DISTURBANCE DUE TO WEAK BUOYANCY ON THE FORCED CONVECTION FIELD OF A SOURCE SINGULARITY OF HEAT*

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1. Introduction. In this paper, we are concerned with the change in the convection field of a uniform stream $\bar{U}$ when small effects of buoyancy are present for a flow in an unbounded region containing a point source of heat at the origin. Density differences established by temperature gradients induce fluid motion by driving less dense fluid elements against the direction of the gravity vector $\bar{g}(\bar{r})$. In such a flow there is, therefore, a coupling of the velocity and temperature fields. Except as they induce buoyancy forces the density differences are otherwise ignored. The fluid is assumed to be a constant-property one, so that the variation of thermal properties and the effects of viscous dissipation on the temperature distribution are neglected.

The problem is treated by a regular perturbation of the classical problem in which buoyancy is totally neglected (see e.g. [3, p. 266]). The first-order perturbation solutions, which are obtained explicitly for a Prandtl number of one, are assumed to provide the dominant behaviour of the self-convection fields of velocity and temperature. The solutions imply a paraboloidal wake-like behaviour of these fields. Such behaviour is consistent with solutions of the free convection problem as obtained by Mahony [7] and Yih [9].

2. Formulation of the problem. The Oseen equations. The appropriate governing system of equations for steady motion is (see e.g. Howarth [6, Chap. II, Eqs. 11, 54, 56, 93]).

$$\begin{aligned} \bar{u}\cdot\text{grad }\bar{u} &= -\text{grad }(p/\rho_0) + \nu\,\Delta\bar{u} - \alpha T\bar{g} \\ \bar{u}\cdot\text{grad }T &= \kappa\,\Delta T + Q\,\delta(\bar{r}) \\ \text{div }\bar{u} &= 0. \end{aligned} \tag{2.1}$$

In the above simplified Boussinesq equations κ, α, ν, are, respectively, the thermometric coefficient, the coefficient of thermal expansion and the kinematic viscosity. The second equation in (2.1) differs from the corresponding equation in Howarth by the inclusion of the term $Q\delta(\bar{r})$ which corresponds to a source singularity of heat delivering Q units of heat per unit time at the origin. The divergence theorem permits this equation to be expressed as

$$\bar{u}\cdot\text{grad }T = \kappa\Delta T \tag{2.2}$$

in any region not containing the source, and

$$\int_S [\bar{u}T - \kappa\text{ grad }T]\cdot\bar{n}\,dS = Q \tag{2.3}$$

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in a region R with surface S which contains the source. Thus, instead of the second equation in (2.1) we employ Eq. (2.2) and retain Eq. (2.3) as a boundary condition.

In a three-dimensional axisymmetric flow, in which the axis of symmetry is in the direction of $\bar{g}$ and in which there is no velocity component in the azimuthal direction $\bar{e}_\varphi$, there is only one component of the vorticity $\bar{\omega} = \omega \bar{e}_\varphi$. This corresponds to an annular vortex filament encircling the axis of symmetry in a circle of radius ρ. The vorticity is proportional to ρ. Following Pillow (see e.g. [2, Chap. I, §3]), we introduce the quantity $l = \omega/\rho$, the ring circulation density, in order to have a quantity which is conserved under convection. The momentum equation in (2.1) when expressed in terms of l becomes

$$\bar{u}\cdot\text{grad } l = \nu \, \Delta l + 2\nu \frac{\text{grad } \rho}{\rho}\cdot\text{grad } l + \alpha\bar{g}\cdot[\rho^{-1} \text{ grad } T \times \bar{e}_\varphi]. \tag{2.4}$$

