "tip the hat" to semigroups when it summarizes the main results in this field.

As stated above, this is a good and valuable exposition of an elegant part of automata and semigroup theory and it will be a rich source book for researchers in this area. I believe that most readers will be impressed by the progress of this field which is summarized in this book. On the other hand, it is perhaps even more impressive to be able to disagree with the editor's claim that a subset of the chapters in this book make "an excellent course in automata theory." There has been so much progress in other parts of automata theory and in the study of formal languages that this book represents only a part of the current work in the very vigorous creation of a theory of computation.

J. HARTMANIS (*Ithaca, N. Y.*)

Thermoelasticity. By Heinz Parkus. Blaisdell Publishing Company, Waltham, Mass., 1968. viii + 112 pp. $5.50.

By avoiding all temptations to stray outside his announced topic, the author has succeeded in presenting a relatively complete discussion of it within a very short compass. He starts with a heuristic presentation of the equations of the linear theory obtained by simple superposition of the strains due to stresses and those due to temperature change. Chapters 2 and 3 are then devoted to methods of solution of boundary-value problems with these field equations. The presentations of these methods, which include use of the thermoelastic potential, Green's functions and complex function techniques, are brief but clear with some illustrative examples. These are followed by a chapter of the same character dealing with thermal bending and buckling of plates. The level of the discussion rises drastically with Chapter 5 in which a thermodynamic derivation of the equations of thermoelasticity, with no kinematic restrictions, is presented. The final chapter deals with wave propagation and includes a neat discussion of the decay of stress discontinuities due to thermoelastic coupling. Problems are posed at the end of each chapter.

JEROME H. WEINER (*Providence, R. I.*)

The economics of uncertainty. By Karl Henrik Borch. Princeton University Press, Princeton, New Jersey, 1968. 224 pp. $8.50.

The theory of games and the development of operations research have served to renew the economists' long-standing interest in the challenging economic problems of risk and uncertainty. These contributions have been scattered over a number of journals, proceedings, multi-author books, Festschriften, and have found their way into the literature of psychology. In *The economics of uncertainty* Karl Borch has assembled this material for the first time in textbook form. The results will be appreciated not only by theoretical and applied economists but also by social scientists, operations researchers, and the growing number of applied mathematicians with an interest in the social sciences. As in Luce and Raiffa, *Games and Decisions,* the emphasis is on ideas rather than techniques, but the scope of the present book is restricted to economics, and its strength is the professional treatment of the economic aspects of uncertainty.

The opening chapters develop basic concepts and the notion, surprising to many, that economists cannot tell management what its preference for risk should be and can therefore never definitively advise for or against a particular venture. The keystone to the economic analysis of risk is the Bernoulli principle that the objective function should be the expected value of utility—sometimes referred to as moral expectation. This principle is developed, discussed and applied in detail. Among the propositions of interest to mathematicians is Pfanzagl's theorem that decisions are independent of the decision maker's initial wealth if and only if the utility of wealth is linear or logarithmic, and a characterisation of those objective functions involving only mean and variance which are consistent with the expected utility hypothesis. Among the applications discussed are portfolio selection, insurance and reinsurance, the latter being one of Professor Borch's research specialties. Next the problems of decision making in the case of unknown probabilities are introduced and the various systems of axioms are reviewed. Economists will find the chapter on market equilibrium under uncertainty particularly stimulating; here it is shown how institutional restrictions on types of bets and securities may seriously limit efficiency in the allocation of risk, ideas which were first developed by Arrow but which deserve to be much better known. Next the principles of game theory are developed and some interesting economic illustrations are given. This up-to-date exposition also of non-zero sum theory will be much appreciated by social scientists. Returning to the main stream of economic thought, the theory of the firm is reconsidered both with regard to objectives under uncertainty, to the survival problem, and finally to group decision making.

Professor Borch has done much more than pull together the most interesting developments in the economic theory of uncertainty from its diverse sources. He has added his own original and penetrating analysis and has supplied thought provoking illustrations and applications. This book is a little masterpiece which pleases the reader with its clarity of exposition, its depth of learning and its fine wit and style.

M. J. Beckmann (*Providence, R. I.*)

Forest biometrics. By Michail Prodan. Translated from the German by Sabine H. Gardiner. Pergamon Press, Oxford, 1968. xi + 447 pp. $20.00.

This book is a translation of the author's *Forstliche Biometrie,* published in 1961. In this book, "the method of mathematical statistics and biometrics which are of most importance to forestry, and other fields related to forestry, are explained with the aid of a large number of practical examples" (Foreword, p. ix). The crucial element of this quotation is, of course, "explained". To state it briefly, the book does *not* accomplish the author's objectives; and whatever it does, it does in a most unsatisfactory way. The book is by no means suitable for its intended audience: "students, practical workers and researchers".

