THE SOLUTION OF SOME INTEGRAL EQUATIONS OF WIENER-HOPF TYPE*

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1. Introduction. The number of functional equations of any type that can be solved explicitly is very small. In this paper, we shall increase it a little by showing how to solve certain integral equations of the form

$$\int_E k(x - t)f(t)\,dt = g(x), \qquad x \in E, \tag{1.1}$$

where E is a finite union of intervals:

$$E = \bigcup_{i=1}^{n} (a_i\,,\, b_i). \tag{1.2}$$

The method to be used is what I have earlier called the general method of Wiener and Hopf [12], [13]. In the present context, the method has points of contact both with the method of separation of variables for the solution of partial differential equations and with Latta's method for solving certain integral equations ([6], see also [10]).

Certain boundary value problems can be reduced to integral equations of the form (1.1). As an example, we have the problem of solving Laplace's equation

$$\varphi_{xx} + \varphi_{yy} = 0 \tag{1.3}$$

with data given on a set E on the x-axis:

$$\varphi(x, 0) = g(x), \qquad x \in E. \tag{1.4}$$

This problem can be solved explicitly when $E = (-\infty, \infty)$ by Fourier transforms. When $E = (0, \infty)$, it can still be solved [8], this time by the Wiener-Hopf method (which I called the *special* Wiener-Hopf method in [13]). When $E = (-1, 1)$, the problem (1.3), (1.4) can also be solved, but now by conformal mapping or, alternatively, by separation of variables in elliptic cylindrical coordinates [3]. I believe that these three cases exhaust the known examples of explicit solutions of Laplace's equation when the boundary values are given on a portion of the x-axis.

But (1.3), (1.4) can be reduced to an integral equation of the form (1.1) for any open set E. In addition, (1.1) can be solved, in principle, by the *general* method of Wiener and Hopf, again, whatever the open set E may be. What we shall do here is describe a method for solving (1.1) for certain simple kernels k when E has the form (1.2). The kernels studied include that associated with the boundary value problem (1.3), (1.4).

The method reduces the solution of (1.1) to a certain eigenvalue problem for an ordinary differential equation with polynomial coefficients and regular singular points.

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That is exactly what the method of separation of variables does when it applies to a boundary value problem like (1.3), (1.4). Thus, when E is the single interval $(-1, 1)$, separation of variables in elliptic cylindrical coordinates reduces (1.3), (1.4) and similar boundary value problems to the solution of an eigenvalue problem. The method we shall describe transforms the problem into *exactly the same eigenvalue problem* when E is a single interval. On the other hand, when E consists of more than one interval, there is no coordinate system in which (1.3) is separable, but our method continues to apply. Therefore, it is fair to say that the method generalizes the method of separation of variables.

The method also applies to some integral equations that are not derived from boundary value problems. Let $k^{\wedge}$ denote the Fourier transform of the kernel k of (1.1). It will be seen below that the method depends primarily on the hypothesis that some power of $k^{\wedge}$ is *rational.* Now, if k is any function whose Fourier transform has this property, it is not hard to prove that k itself satisfies an ordinary differential equation with linear coefficients. This last is exactly the hypothesis made by Latta [6], who also reduced the solution of (1.3) (when E is a single interval) to an ordinary differential equation, but in an entirely different way.

Our entire argument depends on the general Wiener-Hopf method. A description of the method is therefore supplied in Sec. 2. No proofs are given in Sec. 2, however; these can be found in [12] and [13].

The rest of the paper consists of examples of solutions of equations of the form (1.1). Although it seems to be possible to solve (1.1) by the method described here whenever a power of $k^{\wedge}$ is rational, the general solution is complicated. Therefore, I have chosen this method, where the ideas are illustrated by a number of examples, as superior to the derivation of a solution which, while very general, is largely unintelligible.

In the first three examples, solved in Secs. 3, 4, and 5, the set E is a single interval. By choosing E in this way, we are able to describe certain aspects of the method most clearly. In Sec. 3, the kernel is $k(x) = |x|^{-\nu}$, $0 < \nu < 1$. In Sec. 4, the kernel is that associated with the boundary value problem (1.3), (1.4). The problem is solved completely, and it is shown how the separation of variables solution can be recovered from our solution.

The integral equations of Secs. 3 and 4 can be solved by other means. In Sec. 5, in which we still choose E as a single interval, the complicated kernel $|x|^{\nu} K_\nu(\kappa |x|)$ is considered. When $\nu = 0$, this is the kernel that arises from solving a boundary value problem like (1.3), (1.4) but with Laplace's equation replaced by the Helmholtz equation. Since this equation is still separable in elliptic-cylindrical coordinates, the problem can also be solved in that way. Again, our solution has exactly the same form as the separation of variables solution when $\nu = 0$, but our solution remains valid even when $\nu \neq 0$.

This method remains essentially the same when E consists of more than one interval. However, there are enough differences that an example of this situation is warranted. Such an example is supplied in Sec. 6, where the solution of (1.3), (1.4) is determined when E is the union of *two* intervals. Study of this example will make it clear, I hope, that the method imposes no limitation whatever on the number of intervals of which E is composed. The reader who is interested in a problem (1.1) in which E consists of more than two intervals should easily be able to construct the solution for himself.

2. The general method of Wiener and Hopf. The special Wiener-Hopf method is a method for solving equations

$$g(x) = \int_0^\infty k(x - t)f(t)\,dt, \qquad x > 0, \tag{2.1}$$

and certain other equations similar to (2.1). (See [5], [8], [15], [16].)

Let $k\hat{\ }(\xi)$ denote the Fourier transform of $k(x)$:

$$k\hat{\ }(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}k(x)\,dx.$$

Wiener and Hopf showed that if $k\hat{\ }(\xi)$ is positive and does not approach zero too fast at infinity,[1] then $k\hat{\ }(\xi)$ can be factored into a product of two functions having certain desirable analyticity properties. With the aid of these factors, Eq. (2.1) can be solved.

In [12] and [13], I tried to show that it is not the hard analytic details of (2.1) that allow it to be solved. Rather, it is certain general properties that (2.1) has in common with a very large class of equations that are important. To see this, we shall recast (2.1) in a different, more general, form.

Let $f(x)$ be any function defined on $(-\infty, \infty)$. Define an operator P by the equation

$$\begin{aligned}(Pf)(x) &= f(x), &\quad x > 0,\\ &= 0, &\quad x \leq 0.\end{aligned} \tag{2.2}$$

Then, (2.1) is equivalent to

$$(Pg)(x) = P\left[\int_{-\infty}^{\infty} k(x - t)(Pf)(t)\,dt\right]\cdot \tag{2.3}$$

Indeed, if $x \leq 0$, (2.3) becomes just the identity $0 = 0$, while if $x > 0$, (2.3) becomes

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} k(x - t)(Pf)(t)\,dt\\ &= \int_0^\infty k(x - t)f(t)\,dt,\end{aligned}$$

by (2.2).

Now, write

$$(Af)(x) = \int_{-\infty}^{\infty} k(x - t)f(t)\,dt. \tag{2.4}$$

Then, (2.3) can be written in the shorter form

$$Pg = PAPf, \tag{2.5}$$

and this equation is equivalent to (2.1).

