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CANONICAL APPROACH TO BIHARMONIC VARIATIONAL PROBLEMS*

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Abstract. A canonical approach to biharmonic variational problems is presented. It provides a new form of the principle of stationary energy and a new derivation of the principle of minimum potential energy.

1. Introduction. In a recent series of papers [1]–[4], variational principles associated with the canonical equations

$$T\Phi = \partial W/\partial U, \qquad T^*U = \partial W/\partial \Phi$$

have been studied. The work of Noble [1] dealt with the one-dimensional case $T = d/dx$, $T^* = -d/dx$, while the operators $T = \text{grad}$, $T^* = -\text{div}$ have been discussed in papers on diffusion and related topics [2]–[4].

The purpose of this note is to present some results for the operators $T = \nabla^2$, $T^* = \nabla^2$. The theory is used to derive a canonical form of the principle of stationary energy for biharmonic problems and to provide a new derivation of the principle of minimum potential energy.

2. Theory. We consider a physical problem which is described by the pair of canonical Euler equations

$$T\Phi = \frac{\partial}{\partial U} W(\mathbf{r}, \Phi, U) \tag{1}$$

$$\text{in} \quad R,$$

$$T^*U = \frac{\partial}{\partial \Phi} W(\mathbf{r}, \Phi, U) \tag{2}$$

in which T and T^* are linear operators, $\mathbf{r}$ is a position vector and Φ and U are functions of $\mathbf{r}$. The region R is a part of the xy-plane which has a piecewise smooth boundary B. The boundary conditions are taken to be

$$\Phi = \varphi_0 \tag{3}$$

$$\text{on} \quad B$$

$$\partial\Phi/\partial n = f(U) \tag{4}$$

where n is the outward pointing normal to the boundary, and φ_0 and f are given functions. The operator T^* in (2) is the adjoint of T in the sense that

$$(U, T\Phi) = \langle T^*U, \Phi\rangle, \tag{5}$$

where the inner products are defined by

*Received April 8, 1968.

$$(U, T\Phi) = \int_R UT\Phi \, dx \, dy - \int_B U \frac{\partial \Phi}{\partial n} \, ds + \int_B F(U) \, ds, \tag{6}$$

$$\langle T^*U, \Phi \rangle = \int_R (T^*U)\Phi \, dx \, dy - \int_B \frac{\partial U}{\partial n} \Phi \, ds + \int_B F(U) \, ds, \tag{7}$$

with

$$F(U) = \int^U f(U') \, dU'. \tag{8}$$

Definitions (6) and (7) are appropriate to the case $T = \nabla^2$, $T^* = \nabla^2$. If we introduce the functional

$$I(\Phi, U) = \int_R W(\mathbf{r}, \Phi, U) \, dx \, dy - (U, T\Phi), \tag{9}$$

then the following results are obtained:

Stationary property. $I(\Phi, U)$ is stationary at (φ, u) if Eqs. (1)–(4) hold simultaneously at (φ, u).

Extremum principle. Choose a trial function Φ which is equal to φ_0 on B, and determine $U(\Phi)$ so that Eqs. (1) and (4) are satisfied identically. Then, if (2) holds at (φ, u) we have from (9)

$$G(\Phi) \equiv I(\Phi, U(\Phi)) = I(\varphi, u) + \delta^2 I(\Phi) + O(\Phi - \varphi)^3, \tag{10}$$

where

$$\delta^2 I(\Phi) = \frac{1}{2} \int_R \left\{ (\Phi - \varphi)^2 \left[\frac{\partial^2 W}{\partial \Phi^2} \right]_{\varphi, u} - (U(\Phi) - u)^2 \left[\frac{\partial^2 W}{\partial U^2} \right]_{\varphi, u} \right\} dx \, dy + \frac{1}{2} \int_B (U(\Phi) - u)^2 \left[\frac{df}{dU} \right]_{\varphi, u} ds. \tag{11}$$

If terms of third and higher orders can be neglected (or if they vanish), it follows that

$$G(\Phi) \le I(\varphi, u) \quad \text{if } \delta^2 I \le 0, \tag{12}$$

or

$$G(\Phi) \ge I(\varphi, u) \quad \text{if } \delta^2 I \ge 0. \tag{13}$$

Thus we have an upper or a lower bound for $I(\varphi, u)$ depending on the sign of $\delta^2 I$. The pair of functions (φ, u) furnishes the exact solution of the problem in Eqs. (1)–(4).