In justification of this evaluation of this book, I will present four relevant arguments:

1. In a review in Biom. Zeitschrift (1962) of the original German edition, Professor Erna Weber gave the author the following well-meaning advice: "The theoretical foundations must be revised for a second edition". This advice has been completely ignored in the preparation of the English edition. As a result, the present book, similar to the German original, is void of anything that may reasonably be called an acceptable presentation of the probabilistic assumptions underlying today's "methods of mathematical statistics and biometrics".

The presentation reflects throughout the author's conception of the prime role of mathematical statistics: to be a 'technique' for manipulation of large masses of data. Thus, the reader must not be surprised to find that there is very little discussion of such a topic as the mechanism of selecting the sample, to mention but one example. Likewise, the author makes no serious effort to make a distinction between a parameter of a distribution, and the estimate of this parameter.

2. The frequency of plain errors of a variety of kinds is simply exceedingly large. Some examples are illuminating.

a) A great many formulas are erroneous. A *small* sample is: (94), (95), (113), (114), (122), (151), (278), (368), (380), (487).

b) The same symbol often appears with two or even several different meanings. As an example, reference is made to "$F(u)$" in Table 44, p. 104. In many instances (cf. e.g. p. 169), "variance" is used as a synonym of "sum of squares"; in other places, "variance" is used with its usual meaning. A final example of the confusion caused by the inadequate use of symbols is given by formula (371); there, as in (372), "s" is used, while in (367), "σ" is used.

c) Formula (278) convinces the reviewer that the author interprets $\tan^{-1}x$ to mean $(\tan x)^{-1} = 1/\tan x$.

d) On p. 152, "estimation" is used in a misleading way.

e) Page 218 offers one of several examples of a mathematically nonsensical statement: "· · · should give the highest degree of accuracy at lowest costs."

3. The references in the text, and the extensive bibliography, reflect the author's thorough acquaintance with literature now outdated, but fall short of an acceptable standard for an up-to-date textbook. In fact, some of the references are dilettantish.

4. The translation is generally unsatisfactory; especially, it has served to introduce errors into the book, over and above those found in the German edition. Thus,

"· · · im Wahrscheinlichkeitsnetz · · ·" is translated as: "· · · on semi-logarithmic graph paper" (p. 105, 106, 107); "Die Zahl 77 wurde · · · subtrahiert · · ·" is translated as: "The Fig. 77 was subtracted · · ·" (p. 146); "Grössere Streuung innerhalb der einzelnen Stichproben" is translated as: "Larger variance between the individual samples" (p. 210).

It may be added that some very obvious misprints in the formulas in the German edition have been carefully preserved in the English edition. On the other hand, some new errors are introduced; cf. e.g. formula (362), p. 192. Finally, the translator has, in several cases, tried to be helpful to the reader, by inserting additional "explanations" which only serve to distort the original meaning.

In the Foreword, p. x, the author says: "A scientific book always demands critical examination from a cooperating person". In fairness to the author, I want to emphasize his commendable aim of introducing modern methods of mathematical statistics into the forest science. In fairness to the expected audience of this book, I want to make it clear that *Forest biometrics* is the regrettable outcome of the poor translation of an unsatisfactory German book.

TORE DALENIUS (*Stockholm, Sweden*)

Scattering theory. By Peter D. Lax and Ralph S. Phillips. Academic Press, New York and London, 1967. xii + 276 pp. $12.00.

This monograph describes the authors' results on the asymptotic behavior of solutions of linear hyperbolic systems. Scattering theory compares the behavior as time $t \to +\infty$ with that as $t \to -\infty$. The book contains many new results as well as new formulations of old results. The presentation throughout is elegant and practically self-contained, requiring from the reader only some familiarity with Hilbert spaces and generalized functions. The authors include whatever is needed of semigroups and of differential equations.

They are principally concerned with solutions of the wave equation $u_{tt} = \Delta u$ in a domain G. The solutions should vanish on the boundary of G and have finite (time-independent) energy norm:

$$||u||_H^2 = \int_G (u_t^2 + |\nabla u|^2)\, dx < \infty.$$

G is the exterior of a bounded domain (the "obstacle") with C^2 boundary in an *odd*-dimensional Euclidean space. The principal results are the following.

(1) $u(x, t) \to 0$ in local energy norm as $|t| \to \infty$; that is, in the L^2 norms of the first derivatives over bounded subsets of G. (2) The rate of decay is exponential if the obstacle is star-shaped and the solution has Cauchy data of compact support. (3) There is only a discrete set of complex eigenvalues, all of which have negative real parts, of the stationary problem: $\Delta u = \mu^2 u$ in G, $u = 0$ on $bdyG$, u is "μ-outgoing." (4) Convergence of solutions of the inhomogeneous wave equation to a steady state ("limiting amplitude principle").