Next, we note two facts about the operators A and P occurring in (2.5). We consider them both as operators on $L^2(-\infty, \infty)$. Without being too precise about the conditions for the validity of the following operations, we note that

$$\begin{aligned}(Af, f) &\equiv \int_{-\infty}^{\infty} (Af)(x)\cdot \bar{f}(x)\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} k\hat{\ }(\xi)\ |f\hat{\ }(\xi)|^2\,d\xi,\end{aligned} \tag{2.6}$$

[1] The precise condition is $\int_{-\infty}^{\infty} (\log k\hat{\ }(\xi))/(1 + \xi^2)\,d\xi > -\infty$; see [16].

by Parseval's formula. If, as Wiener assumes, $k^{\wedge}(\xi)$ is positive, (2.6) shows that

$$(Af, f) > 0 \text{ if } f \not\equiv 0. \tag{2.7}$$

A selfadjoint, linear operator A satisfying (2.7) for all $f \in D(A)$ is called *positive.* What we have shown is that if $k^{\wedge}(\xi)$ is positive, *then the operator A defined by* (2.4) *is positive.*

The second important fact about (2.5) is that *the operator P* occurring there *is an orthogonal projection.* This is trivial; one only has to see that P is selfadjoint and $P^2 = P$. Thus, the Wiener-Hopf equation (2.1) has the form (2.5) where A is a positive operator and P is a projection. We shall call any equation of the form (2.5) with A a positive operator and P a projection a Wiener-Hopf equation. If A and P have the special forms (2.4) and (2.2), we shall refer to (2.5) as a *special* Wiener-Hopf equation.

It should be noted that (1.1) is a Wiener-Hopf equation with this definition whatever the open set E may be. To reduce (1.1) to the form (2.5), it is only necessary to define A as before, by (2.4), and to define P by

$$\begin{aligned}(Pf)(x) &= f(x), \qquad x \in E,\\ &= 0, \qquad x \notin E.\end{aligned} \tag{2.8}$$

In [13], I showed[2] that whenever A is a positive operator and P is a projection, then A can be factored as in the special Wiener-Hopf method in such a way that (2.5) can be solved. This brought equations such as (1.1) under the purview of the Wiener-Hopf method. A consequence of this fact is a representation for the solution of (2.5). To state the representation theorem, we need one preliminary definition.

Let H be a Hilbert space and P a projection on it. Denote the scalar product in H by parentheses. Denote the range of P by $R(P)$. We say[3] that a sequence $\{\chi_n\}$ is *total* in $R(P)$ if

$$\{\chi_n\} \subset R(P); \tag{2.9}$$

$$(g, \chi_n) = 0 \text{ for all } n \text{ implies } Pg = 0. \tag{2.10}$$

A good deal of [13] is devoted to the question of when the Wiener-Hopf equation (2.5) has a solution. Here, however, we are interested in the question of explicit solutions of Eqs. (1.1) for which we know there is a solution since, for example, the equation may have come from a boundary value problem like (1.3), (1.4) for which a solution is known to exist. Therefore, we shall generally assume that a solution exists. When there is any doubt of this, [13] should be consulted. With this understanding, we can state the basic

THEOREM 1. *Let A be a positive operator on a Hilbert space H. Let P be a projection in H. Let $\{\chi_n\}$ be any sequence total in $R(P)$ such that $\{A^{1/2}\chi_n\}$ is orthonormal. Then, if the Wiener-Hopf equation* (2.5) *has a solution, it is given by*

$$Pf = \sum (Pg, \chi_n)\chi_n . \tag{2.11}$$

This series converges in the following sense: the sequence of partial sums of the series

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[2]With an additional technical hypothesis related to the condition of footnote 1.

[3]The definition given here is slightly different from the one in [13]. The one given here is more convenient for our purposes, and it is not hard to show that the two definitions are equivalent.

$$\sum (Pg, \chi_n)A^{1/2}\chi_n$$

converges in the norm of H.

When we consider integral equations of the form

$$\int_E k(x - t)f(t)\, dt = g(x), \qquad x \in E, \tag{2.12}$$

it is convenient to work in the Hilbert space $H = L^2(-\infty, \infty)$. Theorem 1 applies to this equation, of course, by defining A and P by (2.4) and (2.8). Unfortunately, however, there are a number of interesting equations of the form (2.12) for which the solution exists, but is *not* in L^2.

To deal with this problem, we have to state Theorem 1 in a slightly different way. Let H be $L^2(-\infty, \infty)$, and let P be defined by (2.8) for some open set E. Also, let A be a positive operator of the form (2.4). If f is a function in the domain of A, the quantity $(Af, f)^{1/2}$ is a norm, since A is positive. However, the integral

$$(Af, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - t)f(t)\overline{f(x)}\, dt\, dx \tag{2.13}$$

may make sense even if f is not in the domain of A. An example is the following. Let $k(x) = -(1/\pi) \log |x|$. As we shall see later, the corresponding operator (2.4) is positive. Let $f(x) = 1$ for $|x| < 1$, and let $f(x) = 0$ otherwise. Then f is not in the domain of A, for when $x > 1$,

$$\begin{aligned} g(x) &= -\frac{1}{\pi} \int_{-1}^{1} \log |x - t|\, dt \\ &= (x + 1) \log (x + 1) - (x - 1) \log (x - 1). \end{aligned}$$

Since this grows logarithmically as $x \to \infty$, it cannot be in $L^2(-\infty, \infty)$. On the other hand,

$$\begin{aligned} (Af, f) &= -\frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \log |x - t|\, dt\, dx \\ &= \frac{2}{\pi}(3 - 2 \log 2). \end{aligned}$$

Let H_+ denote the set of all functions such that the right side of (2.13) is finite. For such functions, we define (Af, f) by (2.13). In the same way, we let H_- denote the set of all functions g such that $(A^{-1}g, g)$ is finite. An important point to notice is that if $f \in H_+$ and $g \in H_-$, it makes sense to speak of the scalar product (g, f), for the generalized Schwarz inequality [9] shows that

$$|(g, f)|^2 \leq (A^{-1}g, g)(Af, f).$$

Recall now that P is defined by (2.8). $R(P)$ is a subspace of $L^2(-\infty, \infty)$, of course, since it simply consists of all functions in $L^2(-\infty, \infty)$ that are zero outside of E. Clearly, $R(P)$ can be defined in a natural way as a subspace of either H_+ or H_-. If f is a function in H_+, say, and if $f(x) = 0$ for $x \notin E$, we shall still write $f \in R(P)$. Thus, the symbol $R(P)$ will be used to denote functions that are zero for $x \notin E$, regardless of which of the three spaces H_+, H_-, or $L^2(-\infty, \infty)$ they may lie in. No confusion need result from this practice.

Let $\{\chi_n\}$ be a sequence of functions in H_+. We shall say that the sequence $\{\chi_n\}$ is *A-orthonormal* if

$$(A\chi_m, \chi_n) = \delta_{mn},$$

the Kronecker delta. Clearly, this idea of A-orthonormality generalizes the idea that the sequence $\{A^{1/2}\chi_n\}$ is orthonormal in $L^2(-\infty, \infty)$.

Next, a sequence $\{\chi_n\}$ in H_+ will be called *total in* $R(P)$ if two things are true. First, each function χ_n must be in $R(P)$. Second, it must be true that if a function $g \in H_-$ is in $R(P)$, and if $(g, \chi_n) = 0$ for all n, then g must be zero. Again, this notion clearly generalizes the definition of totality given earlier.

We then have the following generalization of [13, Theorem 1].

THEOREM 2. *Let A be the integral operator* (2.4). *Define the spaces H_+ and H_- as we have just done. Let $\{\chi_n\}$ be any A-orthonormal sequence in H_+ that is total in $R(P)$. Let $g \in H_-$ be an element in $R(P)$, and suppose that the integral equation*

$$g(x) = \int_E k(x - t)f(t)\,dt, \qquad x \in E,$$

has a solution in H_+. Then, this solution is given by

$$f = \sum_{n=1}^{\infty} (g, \chi_n)\chi_n. \tag{2.11}$$

The series converges in the sense that

$$(A(f - f_N), f - f_N) \to 0 \quad \text{as } N \to \infty,$$

where $f_N = \sum_{n=1}^{N} (g, \chi_n)\chi_n$.

The point of Theorem 2 is that the solution need not be in L^2, but only in H_+. Formally, however, all calculations are the same, except that the operator $A^{1/2}$ never appears. Instead of $\|A^{1/2}f\|^2$, say, we always write (Af, f) for $f \in H_+$. Notice that the formula (2.11) appears in both Theorem 1 and Theorem 2. It is the basic formula that we shall use in the sequel.