The equation of continuity can be used to advantage when the scalar stream function ψ is introduced by the identity $\bar{u} = \text{curl } (-\psi\bar{e}_\varphi)$. The relation between ring circulation density and the stream function is then given by the Poisson equation

$$\Delta\psi = \rho l. \tag{2.5}$$

In terms of coordinates (r, μ), where μ is related to the polar coordinate θ by $\mu = \cos\theta$ (see Fig. 1), the governing system (2.2)–(2.5) then becomes

$$[r^2 l_{rr} + (1 - \mu^2)l_{\mu\mu} + 4rl_r - 4\mu l_\mu] - \epsilon[rT_r - \mu T_\mu] = \partial(\psi, l)/\partial(r, \mu), \tag{2.6}$$

$$r^2 T_{rr} + (1 - \mu^2)T_{\mu\mu} + 2rT_r - 2\mu T_\mu = \sigma \, \partial(\psi, T)/\partial(r, \mu), \tag{2.7}$$

$$r^2\psi_{rr} + (1 - \mu^2)\psi_{\mu\mu} = -r^4(1 - \mu^2)l, \tag{2.8}$$

with the condition

$$\int_{-1}^{1} \left[T\psi_\mu + \frac{1}{\sigma} r^2 T_r\right] d\mu = \frac{1}{2\pi}. \tag{2.9}$$

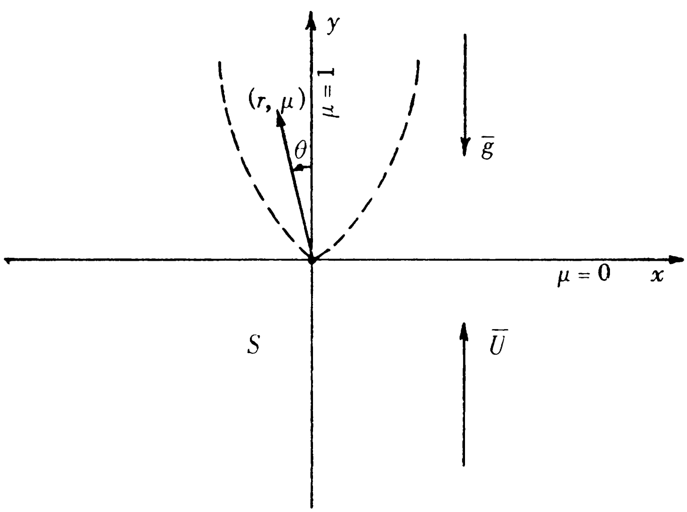

FIG. 1. Flow geometry.

The above equations have been written in terms of nondimensional variables defined by

$$\psi' = U\nu^{-2}\psi, \quad l' = U^{-3}\nu^2 l, \quad T' = U^{-1}\nu^2 Q^{-1}T, \quad r' = U\nu^{-1}r$$

and the primes have been dropped subsequently for convenience. The set of dimensional scales

$$1 = [U^{-1}\nu], \quad t = [U^{-2}\nu], \quad \vartheta = [U\nu^{-2}Q]$$

which has been employed is independent of αg, and therefore is suitable for considering processes in which $\alpha g \to 0$. The two nondimensional parameters $\sigma = \nu\kappa^{-1}$ and $\epsilon = (\alpha g)QU^{-2}\nu^{-1}$ are the Prandtl and Grashof numbers, respectively. In the boundary condition (2.3), S has been chosen to be the surface of a sphere with center at the origin.

The system (2.6)–(2.8) is to be solved subject to the additional boundary conditions

$$\psi \sim \tfrac{1}{2}r^2(1 - \mu^2); \qquad T, l \to 0 \quad \text{as} \quad r \to \infty, \tag{2.10}$$

uniformly in θ,

$$\sqrt{1 - \mu^2}\, T_\mu = o(1), \qquad \sqrt{1 - \mu^2}\, l_\mu = o(1) \qquad \text{as} \quad |\mu| \to 1, \tag{2.11}$$

uniformly in r, and

$$\psi(r, \pm 1) = 0. \tag{2.12}$$

Conditions (2.10) are statements that the flow approaches a uniform stream for large r, and conditions (2.11) are symmetry requirements on the derivatives of T and l normal to the axis of symmetry. The definition of the stream function is completed by the requirement (2.12).

The nonlinearity of the convection terms in Eqs. (2.6), (2.7) precludes closed solutions in all but special cases. An approximate solution is attempted based on an Oseen linearization in which the velocity is assumed to differ only slightly from that of a uniform stream. The effects of buoyancy are therefore regarded as small in the Oseen region. More precisely, the perturbation of the system (2.6)–(2.8) with boundary conditions (2.9)–(2.12) is considered about the state of zero-buoyancy which is represented by writing $\epsilon = 0$ in the above equations. It is assumed that the perturbation is regular. This assumption would appear to be reasonable, at least for points in the region of downstream infinity. Justification of a regular perturbational analysis for large r is given *a posteriori* by employing in Sec. 6 the solutions obtained in preceding sections to calculate directly orders of magnitude of neglected terms at the first stage of the process.