To each solution u there corresponds a pair u_+ and u_- of solutions of the wave equation in free space such that $u - u_\pm$ tends to zero in energy norm as $t \to \pm \infty$. This is an easy consequence of Huygens' principle, the dimension being odd. Define the *scattering operator* S as carrying the Cauchy data of u_- at time zero into that of u_+. S contains most of the physically interesting information in the system and it can be regarded as a unitary operator on the Hilbert space H of pairs of functions in G of finite energy. Then (5) S determines G. (6) H may be represented as $L^2(-\infty, \infty; K)$, K being an auxiliary Hilbert space, in such a way that S is a "multiplication" operator $\mathcal{S}(\sigma)$, each $\mathcal{S}(\sigma)$ being a unitary operator on K; $\mathcal{S}(\sigma)$ has a meromorphic extension $\mathcal{S}(z)$ to the whole complex plane. (7) The poles of $\mathcal{S}(z)$ occur only at $z = -i\mu_n$ where μ_n are the eigenvalues of the outgoing stationary problem. (8) $\mathcal{S}(\sigma)$ equals the identity plus an integral operator with a certain analytic kernel, the "transmission coefficient".

As for the methods, in Chapter II a basic representation theorem is deduced from the abstract setup $\{H, U(t), D_\pm\}$, where $U(t)$ is a group of unitary operators $(-\infty < t < \infty)$ on H, and D_+ and D_- are a pair of orthogonal subspaces related to this group. In the case of the wave equation, $U(t)$ is the propagator and $D_\pm$ consists of the Cauchy data at $t = 0$ of solutions vanishing in $|x| < \pm t$. In Chapter III, the analytic properties of $\mathcal{S}(z)$ are linked to the spectral properties of the *semigroup* $Z(t) = P_+U(t)P_-$, $t > 0$, where $P_\pm$ is the projection onto the orthogonal complement of $D_\pm$. For instance, z is a pole of $\mathcal{S}(z)$ if and only if iz is an eigenvalue of the infinitesimal generator B of $Z(t)$. Roughly speaking, P_- filters out the distant past and P_+ the distant future.

In Chapter IV, the wave equation is discussed and the Sommerfeld radiation conditions at infinity are reformulated in a very natural way. It is shown that a solution of $\Delta v = \mu^2 v$ $(\mathrm{Re}\mu \geq 0)$ satisfies the outgoing [incoming] Sommerfeld condition if and only if the solution of the wave equation with Cauchy data $\{v, \mu v\}$ vanishes for $|x| < t - t_0$ [for $|x| < t_0 - t$] for some t_0. In Chapter V everything is put together. The keystone of the edifice is (1) above, for which three proofs are given! (1) is closely related to the stationary problem (3), whose eigenvalues are precisely the eigenvalues of B. The main analytic properties of $\mathcal{S}(z)$ are deduced from Chapter III and the compactness of $Z(t)\ (\lambda - B)^{-1}$ for λ positive and t large.

In case the obstacle is *star-shaped*, an energy estimate of C. Morawetz is used to obtain (2) above, as well as the existence of $\alpha > 0$ such that $\mathcal{S}(z)$ is holomorphic and bounded in Im $z < \alpha$. The same results would be valid if $Z(t)$ were itself compact for large t. Also, every horizontal strip would contain only a finite number of poles of $\mathcal{S}(z)$, and $Z(t)$ would have an asymptotic expansion in terms of the μ_n. The authors conjecture this compactness in case the obstacle does not trap energy in the sense of geometrical optics. In as yet unpublished work, this has been proved by D. Ludwig and C. Morawetz for *convex* domains.

Other applications of the abstract theory of Chapters II and III are given. *A*. The reflection of particles off an obstacle. *B*. Symmetric hyperbolic systems in odd dimensions which enjoy the unique continuation property, with minimal, energy-preserving, coercive boundary conditions. Maxwell's equations are considered in an appendix by G. Schmidt. *C*. The acoustic equation with an indefinite energy form. In this case, D_+ and D_- do not span all of H and $\mathcal{S}(z)$ may have a finite number of poles on the negative imaginary axis. *D*. Similar results are deduced for the Schrödinger equation using the Birman-Kato invariance principle.

The authors and their associates are presently pushing the theory in several directions. They are interested in the location of the poles of the scattering matrix. Situations in which Huygens' principle is not valid are being investigated by more direct methods. Nonlinear problems constitute an immense challenge.

This monograph beautifully illustrates the interplay between various types of analysis and so would be well-suited for a graduate course or seminar. It demonstrates how artificial are the distinctions commonly made between 'soft' and 'hard' analysis, between 'pure' and 'applied' mathematics.

Walter A. Strauss (*Providence, R. I.*)