3. The kernel $|x|^{-\nu}$ on a single strip. We begin our discussion by considering the integral equation

$$g(x) = \int_{-1}^{1} \frac{f(t)}{|x - t|^{\nu}}\,dt, \qquad -1 < x < 1. \tag{3.1}$$

We assume that $0 < \nu < 1$. As in Sec. 2, to bring (3.1) into the form (2.5) of a general Wiener-Hopf equation, we define an operator A by

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(t)}{|x - t|^{\nu}}\,dt, \tag{3.2}$$

and a projection P by

$$\begin{aligned} (Pf)(x) &= f(x), \qquad \text{if } -1 < x < 1, \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{3.3}$$

Then, (3.1) can be written in the form

$$PAPf = Pg.$$

In order for the theorems of Sec. 2 to apply, we must show that A is positive. This is easy, for the Fourier transform of the kernel $|x|^{-\nu}$ is $c_0\ |\xi|^{\nu-1}$, where c_0 is the positive constant

$$c_0 = \frac{\pi \sec (\pi\nu/2)}{\Gamma(\nu)}. \tag{3.4}$$

(See, *e.g.*, [4].) Therefore, Parseval's equation shows that

$$(Af, f) = \frac{c_0}{2\pi} \int_{-\infty}^{\infty} |\xi|^{\nu-1}\ |f^{\wedge}(\xi)|^2\ d\xi,$$

and this is positive if $f \not\equiv 0$.

According to the Theorem 1, then, to solve (3.1) we need only find a sequence $\{\chi_n\}$, total in $R(P)$, and such that $\{A^{1/2}\chi_n\}$ is orthonormal. One way to do this is simply to pick any total sequence in $R(P)$—say, the powers x^n—and to orthonormalize the sequence $\{A^{1/2}Px^n\}$ by the Gram-Schmidt process. But since it is hard to justify the choice of the sequence $\{x^n\}$ over any other total sequence, we shall proceed differently, and attempt to find a sequence $\{\chi_n\}$ that in some sense is naturally connected with the integral operator (3.2).

Let L be any selfadjoint operator on $L^2(-\infty,\ \infty)$. Suppose the χ_n's can be chosen as the nontrivial solutions of the equation

$$L\chi = \lambda A \chi \tag{3.5}$$

normalized, of course, so that

$$||A^{1/2}\chi|| = 1. \tag{3.6}$$

Then the condition that $\{A^{1/2}\chi_n\}$ be orthonormal is automatic. This is proved in exactly the same way that it is proved that the eigenvectors of a selfadjoint operator are orthonormal.

In order for a sequence $\{\chi_n\}$ to qualify as a sequence for which (2.11) is valid, it must have two properties: it must be total in $R(P)$, and $\{A^{1/2}\chi_n\}$ must be orthonormal. *We shall choose the χ's as solutions of an equation* (3.5) *with L selfadjoint.* This will imply the sequence $\{A^{1/2}\chi_n\}$ is orthogonal.

We must also require that $\{\chi_n\}$ be total in $R(P)$. We shall return to this important condition in a moment. But first, we should like to impose some kind of condition that assures us that (3.5) is in some way easier to study than the original equation (3.1). Because the type of equation most studied in analysis is the ordinary differential equation with polynomial coefficients and regular singular points, *we meet this condition by requiring arbitrarily that* (3.5) *be equivalent to such an equation.* For shortness, we shall call an equation equivalent to an ordinary differential equation with polynomial coefficients and regular singular points a *regular equation.*

To see how to construct an operator L such that (3.5) is regular while the χ's are total in $R(P)$, consider the Fourier transform of (3.5). As we remarked before, the Fourier transform of $A\chi$ is $c_0\ |\xi|^{\nu-1}\ \chi^{\wedge}(\xi)$. Therefore, if we define an operator $L^{\wedge}$ by the equation $L^{\wedge}\chi^{\wedge} = (L\chi)^{\wedge}$, the transform of (3.5) takes the form

$$L^{\wedge}\chi^{\wedge} = \lambda c_0\ |\xi|^{\nu-1}\ \chi^{\wedge}. \tag{3.7}$$

There are any number of selfadjoint[4] operators $L^\wedge$ with the property that at least the associated equation (3.7) is regular. Examples are

$$L^\wedge \chi^\wedge = |\xi|^{1+\nu} \chi^\wedge, \qquad \left(\frac{d}{d\xi} |\xi|^{1+\nu} \frac{d}{d\xi}\right)\chi^\wedge, \qquad \left(\frac{d}{d\xi} |\xi|^{3+\nu} \frac{d}{d\xi}\right)\chi^\wedge, \qquad \left(\frac{d^2}{d\xi^2} |\xi|^{1+\nu} \frac{d^2}{d\xi^2}\right)\chi^\wedge, \tag{3.8}$$

etc. In each case, it can be verified that the corresponding equation (3.7) is regular by a simple computation.

Any linear combination of the operators (3.8) still gives rise to an equation (3.7) that is regular. Let α be a constant, and define

$$L^\wedge \chi^\wedge = \left(\frac{d}{d\xi} |\xi|^{1+\nu} \frac{d}{d\xi}\right)\chi^\wedge + \alpha |\xi|^{1+\nu} \chi^\wedge. \tag{3.9}$$

This operator is formally selfadjoint. In addition, if $L^\wedge$ is defined by (3.9), (3.7) can easily be reduced to the form

$$\xi^2\left(\frac{d^2}{d\xi^2} + \alpha\right)\chi^\wedge + (1 + \nu)\xi \frac{d}{d\xi} \chi^\wedge = \lambda c_0 \chi^\wedge. \tag{3.10}$$

This equation is certainly regular. We should like to be able to say that its nontrivial solutions qualify as the Fourier transforms of functions that can be used in (2.11). For that, we must surely show that (3.10) has solutions which are transforms of functions that are in $R(P)$, and so are zero for $|x| > 1$. Take the inverse transform of (3.10). We find that any solution $\chi^\wedge$ of (3.10) is the transform of a function χ satisfying

$$\frac{d^2}{dx^2} [(x^2 - \alpha)\chi(x)] - (1 + \nu) \frac{d}{dx} [x\chi(x)] = \lambda c_0 \chi(x). \tag{3.11}$$

Suppose that (3.10) has a solution that is the transform of a function that is zero for $|x| > 1$. Then, this function will satisfy (3.11), and taking the *finite* Fourier transform of (3.11) over the interval $(-1, 1)$, we should get (3.10) back. But the finite Fourier transform of (3.11) is

$$\xi^2\left(\frac{d^2}{d\xi^2} + \alpha\right)\chi^\wedge + (1 + \nu)\xi \frac{d}{d\xi} \chi^\wedge = \lambda c_0 \chi^\wedge + R,$$

where the remainder R has the form

$$R = \left\{-\frac{d}{dx} [(x^2 - \alpha)\chi(x)] + (1 + \nu)x\chi(x) + i\xi(x^2 - \alpha)\chi(x)\right\}e^{i\xi x} \Big|_{x=-1}^{x=1}.$$

For the finite transform of (3.11) to agree with (3.10), then, R must vanish, and this yields the four conditions

$$\begin{aligned} &\text{a.} \qquad \lim_{x \to \pm 1} (x^2 - \alpha)\chi(x) = 0, \\ &\text{b.} \qquad \lim_{x \to \pm 1} \left\{\frac{d}{dx} [(x^2 - \alpha)\chi(x)] - (1 + \nu)x\chi(x)\right\} = 0. \end{aligned} \tag{3.12}$$

If $\alpha \neq 1$, (3.12a) gives $\chi(-1) = 0$, and then (3.12b) gives $\chi'(-1) = 0$. Since the point $x = -1$ is a regular point of (3.11) when $\alpha \neq 1$. we conclude that the only solution

[4]Formally; we shall worry about such things as boundary conditions later on.

