ON CONVECTION AND DIFFUSION*

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1. Preliminaries. In a recent paper [1], Marris and Passman developed the theory of the transport of a general solenoidal vector in the motion of a continuum. This constituted a generalization of part of the elegant classical theory of vorticity transport, as set forth by Truesdell [2]. I show here that part of the work of Marris and Passman can be considered to be a special case of a somewhat more general theory, and give another application of that theory. With certain minor exceptions, I use the notations and assumptions of Truesdell and Toupin [3].

2. The general integral formula[1]**.** Let $\boldsymbol{\beta}$ be any twice continuously differentiable vector function with covariant components β_k. Form the material expression $\beta_{k,\alpha}\, x^k{}_{,\beta}$. Then

$$\frac{d}{dt}(\beta_{k,\alpha}x^k{}_{,\beta}) = \dot{\beta}_{k,\alpha}x^k{}_{,\beta} + \beta_{k,\alpha}\dot{x}^k{}_{,\beta}\,. \tag{1}$$

Thus

$$\frac{d}{dt}(\beta_{k,l}x^l{}_{,\alpha}x^k{}_{,\beta}) = \dot{\beta}_{k,\alpha}x^k{}_{,\beta} + \beta_{k,\alpha}\dot{x}^k{}_{,\beta}\,. \tag{2}$$

Let $\beta_{\alpha,\beta}$ represent the values of $\beta_{k,l}$ at $t = 0$. The integration of (2) along the path of a particle[2] yields

$$\beta_{k,l}x^l{}_{,\alpha}x^k{}_{,\beta} = \beta_{\alpha,\beta} + \int_0^t (\dot{\beta}_{k,\alpha}x^k{}_{,\beta} + \beta_{k,\alpha}\dot{x}^k{}_{,\beta})\,dt, \tag{3}$$

which, with appropriate changes, yields

$$\beta_{k,l} = \left[\beta_{\beta,\alpha} + \int_0^t (\dot{\beta}_{i,j} + \beta_{p,j}\dot{x}^p{}_{,i})x^j{}_{,\alpha}x^i{}_{,\beta}\,dt\right] X^\alpha{}_{,l}X^\beta{}_{,k}\,. \tag{4}$$

Eq. (4) is a relationship for the transport of the gradient of an arbitrary vector in the motion of a continuum. The first formula of this type was published in 1948 by Truesdell [5].

It is seen that there are two processes involved in the transport of the gradient of $\boldsymbol{\beta}$. One, represented here by the term

$$\beta_{\beta,\alpha}\, X^\alpha{}_{,k}\, X^\beta{}_{,l},$$

depends only on the value of $\boldsymbol{\beta}$ at $t = 0$ and the initial and final coordinates of the

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[1]The method used here is due originally to Carstoiu [4].

[2]That is, write all of the functions in (2) in terms of $\mathbf{X}$ and t. Fix $\mathbf{X}$ and integrate.

particle, and is called *convection*. The second process, here represented by the term

$$\left[\int_0^t (\dot{\beta}_{i,j} + \beta_{p,j}\dot{x}^p{}_{,i})x^i{}_{,\alpha}x^i{}_{,\beta}\,dt\right] X^\alpha{}_{,k}X^\beta{}_{,l}$$

depends on the motion and the values of $\boldsymbol{\beta}$ between 0 and t. It is called *diffusion*.

The axial-vector of (4) yields the relation (2.11) of Marris and Passman [1] which, in turn, yields relations for the transport of Nth-order vorticity, including Truesdell's formula for transport of first-order vorticity. The symmetric part of (4) with the substitution $\beta_k = \dot{x}_k$ yields the formula for transport of stretching ([3], p. 381).

3. An application. The covariant components of the Nth Rivlin-Ericksen tensor are given by

$$A_{pq}^{(N)} = 2\,d_{pq}^{(N)} + \sum_{K=1}^{N-1} \binom{N}{K} \overset{(N-K)}{x}_{m,p} \overset{(K)}{x}{}^m{}_{,q}\,, \tag{5}$$

where $\overset{(N)}{x}{}^k$ are the contravariant components of the N − 1st acceleration, defined by

$$\overset{(N)}{x}{}^k = \frac{d^N x^k}{dt^N}\,, \tag{6}$$

and $d_{pq}^{(N)}$ are the covariant components of the Nth stretching tensor, given by

$$d_{km}^{(N)} = \overset{(N)}{x}_{(k,m)}\,. \tag{7}$$

The symmetric part of (4) is

$$\beta_{(k,l)} = \left[\beta_{(\beta,\alpha)} + \int_0^t (\dot{\beta}_{(i,j)} + \beta_{k,(j}\dot{x}^p{}_{,i)})x^i{}_{,\alpha}x^i{}_{,\beta}\,dt\right] X^\alpha{}_{,l}X^\beta{}_{,k}\,. \tag{8}$$

We thus have the following relation for transport of the Nth Rivlin-Ericksen tensor:

$$\begin{aligned} A_{pq}^{(N)} = 2\Bigg[& d_{\beta,\alpha}^{(N)} + \int_0^t (d_{ij}^{(N+1)} + d_{r,(j}^{(N)}\dot{x}^r{}_{,i)}x^i{}_{,\alpha}x^i{}_{,\beta}\,dt\Bigg] X^\alpha{}_{,k}X^\beta{}_{,q} \\ & + \sum_{K=1}^{N-1}\binom{N}{K}\left[\left(\overset{(N-K)}{x}_{\beta,\alpha} + \int_0^t\left(\overset{(N-K+1)}{x}_{i,j} + \overset{(N-K)}{x}_{r,j}\dot{x}^r{}_{,i}\right)x^i{}_{,\alpha}x^i{}_{,\beta}\,dt\right) X^\alpha{}_{,m}X^\beta{}_{,k}\right. \\ & \left.\cdot\left(\overset{(N)}{x}_{\delta,\gamma} + \int_0^t\left(\overset{(N+1)}{x}_{k,n} + \overset{(N)}{x}_{r,n}\dot{x}^r{}_{,k}\right)x^n{}_{,\gamma}x^k{}_{,\delta}\,dt\right)g^{ml}X^\gamma{}_{,l}X^\delta{}_{,q}\right]\cdot \end{aligned} \tag{9}$$

It might be speculated that there exists a relation for the transport of a Rivlin-Ericksen tensor which depends only on other Rivlin-Ericksen tensors and the second referential kinestate. I have not been able to derive such a relation. Eq. (9) indicates that the Nth Rivlin-Ericksen tensor at time t is, in general, the result of a much more complex transport mechanism. It is also seen that there exist conditions under which the transport of a Rivlin-Ericksen tensor is due to convection only. It would be interesting to study these conditions.

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References

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