The process begins by expressing ψ, l and T as power series in ϵ, i.e.

$$\psi = \sum_{n=0}^{\infty} \psi_n(r, \mu)\epsilon^n, \qquad l = \sum_{n=0}^{\infty} l_n(r, \mu)\epsilon^n, \qquad T = \sum_{n=0}^{\infty} T_n(r, \mu)\epsilon^n. \tag{2.13}$$

The zero-order equations with the appropriate boundary conditions have the classical solution (see e.g. [3, p. 266])

$$\psi_0 = \tfrac{1}{2}r^2(1 - \mu^2), \qquad l_0 \equiv 0, \qquad T_0 = \frac{\sigma}{4\pi}\,\frac{e^{-\frac{1}{2}\sigma r(1-\mu)}}{r}. \tag{2.14}$$

The equations for the quantities ψ_1, l_1 and T_1 characterize the first-order perturbations due to nonzero buoyancy, and are termed the Oseen equations for this problem.

They are

$$[r^2 l_{1_{rr}} + (1 - \mu^2) l_{1_{\mu\mu}} + 4r l_{1_r} - 4\mu l_{1_\mu}] - [r T_{0_r} - \mu T_{0_\mu}] = \frac{\partial(\psi_0, l_1)}{\partial(r, \mu)} + \frac{\partial(\psi_1, l_0)}{\partial(r, \mu)}, \quad (2.15)$$

$$r^2 T_{1_{rr}} + (1 - \mu^2) T_{1_{\mu\mu}} + 2r T_{1_r} - 2\mu T_{1_\mu} = \sigma\left[\frac{\partial(\psi_0, T_1)}{\partial(r, \mu)} + \frac{\partial(\psi_1, T_0)}{\partial(r, \mu)}\right], \quad (2.16)$$

and

$$r^2 \psi_{1_{rr}} + (1 - \mu^2) \psi_{1_{\mu\mu}} = -r^4(1 - \mu^2) l_1 . \quad (2.17)$$

This set of equations is to be solved subject to the conditions

$$\int_{-1}^{1} \left[T_0 \psi_{1_\mu} + T_1 \psi_{0_\mu} + \frac{1}{\sigma} r^2 T_{1_r}\right] d\mu = 0, \quad (2.18)$$

with

$$\psi_1 = o(r^2), \qquad T_1, l_1 \to 0 \quad \text{as} \quad r \to \infty \text{ uniformly in } \mu, \quad (2.19)$$
$$\sqrt{1 - \mu^2}\, T_{1_\mu} = o(1), \qquad \sqrt{1 - \mu^2}\, l_{1_\mu} = o(1) \qquad \text{as } |\mu| \to 1 \text{ uniformly in } r,$$

and $\psi_1(r, \pm 1) = 0$.

In the remainder of this paper attention is centered on solution of the above system for the quantities ψ_1, l_1 and T_1 as functions of the coordinates (r, μ). The analysis is carried out completely for Prandtl number equal to one. For other values of the Prandtl number suitable Green's functions permit expression of ψ_1, l_1 and T_1 in closed form as multiple integrals to which conditions (2.18), (2.19) apply. The formidable integral boundary condition (2.18) appears to make intractable the complete solution of the perturbation problem for arbitrary values of the Prandtl number.

3. Perturbation solution for ring circulation density. Substitution of the zero-order solutions (2.14) for ψ_0 and T_0 into Eq. (2.15) gives the Oseen equation for l_1 as

$$\begin{aligned} r^2 l_{rr} + (1 - \mu^2) l_{\mu\mu} + 4r l_r - 4_\mu l_\mu - \mu r^2 l_r - r(1 - \mu^2) l_\mu &= -f(r, \mu) \\ &\equiv -\frac{\sigma}{4\pi}\left[\frac{\sigma}{2} + \frac{1}{r}\right] e^{-\frac{1}{2}\sigma r(1-\mu)}, \end{aligned} \quad (3.1)$$

in which the subscript on l_1 has been suppressed temporarily.