of (3.11) and (3.12) is $\chi(x) \equiv 0$. If we are to have any hope that the solution of (3.11) and (3.12) be total, then, we must choose $\alpha = 1$. With this choice of α, (3.11) becomes

$$(1 - x^2)\chi'' - (3 - \nu)x\chi' + \mu\chi = 0, \tag{3.13}$$

where we have written $\mu = \lambda c_0 + \nu - 1$. Also, the boundary conditions (3.12) turn into

$$\begin{aligned} &\text{a.} \quad \lim_{x \to \pm 1} (1 - x^2)\chi(x) = 0, \\ &\text{b.} \quad \lim_{x \to \pm 1} [(1 - x^2)\chi'(x) - (1 - \nu)x\chi(x)] = 0. \end{aligned} \tag{3.14}$$

There are four boundary conditions here. But two of them are irrelevant, for a very simple reason. The indicial equation of (3.13) has the roots 0 and $-(1 - \nu)/2$ at both singular points $x = \pm 1$. Therefore, in a neighborhood of $x = 1$, say, the general solution of (3.13) has the form

$$\sum a_n (1 - x)^n + \sum b_n (1 - x)^{n - (1-\nu)/2}.$$

But *any* function of this form satisfies the boundary condition (3.14a) at $x = 1$ since $(1 - x^2)$ goes to zero linearly and $\nu > 0$. Indeed, this would be so even if ν were merely greater than -1. Thus, we can simply ignore the condition (3.14a).

We now consider the eigenvalue problem (3.13), (3.14b). If the argument leading to these equations is carried backwards, it will be seen that this problem can be written in the form (3.5), with L at least formally selfadjoint and, indeed, the boundary conditions (3.14b) are such that L is selfadjoint. We conclude, then, that the solutions of (3.13), (3.14b) have the orthogonality property

$$(A^{1/2}\chi_m, A^{1/2}\chi_n) = 0 \qquad \text{if } m \neq n.$$

In addition, (3.13) is clearly a regular equation, as desired.

It remains to show that there are enough solutions of (3.13), (3.14b) that the totality hypothesis (2.10) is satisfied. But this is a standard matter since, after all, (3.13), (3.14b) is a Sturm-Liouville problem, and the completeness of the eigenfunctions is well known [1]. Therefore, we can apply Theorem 1 to derive the following result.

THEOREM. *Let $\{\chi_n(x)\}$ denote the solutions of the eigenvalue problem*

$$(1 - x^2)\chi'' - (3 - \nu)x\chi' + \mu\chi = 0,$$

$$\lim_{x \to \pm 1} [(1 - x^2)\chi'(x) - (1 - \nu)x\chi(x)] = 0.$$

Define $\chi_n(x) \equiv 0$ for $|x| > 1$, and normalize $\chi_n(x)$ by the condition that

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{1} \frac{\chi_n(x)\chi_n(t)}{|x - t|^\nu}\, dx\, dt &\equiv \frac{\sec(\nu\pi/2)}{2\Gamma(\nu)} \int_{-\infty}^{\infty} |\xi|^{\nu-1}\, |\hat{\chi}_n(\xi)|^2\, d\xi \\ &= 1. \end{aligned} \tag{3.15}$$

Then, the solution of the integral equation (3.1) *is given by*

$$f(t) = \sum \chi_n(t) \int_{-1}^{1} \chi_n(x)g(x)\, dx,$$

if it exists.

Until now, it has not been mentioned that the eigenvalue problem (3.13), (3.14b) can be solved in closed form, although this is in fact the case. The reason for not mentioning it is that the derivation of the preceding theorem is meant to illustrate a general method rather than to solve a special problem, and in general, of course, the corresponding eigenvalue problem will not be so simple as to have elementary solutions. Having derived the above result without using the explicit solution, however, we now finish the job by solving (3.13), (3.14b).

Let $C_n^{\nu/2}$ denote the (nonnormalized) Gegenbauer polynomial defined by

$$C_n^{\nu/2}(x) = (1 - x^2)^{(1-\nu)/2} \frac{d^n}{dx^n} (1 - x^2)^{n+(\nu-1)/2}. \tag{3.16}$$

It is known [2] that the function

$$\chi_n(x) = \gamma_n(1 - x^2)^{(\nu-1)/2} C_n^{\nu/2}(x), \tag{3.17}$$

where γ_n is a constant, satisfies the differential equation (3.13) with

$$\mu = (n + 1)(n + \nu - 1).$$

Moreover, with χ_n defined by (3.17), the boundary conditions (3.14b) becomes simply

$$\lim_{x\to\pm1} (1 - x^2)^{(1+\nu)/2} C_n^{\nu/2}(x) = 0,$$

and this is surely correct since $C_n^{\nu/2}(x)$ is a polynomial.

It remains to impose the normalization condition (3.15). The finite Fourier transform of (3.17) is known [4]. It is simply

$$\hat{\chi}_n(\xi) = \gamma_n \pi^{1/2} 2^{n+\nu/2} e^{-in\pi/2} \Gamma\left(n + \frac{\nu + 1}{2}\right) \xi^{-\nu/2} J_{n+\nu/2}(\xi).$$

Therefore, (3.15) reads

$$1 = \gamma_n^2 \pi 2^{2n+\gamma} \Gamma^2\left(n + \frac{\nu + 1}{2}\right) \frac{\sec(\pi\nu/2)}{2\Gamma(\nu)} \int_{-\infty}^{\infty} |\xi|^{-1} [J_{n+\nu/2}(\xi)]^2 \, d\xi$$

$$= \frac{\gamma_n^2 \pi 2^{2n+\nu} \Gamma^2\left(n + \frac{\nu + 1}{2}\right) \sec(\pi\nu/2)}{(2n + \nu)\Gamma(\nu)}.$$

(See, *e.g.* [2, p. 92].) Thus γ_n must be taken to be

$$\gamma_n = \Gamma\left(n + \frac{\nu + 1}{2}\right)\left(\frac{(2n + \nu)\Gamma(\nu) \cos(\pi\nu/2)}{\pi 2^{2n+\nu}}\right)^{1/2} \tag{3.18}$$

and we have the

Corollary. *Define*

$$\chi_n(x) = \gamma_n \frac{d^n}{dx^n} (1 - x^2)^{n+(\nu-1)/2}$$

where γ_n is given by (3.18). *Then, the solution of the integral equation* (3.1) *is*

$$f(t) = \sum \chi_n(t) \int_{-1}^{1} \chi_n(x) g(x) \, dx,$$

if it exists.

4. The potential of a strip. Consider Laplace's equation

$$\varphi_{xx} + \varphi_{yy} = 0 \tag{4.1}$$

where φ is given on a line segment on the x-axis, say

$$\varphi(x, 0) = g(x), \qquad -1 < x < 1. \tag{4.2}$$

We suppose, as usual, that φ has at most logarithmic growth at infinity.

The function

$$\varphi(x, y) = -\frac{1}{\pi} \int_{-1}^{1} f(t) \log ((x - t)^2 + y^2)^{1/2} \, dt \tag{4.3}$$

has logarithmic growth and satisfies (4.1) whatever $f(t)$ may be. Therefore, (4.3) will be a solution of the entire boundary value problem if f is chosen to satisfy

$$g(x) = -\frac{1}{\pi} \int_{-1}^{1} f(t) \log |x - t| \, dt, \qquad -1 < x < 1. \tag{4.4}$$

(4.4) is again a Wiener-Hopf equation, having the form (2.5). We define

$$(Af)(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \log |x - t| \, dt$$

whenever this makes sense, and

$$\begin{aligned} (Pf)(x) &= f(x), \qquad -1 < x - 1 \\ &= 0, \qquad \text{otherwise} \end{aligned}$$

The kernel

$$k(x) = -\frac{1}{\pi} \log |x|$$

has generalized Fourier transform $1/|\xi|$, in the sense that [4]

$$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} e^{-\epsilon |x|} e^{i\xi x} \log |x| \, dx = -\frac{\pi}{|\xi|}.$$

Thus, A is positive, and the theorems of Sec. 2 apply.

