NOTE ON ELECTROHYDRODYNAMIC STABILITY*

BY

D. H. MICHAEL

University of Melbourne

Problems of small oscillations of fluid systems subject to electrostatic forces have been studied by many authors, for example Rayleigh [1], Bassett [2], Taylor and McEwan [3] and Melcher [4]. Such problems can usually be analysed with the simplification that the electric field may be treated as an electrostatic field on the time scales which are of interest for mechanical vibrations. The time scale of decay of charge in the interior of a conducting body is $\kappa/4\pi\eta$ (see Jeans [5]), where κ is the dielectric constant and η the electrical conductivity. We find that with water, for example, this decay time is of order 10^{-4} seconds, so that only for oscillations with periods as small as this need we consider the departure of the electric field from its equilibrium configuration for waves on water in the presence of an electric field.

In considering perturbations of an electrohydrodynamic system from equilibrium it is possible for the electrostatic field to be perturbed in two distinct ways, one in which electrostatic potentials of conductors are maintained constant in the oscillation by connections with batteries, and the other in which the conductors are electrically insulated from their surroundings, in which case their charges are maintained constant. In studying several problems of this kind recently the author has found that the normal modes of oscillation have turned out to be the same in both these cases. For example it was shown (Michael [6]) that this is the case for small oscillations of an incompressible conducting circular jet which is projected along the axis of a concentric electrified conducting cylinder. The intention of this paper is to show that this result is not valid in general. An attempt is made to classify the problems in which the result is true, and we give finally a simple illustration of a case in which the normal modes of oscillation are different in the two cases.

It is not difficult to identify formally the situations in which charge-maintained and potential-maintained oscillations are the same. If we represent the system in general by a number of charged conductors, in which the charge is Q_i and potential V_i on the ith conductor, then

$$V_i = p_{ij}Q_j , \tag{1}$$

$$Q_i = c_{ij}V_j , \tag{2}$$

where p_{ij} and c_{ij} are the coefficients of potential and capacity respectively. These coefficients are functions of the geometrical shape and positions of the conducting bodies. We also know from the reciprocal theorem in electrostatics that

$$p_{ij} = p_{ji}, \quad \text{and} \quad c_{ij} = c_{ji} . \tag{3}$$

*Received February 3, 1969.

The electrostatic energy W is given by

$$W = \tfrac{1}{2}Q_i V_i = \tfrac{1}{2}p_{ij}Q_i Q_j = \tfrac{1}{2}c_{ij}V_i V_j ,$$

so that

$$\Phi = Q_i V_i - \tfrac{1}{2}p_{ij}Q_i Q_j - \tfrac{1}{2}c_{ij}V_i V_j \equiv 0.$$

We are interested in small changes in the electrostatic field in situations in which the conducting bodies may be fluids which may change their shape, size and relative positions. Let ΔQ_i , ΔV_i , Δp_{ij} , and Δc_{ij} represent small changes in Q_i , V_i , p_{ij} and c_{ij} , respectively. Then

$$\Delta\Phi = Q_i \Delta V_i + V_i \Delta Q_i - \tfrac{1}{2}\Delta p_{ij}Q_i Q_j - \tfrac{1}{2}p_{ij}(Q_i \Delta Q_j + Q_j \Delta Q_i) - \tfrac{1}{2}\Delta c_{ij}V_i V_j - \tfrac{1}{2}c_{ij}(V_i \Delta V_j + V_j \Delta V_i) = 0,$$

to the first order, and using (1), (2) and (3) we find

$$\tfrac{1}{2}\ \Delta p_{ij}Q_i Q_j = -\tfrac{1}{2}\ \Delta c_{ij}V_i V_j\ . \tag{4}$$