3. The biharmonic equation. The equation which governs the small deflection bending of a thin elastic plate is [5], [6]

$$\nabla^4 \Phi = q/D \quad \text{in R}, \tag{14}$$

where Φ is the deflection normal to the surface, $q(x, y)$ is the distribution of normal loading and D is the flexural rigidity. The boundary conditions [5], [6] for a plate which is clamped on part B_1 and simply supported on $B - B_1$ are

$$\Phi = \partial \Phi / \partial n = 0 \quad \text{on } B_1, \tag{15}$$

and

$$\Phi = \partial\Phi/\partial n - 1/\kappa(1-\nu)\,\nabla^2\Phi = 0 \quad \text{on } B - B_1\,. \tag{16}$$

Here κ is the local curvature of B and ν is Poisson's ratio. In addition we assume that these deflection conditions satisfy Eq. (14) explicitly. We now write (14) as the pair of equations

$$\nabla^2\Phi = U \qquad \text{in} \quad R. \tag{17}$$

$$\nabla^2 U = q/D \tag{18}$$

This way of writing (14) is equivalent to that used by Morley [5], but the canonical form of (17) and (18) does not seem to have been emphasized previously. The boundary value problem in (15)–(18) is a special case of that discussed in Sec. 2 and corresponds to

$$W(\mathbf{r}, \Phi, U) = \tfrac{1}{2}U^2 + q\Phi/D, \tag{19}$$

$$T = \nabla^2, \qquad T^* = \nabla^2, \tag{20}$$

$$\varphi_0 = 0 \quad \text{on } B, \tag{21}$$

$$\begin{aligned} f(U) &= 0 && \text{on} \quad B_1\,, \\ &= U/\kappa(1-\nu) && \text{on} \quad B - B_1\,. \end{aligned} \tag{22}$$

Putting these in (9) we obtain

$$I(\Phi, U) = \int_R \left\{\frac{1}{2}U^2 + \frac{q}{D}\Phi - U\nabla^2\Phi\right\} dx\,dy + \int_{B-B_1} \left\{U\frac{\partial\Phi}{\partial n} - \frac{U^2}{2\kappa(1-\nu)}\right\} ds. \tag{23}$$

From Sec. 2 we see that $I(\Phi, U)$ in (23) is stationary at (φ, u) where φ, u are the exact solutions of (15)–(18). This is a canonical form of the principle of stationary energy.

When $\Phi = \varphi$ and $U = u$ we find that (23) gives

$$I(\varphi, u) = \frac{1}{2}\int_R u^2\,dx\,dy - \frac{1}{2(1-\nu)}\int_{B-B_1} \frac{1}{\kappa}u^2\,ds. \tag{24}$$

The functional $G(\Phi)$ given by (10) is found to be

$$G(\Phi) = \int_R \left\{-\frac{1}{2}(\nabla^2\Phi)^2 + \frac{q}{D}\Phi\right\} dx\,dy + \frac{1}{2}\int_{B-B_1} \kappa(1-\nu)\left(\frac{\partial\Phi}{\partial n}\right)^2 ds, \qquad \Phi = 0 \quad \text{on} \quad B. \tag{25}$$

In addition Eq. (11) becomes

$$\delta^2 I(\Phi) = -\frac{1}{2}\int_R (U(\Phi) - u)^2\,dx\,dy + \frac{1}{2(1-\nu)}\int_{B-B_1} \frac{1}{\kappa}(U(\Phi) - u)^2\,ds. \tag{26}$$

If

$$\kappa(1-\nu) < 0, \tag{27}$$

then

$$\delta^2 I(\Phi) \le 0, \tag{28}$$

and hence, by (12), we have the variational principle

$$G(\Phi) \le I(\varphi, u). \tag{29}$$

Eq. (29) is equivalent to the principle of minimum potential energy [5], [6].

Acknowledgment. I should like to thank Sir H. Jeffreys for valuable comments on this work.

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