In order to obtain the solution of Eq. (3.1) an appropriate Green's function valid in the plane region $0 \le r \le \infty$, $-1 \le \mu \le 1$ is employed. This fundamental solution, denoted by $G(r, \mu; r_0, \mu_0)$ and satisfying the equation

$$r^2 G_{rr} + (1 - \mu^2) G_{\mu\mu} + 4r G_r - 4\mu G_\mu - \mu r^2 G_r - r(1 - \mu^2) G_\mu = -\frac{\delta(r - r_0)\, \delta(\mu - \mu_0)}{2\pi r^2}, \quad (3.2)$$

has been given by Breach [2, Chap. 3] along with the fundamental solution of Eq. (2.17) for ψ_1. We will employ both of these along with the corresponding fundamental solution of Eq. (2.16) appropriate for Oseen flow obtained by the present author. With the Green's function taken in the form

$$\begin{aligned} G(r, \mu; r_0, \mu_0) = \sum_0^\infty \frac{(n + \frac{3}{2})}{4\pi(n + 1)(n + 2)} \frac{e^{\frac{1}{2}r\mu - \frac{1}{2}r_0\mu_0}}{(rr_0)^{3/2}} C_n^{3/2}(\mu) C_n^{3/2}(\mu_0) \\ \cdot \int_0^\infty \exp\left[-\frac{1}{2}\left(t + \frac{r^2 + r_0^2}{4t}\right)\right] I_{n+3/2}\left[\frac{rr_0}{4t}\right] \frac{dt}{t} \end{aligned} \quad (3.3)$$

then the solution for l_1, considered as a function of the coordinates (r, μ) and parameter σ and satisfying all the boundary conditions, is

$$l_1(r, \mu, \sigma) = \frac{\sigma}{64}\left(\frac{2}{\pi}\right)^{3/2} e^{\frac{1}{2}r\mu} \int_0^\infty \frac{e^{-\frac{1}{2}(t+r^2/4t)}}{s^2 t^{5/2}} \left[-2 + \left(\frac{t}{2}\right)^{1/2}\left\{\left(\sigma + \frac{1}{st}\right)\Psi(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}(\sigma - s)^2 t)\right.\right.$$
$$\left.\left. + \left(\sigma - \frac{1}{st}\right)\Psi(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}(\sigma + s)^2 t)\right\}\right] dt, \qquad (3.4)$$

where the Ψ function is the second solution of the confluent hypergeometric equation, and $s^2 = [r/2t + (\sigma - 1)\mu]^2 + (\sigma - 1)^2(1 - \mu^2)$. It appears improbable that the integration with respect to t can be performed for arbitrary σ.

The integral in (3.4) may be evaluated explicitly for the value $\sigma = 1$. In this case,

$$l_1(r, \mu) = \frac{1}{8\pi}\frac{e^{-\frac{1}{2}r(1-\mu)}}{r} \qquad (\sigma = 1). \qquad (3.5)$$

The above solution may be verified to satisfy Eq. (2.15) directly by noting that it is equal to $\frac{1}{2}T_0(r, \mu, 1)$.

The perturbation ring circulation density l_1 decays exponentially everywhere, except along the axis of symmetry where it decays algebraically. The solution has the expected feature of wake-like behaviour. That is, because of the exponential factor in (3.5), the disturbance to the uniform stream due to l_1 is slight, becoming important only when $r(1 - \mu) = O(1)$ for all r, μ. This defines the paraboloidal region $y = O(x^2)$ in which l_1 is significant while outside it l_1 rapidly tends to zero. Further interpretation of the result for the perturbation ring circulation density is reserved until the discussion in Sec. 6.