In this case, it will have to be Theorem 2 that is used rather than Theorem 1, for it is known ([7]; see also [6], [11]) that if g is continuous, f tends to look like $(1 - x^2)^{-1/2}$ near the ends of the interval $(-1, 1)$ and so will not be in L^2 in general. What is important about this is not the warning that Theorem 2 must be used instead of Theorem 1, but that the following calculations are the same no matter which of the two theorems is used. Thus, one need not know in advance how the solution of the integral equation behaves.

We proceed just as in Sec. 3, starting with Eq. (3.5). If L is selfadjoint in L^2 and the functions $\{\chi_n\}$ are chosen as the nontrivial solutions of the equation

$$L\chi = \lambda A\chi, \tag{4.5}$$

then the sequence $\{\chi_n\}$ will be A-orthonormal.

Next, set $(L\chi)\hat{} = L\hat{}\chi\hat{}$. Since the Fourier transform of the kernel of A is $1/|\xi|$, the transform of (4.5) is

$$\hat{L}\hat{\chi} = \lambda\hat{\chi}/|\xi|. \tag{4.6}$$

We wish (4.5) and (4.6) to be regular. A convenient choice of a selfadjoint operator that makes (4.6) regular is

$$L\hat{\ }\chi\hat{\ } = \left(\frac{d}{d\xi}\,|\xi|\,\frac{d}{d\xi}\right)\chi\hat{\ } + \alpha\,|\xi|\,\chi\hat{\ },\tag{4.7}$$

where α is a constant. Again, (4.7) is only one operator out of many that will make (4.6) regular.

With $L\hat{\ }$ given by (4.7), (4.6) becomes

$$\xi^2\left(\frac{d^2}{d\xi^2} + \alpha\right)\chi\hat{\ } + \xi\,\frac{d}{d\xi}\,\chi\hat{\ } = \lambda\chi\hat{\ }.\tag{4.8}$$

The inverse transform of any solution of this equation will satisfy

$$\frac{d^2}{dx^2}\,[(x^2 - \alpha)\chi] - \frac{d}{dx}\,(x\chi) = \lambda\chi.\tag{4.9}$$

Now, suppose that (4.8) has a solution that is the transform of a function $\chi(x)$ which is zero for $|x| > 1$. Then, (4.8) will be recoverable from (4.9) by taking the finite Fourier transform over the interval $(-1, 1)$. If the finite transform of (4.9) is taken, however, the result is (4.8) with a remainder term added on. This remainder term vanishes if and only if χ satisfies the boundary conditions

$$\begin{aligned}&\lim_{x\to\pm 1}\,[(\alpha - x^2)\chi(x)] = 0,\\ &\lim_{x\to\pm 1}\left\{\frac{d}{dx}\,[(\alpha - x^2)\chi(x)] + x\chi(x)\right\} = 0.\end{aligned}\tag{4.10}$$

As in Sec. 3, if $\chi(x)$ is not to be identically zero, α must be unity. Therefore, (4.9) and (4.10) become

$$\begin{aligned}&\text{a.}\quad (1 - x^2)\chi'' - 3x\chi' + (\lambda - 1)\chi = 0,\\ &\text{b.}\quad \lim_{x\to\pm 1}\,(1 - x^2)\chi(x) = 0,\\ &\text{c.}\quad \lim_{x\to\pm 1}\,[(1 - x^2)\chi'(x) - x\chi(x)] = 0.\end{aligned}\tag{4.11}$$

The indicial equation of (4.11a) has roots 0 and $(-1/2)$ at both singular points; therefore, (4.11b) is satisfied by *every* solution of (4.11a) and is irrelevant.

The eigenvectors of (4.11a, c) are total. Therefore, if we define the functions $\chi_n(x)$ to be the nontrivial solutions of (4.11a, c) that are identically zero outside the interval $(-1, 1)$, Theorem 2 will apply if we can show that these functions are in the space H_+.

It should be noted that this is *not* automatic. The function

$$A\chi = \frac{1}{\pi}\int_{-1}^{1} \chi(t)\,\log\,|x - t|\,dt$$

might make sense, and χ might be a solution of (4.11), but in principle, we might have $(A\chi, \chi) = \infty$, so that χ would not be in H_+.

However, the roots of the indicial equation associated with (4.11a) are 0 and $(-1/2)$. Therefore, every solution of (4.11a) satisfies

$$|\chi(x)| \le c/(1 - x^2)^{1/2}$$

in the interval $(-1, 1)$ for some positive constant c. Let $1 < p < 2$, and let q be the index dual to p, so that $1/p + 1/q = 1$. Then for $-1 < x < 1$ we have, by Hölder's inequality,

$$\begin{aligned}
|(A\chi)(x)| &\le \frac{c}{\pi}\int_{-1}^{1}\frac{|\log|x-t||}{(1-t^2)^{1/2}}\,dt \\
&\le \frac{c}{\pi}\left(\int_{-1}^{1}\frac{dt}{(1-t^2)^{p/2}}\right)^{1/p}\left(\int_{-1}^{1}|\log|x-t||^q\,dt\right)^{1/q} \\
&= \frac{c}{\pi}\left(\int_{-1}^{1}\frac{dt}{(1-t^2)^{p/2}}\right)^{1/p}\left(\int_{x-1}^{x+1}|\log|t||^q\,dt\right)^{1/q} \\
&\le \frac{c}{\pi}\left(\int_{-1}^{1}\frac{dt}{(1-t^2)^{p/2}}\right)^{1/p}\left(\int_{-2}^{2}|\log|t||^p\,dt\right)^{1/q}.
\end{aligned}$$

Therefore, $(A\chi)(x)$ is bounded in $(-1, 1)$, and $A\chi\cdot\chi$ is integrable. Thus, $\chi \in H_+$, and we have the

THEOREM. *Let $\{\chi_n(x)\}$ denote the sequence of solutions of the eigenvalue problem*

$$\begin{aligned}
&(1-x^2)\chi'' - 3x\chi' + (\lambda - 1)\chi = 0, \\
&\lim_{x\to\pm 1}[(1-x^2)\chi'(x) - x\chi(x)] = 0.
\end{aligned} \tag{4.12}$$

Define $\chi_n(x)$ as zero for $|x| > 1$, and normalize this function by the condition

$$(A\chi_n, \chi_n) \equiv -\frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1}\chi_n(x)\chi_n(t)\log|x-t|\,dt = 1. \tag{4.13}$$

Then, the solution of the integral equation

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} f(t)\log|x-t|\,dt, \qquad -1 < x < 1,$$

is

$$f(t) = \sum \chi_n(t)\int_{-1}^{1} g(x)\chi_n(x)\,dx,$$

whenever it exists.

As in Sec. 3, the solutions of (4.12) are elementary. In fact, (4.12a) has as one of its solutions the function

$$\chi(x) = (1-x^2)^{-1/2}\quad\cos(\lambda^{1/2}\arccos x).$$

When λ is the square of an integer, this function also satisfies the boundary condition (4.12b). Therefore, we shall write

$$\chi_n(x) = \frac{\alpha_n}{(1-x^2)^{1/2}}\cos(n\arccos x), \qquad n = 0, 1, \cdots, \tag{4.14}$$

where α_n is an appropriate normalization constant.

We now evaluate α_n, using (4.13). When $n \ne 0$, an elementary integration shows that

$$\int_{-1}^{1}\chi_n(t)\,dt = 0.$$

Therefore,

$$
\begin{aligned}
(A\chi_n)(x) &= -\frac{1}{\pi}\int_{-1}^{1} \chi_n(t) \log |x - t| \, dt \\
&= -\frac{1}{\pi}\int_{-1}^{1} \chi_n(t)[\log |x - t| - \log |x|] \, dt \\
&= -\frac{1}{\pi}\int_{-1}^{1} \chi_n(t) \log \left|1 - \frac{t}{x}\right| dt \\
&= O(1/|x|) \qquad \text{as} \quad |x| \to \infty .
\end{aligned}
$$

This shows that $A\chi_n$ has a Fourier transform when $n \neq 0$, and we may use Parseval's formula to evaluate the left side of (4.13).