The left-hand side of (4) represents the increase in electrostatic energy due to a change in shape and position of the conductors when the charges are kept constant. We describe as the available energy the energy released by the electrostatic field in a small change, which is available to build up the kinetic energy of the fluid, say. In this case the available energy is $-\tfrac{1}{2}\Delta p_{ij}Q_i Q_j$ since the conductors are electrically insulated from their surroundings. When a small change at constant V occurs the increase in electrostatic energy is $\tfrac{1}{2}\Delta c_{ij}V_i V_j$, but in this case a charge $\Delta Q_i = \Delta c_{ij}V_j$ is added at potential V_i to the ith body so that additional energy $V_i \Delta Q_i = \Delta c_{ij}V_i V_j$ is added to the system. Thus the available energy is now $\tfrac{1}{2}\Delta c_{ij}V_i V_j$. Equation (4) therefore shows that the available energy is the same in both cases. However, an important proviso concerning this result is that it applies only to the lowest order in the small changes in position of the conducting surfaces.

The significance of this result for the problem of small oscillations depends on whether energy changes of this order determine the behaviour of small oscillations. If ϵ denotes a small dimensionless parameter representing the amplitude of a small allowable disturbance, we need to distinguish the two cases in which the electrostatic energy changes are $O(\epsilon)$ and $O(\epsilon^2)$, respectively. In the first of these a steady state is achieved by the interplay of electrostatic forces with other force fields, as for example the pressure field of internal motion of a fluid conductor or of the motion of an insulating fluid surrounding the conductors. When small oscillations occur in such a system the available energy is the same, in charge- and potential-maintained oscillations, only to the order ϵ. To the order ϵ^2, which determines the form of the oscillations, the available energy may be different in the two cases. When the electrostatic energy change is of order ϵ^2 the electric field and other force fields are separately in equilibrium and the result above shows that small oscillations then have the same dispersion relation in the two cases.

We need therefore to classify problems of electrohydrodynamic stability according to whether the electrostatic energy is itself stationary to the allowable displacements or not. The displacements allowable will of course depend on the material of the conducting bodies. Clearly there is an important distinction between oscillations of an incompressible fluid conductor in which the volume is conserved, and a conducting gas, say, in which an expansion or contraction in volume may occur in the oscillations.

In attempting a classification we examine first the electrostatic energy of the field associated with incompressible charged conductors. In the first place, if we have one simply connected incompressible conductor of finite volume and of arbitrary shape with charge Q, for what shape is the electrostatic energy of the field produced stationary to small changes in the shape which preserve the volume of the conductor? This is equivalent to the requirement that the conductor, treated as an ideal incompressible fluid which cannot store internal energy, should be in equilibrium under the action of the electrostatic field. It clearly requires that the surface-charge density σ should be the same at all points, so that the electrostatic stress shall be constant over the surface. This cannot be so in general since it requires V and its normal derivative $\partial V/\partial n$ to be constant on the conductor, and these conditions taken together overprescribe the boundary conditions for V. It seems clear that the only exception is the case in which these conditions are satisfied on account of the symmetry the body, that is, the spherical conductor. Similar remarks apply to a single cylindrical conductor, which needs to be of circular cross-section. For a conducting fluid which extends to infinity in two directions, a plane layer of uniform thickness also satisfies this criterion. We may also add to these the configurations which have the same degrees of symmetry, such as a system of concentric spherical shells, coaxial circular cylindrical shells, or plane parallel layers of conductors. These categories include the problems already discussed by the author [6], [7]. All other systems of incompressible fluid conductors which have occurred to the author lack sufficient symmetry to satisfy this criterion.

The situations in which a system of compressible conductors satisfies the criterion can be at most the same as those for the incompressible case. But in fact the possibility of expansion of a conductor in small oscillations appears to reduce the number of cases to zero. In the small displacement of a single spherical conductor we find that the increment ΔC in capacity C is such that $\Delta C/C$ is $O(\epsilon^2)$ only for displacements in which the volume is unchanged, but when the volume changes $\Delta C/C$ is $O(\epsilon)$. This is simply seen in the case of a radial expansion from radius a to $a(1 + \epsilon)$ since the capacity is equal to the radius. In this case if the charge Q is maintained in the expansion the available energy is

$$\frac{Q^2}{2a}\left\{1 - \frac{1}{(1+\epsilon)}\right\} = \frac{Q^2}{2a}(\epsilon - \epsilon^2 + \epsilon^3 - \cdots),$$

whereas if the potential of the conductor is maintained the available energy is $(Q^2/2a)\epsilon$. The fact that the term in ϵ is the same in the two cases is in accord with our general result. Changes in energy to this order will of course be balanced out from other sources in a displacement from equilibrium, but the example illustrates that the ϵ^2 terms are different, so that different sets of normal modes would result.