4. The perturbation stream function. The perturbation stream function $\psi_1(r, \mu)$ is obtained by solving the Eq. (2.17) with l_1 given by Eq. (3.4). The function

$$g(r, \mu; r_0, \mu_0)$$
$$= \sum_1^\infty \frac{n(n+1)}{4\pi}(1 - \mu^2)^{1/2}(1 - \mu_0^2)^{-1/2}P_n^{-1}(\mu)P_n^{-1}(\mu_0)\frac{r^{1/2}}{r_0^{7/2}}\begin{cases}\left(\dfrac{r}{r_0}\right)^{n+1/2} & (r < r_0),\\[2ex] \left(\dfrac{r_0}{r}\right)^{n+1/2} & (r > r_0),\end{cases} \qquad (4.1)$$

satisfying the equation

$$r^2 g_{rr} + (1 - \mu^2)g_{\mu\mu} = -\,\delta(r - r_0)\,\delta(\mu - \mu_0)/2\pi r^2, \qquad (4.2)$$

is a suitable fundamental solution for Eq. (2.17) valid in the region $0 \le r \le \infty$, $-1 \le \mu \le 1$ and vanishing along the axis of symmetry.

With l_1 given by Eq. (3.4) and $g(r, \mu; r_0, \mu_0)$ given by Eq. (4.1), a particular integral of Eq. (2.17) for arbitrary values of σ is expressible in closed form as a multiple integral. That is,

$$\psi_1(r, \mu) = \int_0^\infty dr_0 \int_{-1}^1 d\mu_0\, 2\pi r_0^2 \cdot r_0^4(1 - \mu_0^2)l_1(r_0, \mu_0)g(r, \mu; r_0, \mu_0). \qquad (4.3)$$

The problem of simplifying the above multiple integral significantly for arbitrary values of the Prandtl number appears intractable. We confine ourselves henceforth, therefore, to a Prandtl number of one, for which value l_1 has the relatively simple form (3.5) and the integrals in (4.3) can be evaluated explicitly. Thus, when $\sigma = 1$ we have

$$\psi_1(r, \mu) = \frac{1}{4\pi} r(1 + \mu)[1 - e^{-\frac{1}{2}r(1-\mu)}]. \tag{4.4}$$

The perturbation velocity $\bar{u}_1 = (u_r, u_\theta)$ is readily obtained from the stream function, and the components are

$$\begin{aligned} u_r &= -\frac{1}{r^2}\frac{\partial \psi_1}{\partial \mu} = \frac{1}{4\pi}\left[\frac{e^{-\frac{1}{2}r(1-\mu)} - 1}{r} + \tfrac{1}{2}(1 + \mu)e^{-\frac{1}{2}r(1-\mu)}\right], \\ u_\theta &= -\frac{1}{r(1 - \mu^2)^{1/2}}\frac{\partial \psi_1}{\partial r} = \frac{1}{4\pi}\left(\frac{1 + \mu}{1 - \mu}\right)^{1/2}\left[\frac{e^{\frac{1}{2}r(1-\mu)} - 1}{r} - \tfrac{1}{2}(1 - \mu)e^{-\frac{1}{2}r(1-\mu)}\right]. \end{aligned} \tag{4.5}$$

Off the axis, the components u_r, u_θ of the perturbation velocity are $O(r^{-1})$ for r tending to infinity. However, near the axis of symmetry the exponential terms must be taken into account and in fact, $\lim_{\mu\to 1} u_r = (4\pi)^{-1}$ for all nonzero r, while u_θ is zero for $\mu = \pm 1$ and $r \neq 0$. The above implies there is a concentration of momentum along the axis. This may be seen more clearly by expressing the perturbation velocity in terms of components u_z parallel to the axis and u_ρ perpendicular to the axis. In terms of cylindrical coordinates (ρ, z) in the plane,

$$\begin{aligned} u_z &= -\frac{1}{4\pi}(\rho^2 + z^2)^{-1/2}[\exp(-\tfrac{1}{2}(\rho^2 + z^2)^{1/2} + \tfrac{1}{2}z) - 1] \\ &\qquad + \frac{1}{4\pi}\left[\frac{1}{2} + \frac{z}{2}(\rho^2 + z^2)^{-1/2}\right]\exp(-\tfrac{1}{2}(\rho^2 + z^2)^{1/2} + \tfrac{1}{2}z), \\ u_\rho &= \frac{1}{4\pi}[\rho^{-1} + \rho^{-1}z(\rho^2 + z^2)^{-1/2}][\exp(-\tfrac{1}{2}(\rho^2 + z^2)^{1/2} + \tfrac{1}{2}z) - 1] \\ &\qquad + \frac{1}{4\pi}[\tfrac{1}{2}(\rho^2 + z^2)^{1/2} - \tfrac{1}{2}\rho^{-1}z^2(\rho^2 + z^2)^{-1/2}]\exp(-\tfrac{1}{2}(\rho^2 + z^2)^{1/2} + \tfrac{1}{2}z). \end{aligned} \tag{4.6}$$