According to [4], the Fourier transform of (4.14) is

$$\hat{\chi_n}(\xi) = \pi\alpha_n e^{in\pi/2} J_n(\xi).$$

Also, as we pointed out before,

$$(A\chi_n)\hat{}\,(\xi) = \hat{\chi_n}(\xi)/|\xi|.$$

Therefore, Parseval's equation gives

$$
\begin{aligned}
(A\chi_n , \chi_n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (A\chi_n)\hat{}\,(\xi)\cdot\overline{\hat{\chi_n}(\xi)} \, d\xi \\
&= \pi\alpha_n^2 \int_0^{\infty} [J_n(\xi)]^2 \frac{d\xi}{\xi} \\
&= \frac{\pi\alpha_n^2}{2n}, \qquad n \neq 0.
\end{aligned}
$$

(See [2, p. 92].) If (4.13) is to be satisfied, we must choose

$$\alpha_n = (2n/\pi)^{1/2}, \qquad n = 1, 2, \cdots . \tag{4.15}$$

When $n = 0$, we can evaluate $(A\chi_n , \chi_n)$ directly. In fact,

$$
\begin{aligned}
(A\chi_0 , \chi_0) &= -\frac{\alpha_0^2}{\pi}\int_{-1}^{1}\int_{-1}^{1} \log |x - t| \frac{dt\, dx}{((1 - t^2)(1 - x^2))^{1/2}} \\
&= -\frac{\alpha_0^2}{\pi}\int_0^{\pi}\int_0^{\pi} \log |\cos x - \cos t| \, dt\, dx.
\end{aligned}
$$

Elementary manipulations show that this integral can be reduced to the form

$$(A\chi_0 , \chi_0) = -\alpha_0^2\left[\pi \log 2 + 4\int_0^{\pi/2} \log \sin x \, dx\right].$$

The integral can be found in any table of definite integrals, and the result is

$$(A\chi_0 , \chi_0) = \alpha_0^2\pi \log 2.$$

Thus, we must choose

$$\alpha_0 = \frac{1}{(\pi \log 2)^{1/2}}. \tag{4.16}$$

Using (4.14), (4.15), and (4.16) in the main theorem, we obtain the following result:

COROLLARY. *The solution of the integral equation*

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} f(t) \log |x - t| \, dt, \qquad -1 < x < 1,$$

when it exists, is given by

$$f(t) = \frac{1}{\pi \log 2}\int_{-1}^{1} \frac{g(x)}{((1 - x^2)(1 - t^2))^{1/2}} \, dx + \sum_{n=1}^{\infty} \frac{2n}{\pi}\int_{-1}^{1} g(x) \frac{\cos (n \arccos x)}{(1 - x^2)^{1/2}} \frac{\cos (n \arccos t)}{(1 - t^2)^{1/2}} \, dx \tag{4.17}$$

One further comment before leaving this example. The basic boundary value problem (4.1), (4.2) that we have solved by this theorem can also be solved by separating variables in the coordinate system (ξ, η) defined by the equations

$$\begin{aligned} x &= \cosh \eta \cos \xi, \\ y &= \sinh \eta \sin \xi. \end{aligned} \tag{4.18}$$

If one does this, he is led to exactly the functions (4.14) in terms of which we have solved the problem.

Indeed, in the (ξ, η) coordinate system, we want to solve Laplace's equation

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0$$

in the domain $0 < \xi < \pi$, $\eta > 0$, with the boundary conditions

$$\begin{aligned} \left.\frac{\partial \varphi}{\partial \xi}\right|_{\xi=0} &= \left.\frac{\partial \varphi}{\partial \xi}\right|_{\xi=\pi} = 0, \\ \left.\varphi\right|_{\eta=0} &= g(\cos \xi). \end{aligned} \tag{4.19}$$

Separating variables, we find that the solution has the form

$$\varphi = \sum a_n e^{-n\eta} \cos n\xi. \tag{4.20}$$

The coefficients $\{a_n\}$ are determined by (4.19). Setting $\eta = 0$ and noting that $x = \cos \xi$ when $\eta = 0$, we see that the coefficients must be determined from the equation

$$\begin{aligned} g(x) &= \sum a_n \cos (n \arccos x), \\ &= \sum \frac{a_n}{\alpha_n} (1 - x^2)^{1/2} \chi_n(x). \end{aligned} \tag{4.21}$$

Thus, the same functions χ_n appear here also.

Now, if $\varphi(x, y)$ is given by (4.3), a standard argument shows that

$$f(t) = -\varphi_y(t, 0).$$

One can use (4.21) to determine the a_n's and then (4.20) to determine φ_y, and so f. The result is exactly (4.17).

5. The kernel $|x|^{\nu} K_\nu(\kappa\, |x|)$. Our next example is the equation

$$g(x) = \int_{-1}^{1} |x - t|^{\nu} K_{\nu}(\kappa |x - t|)f(t)\, dt, \qquad -1 < x < 1. \tag{5.1}$$

Here, K_{ν} denotes the modified Hankel function of the third kind which can be defined by [2]

$$x^{\nu}K_{\nu}(x) = \frac{2^{\nu}\Gamma(\nu + 1/2)}{\Gamma(1/2)} \int_{0}^{\infty} \frac{\cos x\xi}{(1 + \xi^2)^{\nu+1/2}}\, d\xi \qquad (x > 0). \tag{5.2}$$

We assume that $-1/2 < \nu < 1/2$.

If $\nu = 0$ and $f(t)$ satisfies (5.1), we can define a function $\varphi(x, y)$ by the formula

$$\varphi(x, y) = \int_{-1}^{1} K_0(\kappa((x - t)^2 + y^2)^{1/2})f(t)\, dt.$$

This function will satisfy the differential equation

$$\varphi_{xx} + \varphi_{yy} - \kappa^2\varphi = 0 \tag{5.3}$$

and the boundary condition

$$\varphi(x, 0) = g(x), \qquad -1 < x < 1. \tag{5.4}$$

The boundary value problem (5.3), (5.4) can be solved by separation of variables in the coordinate system (4.18). This fact allows one to solve (5.1) with $\nu = 0$ in the same way that we solved (4.4) at the end of Sec. 4 by separating variables in (4.1), (4.2). However, (5.1) cannot be solved for $\nu \neq 0$ by this technique. We shall solve (5.1) assuming only that $\frac{1}{2} < \nu < \frac{1}{2}$. The solution that we obtain will agree with the solution by separation of variables when $\nu = 0$. I do not believe that (5.1) has been solved before for general ν.

The idea of the solution should be familiar by now. Define

$$(Af)(x) = \int_{-\infty}^{\infty} |x - t|^{\nu} K_{\nu}(\kappa |x - t|)f(t)\, dt.$$

We look for a selfadjoint operator L such that the equation $L\chi = \lambda A\chi$ is regular, while its solutions satisfy the conditions of either Theorem 1 or Theorem 2. The method proceeds exactly as in Sec. 3 or Sec. 4, and our description will be brief.

(5.2) shows that the transform of $k(x) = |x|^{\nu}K_{\nu}\,(\kappa\,|x|)$ is

$$k\hat{}(\xi) = \frac{c_0}{(\kappa^2 + \xi^2)^{\nu+1/2}},$$

where c_0 is the constant

$$c_0 = (2\kappa)^{\nu}\Gamma(1/2)\Gamma(\nu + 1/2).$$

We take the operator $L\hat{}$ to be

$$L\hat{}\chi\hat{} = \left[\frac{d}{d\xi}(\kappa^2 + \xi^2)^{1/2-\nu}\frac{d}{d\xi}\right]\chi\hat{} + (\kappa^2 + \xi^2)^{1/2-\nu}\chi\hat{}.$$

The solutions of $L\hat{}\chi\hat{} = \lambda k\hat{}\chi\hat{}$ are transforms of solutions of

$$\frac{d^2}{dx^2}[(1 - x^2)\chi(x)] + (1 - 2\nu)\frac{d}{dx}[x\chi(x)] + (\mu + \kappa^2x^2)\chi(x) = 0, \tag{5.5}$$

where $\mu = c_0\lambda - \kappa^2$. Moreover, the finite transform of (5.5) is $L^\wedge\chi^\wedge = \lambda k^\wedge\chi^\wedge$ again if and only if

$$\lim_{x\to\pm1} (1 - x^2)\chi(x) = 0,$$
$$\lim_{x\to\pm1} [(1 - x^2)\chi'(x) - (1 + 2\nu)x\chi(x)] = 0. \tag{5.6}$$

The indicial equation associated with (5.5) has roots 0 and $[-(1 + 2\nu)/2]$. Since $\nu < 1/2$, then, (5.6a) is satisfied by every solution of (5.5). It is for this reason that we assumed $\nu < 1/2$. (The condition $\nu > -1/2$ is assumed in order to make $k(x)$ integrable.)