As a further simple illustration, the capacity C, per unit length, of two parallel circular cylinders of radii a and b, eccentrically placed so that the distance between their axes is d, is given by

$$\frac{1}{C} = 2\cosh^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right).$$

If we make an incompressible translation of one cylinder so that $d \to d(1 + \epsilon)$, the change in capacity is $O(\epsilon)$, except in the case where $d = 0$, when the change becomes $O(\epsilon^2)$ if $d \to \epsilon a$, say. But even in the case where $d = 0$, if we make a compressible displacement, as for example when $a \to a(1 + \epsilon)$, $\Delta C/C$ is again $O(\epsilon)$.

We end this discussion with the derivation of stability criteria for radial oscillations of a gaseous cylindrical conducting jet of radius a, as an illustration of a situation in which different stability characteristics are obtained. Suppose a jet of conducting gas occupies the region $0 < r < a$ and is surrounded by a nonconducting gas in the region $a < r < b$ with a solid conducting outer boundary at $r = b$. We assume each of the gases to be a perfect gas, and denote by p, τ, γ the pressure, volume per unit length, and the ratio of specific heats respectively. Also we use suffixes 0 and 1 to distinguish the inner and outer gases respectively.

The internal energy U is easily seen to be given by

$$U = \frac{p_* \tau_*^\gamma}{(\gamma - 1)} \tau^{(1-\gamma)}, \tag{5}$$

where p_*, τ_* represent a reference state which can be taken as the equilibrium state p_0, τ_0 and p_1, τ_1 for the respective gases. When the interface at $r = a$ is displaced to $r = a(1 + \epsilon)$ we find easily that the increments ΔU_0 and ΔU_1 are given to the order ϵ^2 by

$$\Delta U_0 = \frac{p_0 \tau_0}{(\gamma_0 - 1)} \{(1 - \gamma_0)2\epsilon + (1 - \gamma_0)(3 - 2\gamma_0)\epsilon^2 + O(\epsilon^3)\},$$

$$\Delta U_1 = \frac{p_1 \tau_1}{(\gamma_1 - 1)} \left\{(1 - \gamma_1) \frac{(-2a^2)\epsilon}{(b^2 - a^2)} + \left[(1 - \gamma_1)\left(\frac{-a^2}{(b^2 - a^2)}\right) + 2\, \frac{(1 - \gamma_1^2)a^4}{(b^2 - a^2)^2}\right]\epsilon^2 + O(\epsilon^3)\right\}.$$

In the equilibrium state let the electrostatic potential V be such that $V = 0$ at $r = a$ and $V = \bar{V}$ at $r = b$. The surface charge per unit length is then $Q = \bar{V}/2 \log (b/a)$ at $r = b$, and the electrostatic energy $W = Q^2 \log b/a$. We then find that the available energy in the two cases (i) in which Q is conserved and (ii) in which the potential difference $\bar{V}$ is conserved, are $Q^2\{\epsilon - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\}$, and $Q^2\{\epsilon - \epsilon^2(\frac{1}{2} - 1/\log (b/a)) + O(\epsilon^3)\}$, respectively. The conditions for stability are, respectively,

$$\Delta U_0 + \Delta U_1 > Q^2(\epsilon - \epsilon^2/2), \tag{6}$$

and

$$\Delta U_0 + \Delta U_1 > Q^2\{\epsilon - \epsilon^2(\tfrac{1}{2} - 1/\log (b/a))\}. \tag{7}$$