The method now goes through as before. The result is the

THEOREM. *Let $\{\chi_n(x)\}$ denote the sequence of nontrivial solutions of the boundary-value problem*

$$\frac{d^2}{dx^2}[(1 - x^2)\chi(x)] + (1 - 2\nu)\frac{d}{dx}[x\chi(x)] + (\mu + \kappa^2x^2)\chi(x) = 0,$$
$$\lim_{x\to\pm1} [(1 - x^2)\chi'(x) - (1 + 2\nu)x\chi(x)] = 0. \tag{5.7}$$

Define $\chi_n(x) = 0$ for $|x| > 1$, *and normalize this function by*

$$c_0 \int_{-\infty}^{\infty} \frac{|\chi_n^\wedge(\xi)|^2}{(\kappa^2 + \xi^2)^{\nu+1/2}}\, d\xi = 1,$$

where

$$c_0 = (2\kappa)^\nu\Gamma(1/2)\Gamma(\nu + 1/2).$$

If $f(t)$ satisfies (5.1), *then*

$$f(t) = \sum \chi_n(t) \int_{-1}^{1} g(x)\chi_n(x)\, dx.$$

When $\nu = 0$, the substitution

$$\chi(x) = \frac{\omega(x)}{(1 - x^2)^{1/2}}$$

reduces (5.7) to the Mathieu equation. Solving (5.1) by separation of variables in (5.3) leads exactly to the same form for the solution, involving Mathieu functions divided by $(1 - x^2)^{1/2}$.

6. The potential of two strips. As our last example, we consider an integral equation

$$g(x) = \int_E k(x - t)f(t)\, dt, \qquad x \in E,$$

where E is not a single interval. The equation we shall discuss arises when one tries to solve Laplace's equation with boundary data on two line segments. Since it is easy to map the domain bounded by two arbitrary line segments conformally onto the domain bounded by two specific line segments on the x-axis, there is no loss in generality if we assume the segments to be the intervals $(-2, -1)$ and $(1, 2)$ on the x-axis.

Thus, we shall solve the problem consisting of Laplace's equation

$$\varphi_{xx} + \varphi_{yy} = 0$$

with

$$\varphi(x, 0) = g(x), \qquad x \in E,$$

where

$$E = (-2, -1) \cup (1, 2).$$

As in Sec. 4, the function

$$\varphi(x, y) = -\frac{1}{\pi} \int_E f(t) \log ((x - t)^2 + y^2)^{1/2} \, dt$$

is a solution of this problem if f satisfies the equation

$$g(x) = -\frac{1}{\pi} \int_E f(t) \log |x - t| \, dt, \qquad x \in E. \tag{6.1}$$

As in the earlier sections, we define

$$(Af)(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \log |x - t| \, dt$$

and

$$\begin{aligned}(Pf)(x) &= f(x), \qquad x \in E, \\ &= 0, \qquad x \notin E.\end{aligned}$$

The operator A defined in this way is positive, and P is a projection. Consequently, Theorem 2 applies.

The method for solving (6.1) is much the same as in Sec. 4, but there are one or two additional features worth observing due to the fact that E is not a single interval.

To solve (6.1) by means of Theorem 2, what is required is a sequence of functions χ_n such that

$$\{\chi_n\} \text{ is } A\text{-orthonormal}, \tag{6.2}$$

$$\{\chi_n\} \text{ is total in } R(P). \tag{6.3}$$

To find such a sequence, we try to choose the functions χ_n as solution of an eigenvalue problem of the form

$$L\chi = \lambda A \chi \tag{6.4}$$

where L satisfies the two conditions:

$$L \text{ is selfadjoint}; \tag{6.5}$$

$$\text{Eq. (6.4) is regular.} \tag{6.6}$$

(6.5) assures us that the solutions of (6.4) are A-orthogonal. (6.6), on the other hand, is imposed entirely for our convenience. We insist that (6.4) be regular because regular equations are the ones we like best and know the most about. The regularity of (6.4) also allows us to prove (6.3).

As in the earlier sections, it is easiest to begin with the operator $L^\wedge$ obtained by Fourier transforms: $L^\wedge \chi^\wedge = (L\chi)^\wedge$. Since the Fourier transform of $(-1/\pi) \log |x|)$ is $1/|\xi|$, (6.4) can be written in the alternative form

$$L^\wedge \chi^\wedge = \lambda \chi^\wedge / |\xi|. \tag{6.7}$$

As a first step toward achieving (6.6), we require that (6.7) be regular.

Naturally, in order for (6.5) to hold, $L^\wedge$ must be selfadjoint. But, again as before, we have a wide selection of selfadjoint operators $L^\wedge$ such that (6.7) is selfadjoint. Any of the operators

$$\frac{d^m}{d\xi^m}\,|\xi|^{\pm 2n+1}\,\frac{d^m}{d\xi^m}\,, \qquad m, n = 0, 1, 2, \cdots, \tag{6.8}$$

will do. To decide which of these to choose, we draw on our experience from Secs. 3, 4, and 5. In those sections, it was found that the equation corresponding to (6.4) had to have singular points at the ends of the interval over which the equation is valid. If that experience is a guide, we must expect (6.4) to have singular points at $x = \pm 1$ and $x = \pm 2$, the endpoints of the intervals constituting E. This means that the highest derivative occurring in (6.4) will have to be multiplied by some power of the polynomial $(1 - x^2)(4 - x^2)$. To keep things as simple as possible, we shall attempt to keep the power equal to unity. Thus, the highest derivative in (6.4) will be multiplied by a fourth degree polynomial in x. Since multiplication by x amounts to differentiation with respect to ξ, the means that the transformed equation (6.7) will be a differential equation of order four. This means that the values of m we may use in (6.8) are $m = 0$, 1, and 2.

To continue with the attempt to keep things as simple as possible, we shall try to restrict the order of the differential equation (6.4) to two. This means that when (6.7) is written as an equation with polynomial coefficients, no powers of ξ higher than two may appear. This requirement limits the values of n in (6.8). In fact, the only operators of the form (6.8) satisfying both restrictions we have imposed are

$$|\xi|, \qquad \frac{1}{|\xi|}\,, \qquad \frac{d}{d\xi}\,|\xi|\,\frac{d}{d\xi}\,, \qquad \frac{d}{d\xi}\,\frac{1}{|\xi|}\,\frac{d}{d\xi}\,, \quad \frac{d^2}{d\xi^2}\,|\xi|\,\frac{d^2}{d\xi^2}.$$

The second of these can be absorbed into the right-hand side of (6.7), while the third interferes with the others. Therefore, we see that the simplest choice possible for $L^\wedge$ is the linear combination

$$L^\wedge\chi^\wedge = \left(\frac{d^2}{d\xi^2}\,|\xi|\,\frac{d^2}{d\xi^2} + \alpha\,\frac{d}{d\xi}\,|\xi|\,\frac{d}{d\xi} + \beta\,|\xi|\right)\chi^\wedge,$$

where α and β are constants.