The terms in ϵ cancel out of these conditions on account of the equilibrium condition, which is that

$$p_1 - p_0 = Q^2/2\pi a^2. \tag{8}$$

Using (8) we may write the conditions (6) and (7) in terms of p_0 and p_1 and they become, after simplification,

$$p_0(2 - \gamma_0) < p_1\left[1 - \frac{(1 - \gamma_1)}{(b^2/a^2 - 1)}\right], \tag{9}$$

and

$$p_0\left[2 - \gamma_0 - \frac{1}{\log b/a}\right] < p_1\left[1 - \frac{1}{\log b/a} - \frac{(1 - \gamma_1)}{(b/a)^2 - 1}\right]. \tag{10}$$

To examine the significance of (8), (9) and (10) we have put $\gamma_0 = \gamma_1 = 1.4$. It is easily seen that in case (i) the equilibrium is stable to radial oscillations for any Q. But in

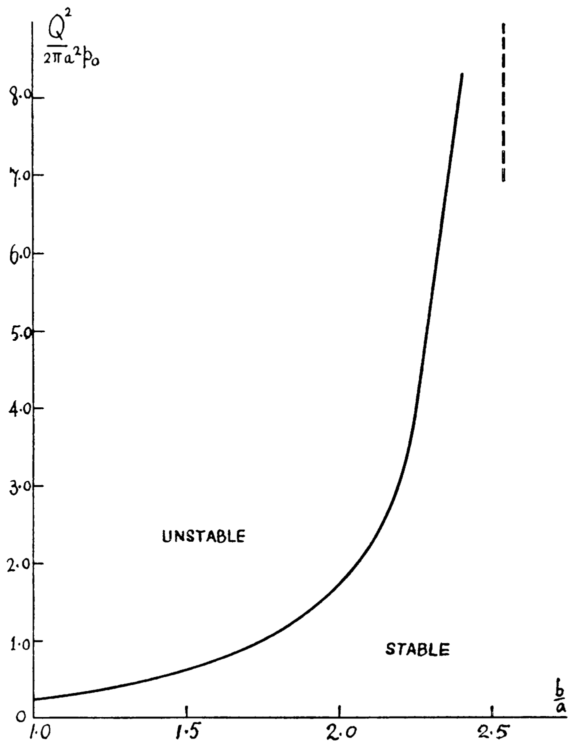

FIG. 1. Stability characteristics for potential maintainance of radial oscillations of a gaseous electrified jet.

case (ii) such oscillations are unstable when $(Q^2/2\pi a^2 p_0) > \frac{1}{4}$. Stable and unstable conditions in case (ii) are shown in Fig. 1.

Under conditions in which distinct normal modes arise, it is to be expected that potential-maintained oscillations will be more unstable than charged-maintained oscillations, since in the former case the system can avail itself of additional energy from external sources.

We should note finally that the effect of electrostatic forces on the vibrations of a system of fluid conductors depends on the geometry of the conducting surfaces only, and not on the fluid motion produced. It follows that the distinctions made in this paper, between cases in which charge-maintained and potential-maintained oscillations are the same or different, will apply equally to oscillations of viscous as to inviscid highly conducting fluids.

REFERENCES

[1] J. W. S. Rayleigh, *Theory of sound*, Vol. 2, 2nd ed., Dover, New York, 1945
[2] Bassett, Amer. J. Math. **26** (1894)
[3] G. I. Taylor and A. D. McEwan, J. Fluid Mech. **22**, 1 (1965)
[4] J. R. Melcher, *Field coupled surface waves*, M.I.T. Press, Cambridge, Mass., 1963
[5] J. H. Jeans, *The mathematical theory of electricity and magnetism*, 5th ed., Cambridge Univ. Press, New York, 396, 1925
[6] D. H. Michael, Canad. J. Physics **45**, 3179 (1967)
[7] D. H. Michael, Proc. Cambridge Philos. Soc. **63**, 88 (1967)