With this choice for $L^\wedge$, (6.7) can be reduced easily to the form

$$\xi^2\left(\frac{d^4}{d\xi^4} + \alpha\,\frac{d^2}{d\xi^2} + \beta\right)\chi^\wedge + \xi\,\frac{d}{d\xi}\left(2\,\frac{d^2}{d\xi^2} + \alpha\right)\chi^\wedge = \lambda\chi^\wedge.$$

The values of α and β can be read off immediately from this equation. The only term involving second derivatives *with respect to* x is the first. Its inverse transform has the form

$$-d^2/dx^2[(x^4 - \alpha x^2 + \beta)\chi].$$

If, as expected, (6.4) has singularities at the endpoints of E, the polynomial $x^4 - \alpha x^2 + \beta$ must have zeroes at $x = \pm 1, \pm 2$. This means that we must have $\alpha = 5$, $\beta = 4$. We now take these values as given and prove, after the fact, that the associated operator L has all the desired properties.

This means that we shall consider the equation

$$\xi^2\left(\frac{d^4}{d\xi^4} + 5\,\frac{d^2}{d\xi^2} + 4\right)\chi^\wedge + \xi\,\frac{d}{d\xi}\left(2\,\frac{d^2}{d\xi^2} + 5\right)\chi^\wedge = \lambda\chi^\wedge \tag{6.9}$$

and prove that it has solutions that are transforms of functions χ_n satisfying (6.2) and (6.3). The first property the solutions of (6.9) must have is part of (6.3): the inverse transforms of solutions of (6.9) must be zero for $x \notin E$. Suppose $\hat{\chi}$ were the transform of a function χ that is zero for $x \notin E$. χ itself must satisfy the equation

$$\frac{d^2}{dx^2}(1 - x^2)(4 - x^2)\chi + \frac{d}{dx}x(5 - 2x^2)\chi + \lambda\chi = 0. \tag{6.10}$$

Before going on, we note that (6.10) is regular, so that (6.6) is satisfied.

If $\hat{\chi}$ has the property assumed of it, (6.9) must be recoverable from (6.10) by transforming (6.10) over E. If one transforms (6.10) over E and insists that the result be (6.9), then $\chi(x)$ must satisfy the boundary conditions

$$\begin{aligned} &\lim_{x\to\partial E} (1 - x^2)(4 - x^2)\chi(x) = 0, \\ &\lim_{x\to\partial E} [(1 - x^2)(4 - x^2)\chi'(x) + x(2x^2 - 5)\chi(x)] = 0, \end{aligned} \tag{6.11}$$

where ∂E denotes the boundary of E.

As before, (6.11a) is satisfied by *every* solution of (6.10) since the indicial equation of (6.10) has the roots 0 and $(-1/2)$ at all the singular points. Thus, we need only consider (6.11b). However, ∂E consists of the four points ± 1, ± 2, so there are already *four* boundary conditions (6.11b).

Let $E = E^- \cup E^+$, where $E^- = (-2, -1)$ and $E^+ = (1, 2)$. Denote the functions that are zero outside E^- by $R(P^-)$ and the functions zero outside E^+ by $R(P^+)$. Clearly, both $R(P^-)$ and $R(P^+)$ are subsets of $R(P)$. Let $\omega^-(x)$ be a solution of (6.10) satisfying the *two* boundary conditions

$$\lim_{x\to\partial E^-} [(1 - x^2)(4 - x^2)\omega'(x) + x(2x^2 - 5)\omega(x)] = 0 \tag{6.12}$$

and

$$\omega^-(x) = 0 \quad \text{for } x \notin E^-. \tag{6.13}$$

The same argument that was used in the earlier sections shows that there is an infinite sequence $\{\omega_n^-\}$ of solutions of (6.10), (6.12), and (6.13) that is total in $R(P^-)$.

The eigenvalues associated with the problem just described are simple. To see this, make the substitution

$$\tau^-(x) = \omega^-(x)((1 - x^2)(4 - x^2))^{1/2}.$$

Since the indicial equation associated with (6.10) has roots 0 and $(-1/2)$, the indicial equation associated with the equation satisfied by τ^- has roots 1/2 and 0. Moreover, in terms of τ^-, the boundary condition (6.12) becomes

$$\lim_{x\to\partial E^-} \tau'(x) = 0.$$

Therefore, near $x = -1$, for instance, $\tau^-(x)$ is regular. Suppose that associated with an eigenvalue λ there were two eigenfunctions τ_1^- and τ_2^-. Then we could find a linear combination $\tau^- = c_1\tau_1^- + c_2\tau_2^-$ that is zero at $x = -1$, while the boundary conditions give that the derivative of τ^- is also zero at $x = -1$. Since τ^- is regular, this implies that τ^- is identically zero, so that τ_1^- and τ_2^- are linearly dependent.

In the same way that we constructed the sequence $\{\omega_n^-\}$, we also construct a sequence $\{\omega_n^+\}$ of solutions of (6.10) satisfying the boundary conditions

$$\lim_{x \to \partial E^+} [(1 - x^2)(4 - x^2)\omega'(x) + x(2x^2 - 5)\omega(x)] = 0 \tag{6.14}$$

and

$$\omega^+(x) = 0 \quad \text{for } x \notin E^+. \tag{6.15}$$

This sequence is total in $R(P^+)$, and the corresponding eigenvalues are simple.

We now define a sequence $\{\omega_n\}$ by the formulas

$$\begin{aligned} \omega_{2n}(x) &= \omega_n^-(x), \\ \omega_{2n+1}(x) &= \omega_n^+(x). \end{aligned} \tag{6.16}$$

The entire sequence $\{\omega_n\}$ includes all the functions ω_n^- and ω_n^+ and is therefore total in $R(P)$. Two functions ω_m and ω_n associated with different eigenvalues are A-orthogonal since these functions each satisfy an equation (6.4) with L selfadjoint. Because of the simplicity of the eigenvalues proved earlier, no two eigenfunctions can have the same eigenvalue except possibly for two successive functions ω_{2n} and ω_{2n+1}. These two functions do correspond to the same eigenvalue since, as is easily seen, $\omega_{2n}(x) = \omega_{2n+1}(-x)$.

On the other hand, ω_{2n} and ω_{2n+1} are obviously linearly independent, since each is zero wherever the other is not. Also, we can prove in the same way that we did in Sec.4 that $(A\omega_n, \omega_n) < \infty$. Therefore, the two functions can be A-orthonormalized by the Gram-Schmidt process. We call the resulting two functions χ_{2n} and χ_{2n+1}. It is easily seen that the sequence $\{\chi_n\}$ so defined satisfies all the conditions of Theorem 2. Thus, we have the

THEOREM. *Define two sequences* $\{\omega_n^-\}$ *and* $\{\omega_n^+\}$ *as the solutions of* (6.10), (6.12), *and* (6.13) *and of* (6.10), (6.14), *and* (6.15), *respectively. For each* n, *let* χ_{2n} *and* χ_{2n+1} *be linear combinations of* ω_n^- *and* ω_n^+ *chosen in such a way that*

$$-\frac{1}{\pi}\left(\int_{-2}^{-1} + \int_1^2\right)\chi_{2n}(x)\chi_{2n}(t) \log |x - t| \, dt \, dx = 1,$$

$$-\frac{1}{\pi}\left(\int_{-2}^{-1} + \int_1^2\right)\chi_{2n+1}(x)\chi_{2n+1}(t) \log |x - t| \, dt \, dx = 1,$$

$$-\frac{1}{\pi}\left(\int_{-2}^{-1} + \int_1^2\right)\chi_{2n}(x)\chi_{2n+1}(t) \log |x - t| \, dt \, dx = 0.$$

Consider the integral equation

$$g(x) = -\frac{1}{\pi}\left(\int_{-2}^{-1} + \int_1^2\right)f(t) \log |x - t| \, dt, \qquad 1 < |x| < 2.$$

If there is a solution, it is given by the formula

$$f(t) = \sum \chi_n(t)\left(\int_{-2}^{-1} + \int_1^2\right)g(x)\chi_n(x) \, dx.$$

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