The wake-like behaviour superimposed on the uniform stream can now be inferred as follows. The component u_ρ of the velocity tends to zero as ρ tends to infinity, for all z. Moreover, for all ρ, u_ρ tends to zero as z tends to infinity. The downstream component of velocity u_z approaches zero as ρ tends to infinity, for all z; however, it approaches the constant value $(4\pi)^{-1}$ as z tends to infinity for finite ρ.

This wake region in which the perturbation velocity is significant is paraboloidal in shape. The form of the wake is deduced from the observation that exponential terms appearing in the velocity components (4.5) are significant only when $(r/2)(1 - \mu) = O(1)$ for all r, μ, i.e. $y = O(x^2)$.

The mass flux flowing downstream may be compared with that flowing into the paraboloidal wake region as follows. In the boundary layer, the former is almost entirely the mass convected across a disc of radius ρ at a height z. This flux is given by

$$\rho_0 \int_0^{2\pi}\int_0^{\rho} u_z \rho \, d\rho \, d\phi,$$

where ρ_0 is the density. The mass of fluid directed into the wake is given by

$$\rho_0 \int_0^{2\pi}\int_0^{z} (-u_\rho)\rho \, dz \, d\phi.$$

With the aid of (4.6), the preceding expressions both become

$$\tfrac{1}{2}\rho_0[(\rho^2 + z^2)^{1/2} + z]\cdot[1 - \exp(-\tfrac{1}{2}(\rho^2 + z^2)^{1/2} + \tfrac{1}{2}z)],$$

and it is seen that the increasing mass flux flowing downstream is balanced by that entrained at the edge of the wake. It may be noted further that each is $O(z)$ for z tending to infinity, a result which is indicated by similarity solutions of the free convection problem (see e.g. Batchelor [1]).

5. Perturbation temperature. The perturbation to T_0 is obtained by solving Eq. (2.16) with ψ_0, ψ_1 and T_0 known. The solution of this equation for arbitrary values of σ is expressible in closed form as a multiple integral analogous to that in (4.3). Since the perturbation stream function and ring circulation density have been evaluated explicitly for $\sigma = 1$ we concern ourselves with this value of σ only in solving for T_1. By virtue of the expressions (2.14) and (4.4), Eq. (2.16) with the subscript on T_1 suppressed is

$$r^2T_{rr} + (1 - \mu^2)T_{\mu\mu} + 2rT_r - 2\mu T_\mu - \mu r^2 T_r - r(1 - \mu^2)T_\mu$$
$$= f_1(r, \mu) + f_2(r, \mu) + f_3(r, \mu), \tag{5.1}$$

where,

$$f_1 = \frac{1}{16\pi^2}(1 + 1/r)e^{-\frac{1}{2}r(1-\mu)}, \quad f_2 = \frac{1}{32\pi^2}(1 - \mu)e^{-r(1-\mu)}, \quad f_3 = -\frac{1}{16\pi^2}(2 + 1/r)e^{-r(1-\mu)}.$$

It is convenient to express the right-hand side as above and to obtain particular integrals corresponding to f_1, f_2 and f_3 separately in order to simplify the task of applying the relatively complicated boundary condition (2.18).

The solution of this equation is obtained in essentially the same manner as the solutions for l_1 and ψ_1. The complementary function is

$$T_c(r, \mu) = e^{\frac{1}{2}r\mu} \sum_0^\infty \left[A_n \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} + B_n \frac{K_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \right][C_nP_n(\mu) + D_nQ_n(\mu)]. \tag{5.2}$$

By employing the fundamental solution

$$\mathcal{G}(r, \mu; r_0, \mu_0) = \frac{1}{2\pi}\frac{e^{\frac{1}{2}r\mu - \frac{1}{2}r_0\mu_0}}{r^{1/2}r_0^{5/2}} \sum_0^\infty (n + \tfrac{1}{2})P_n(\mu)P_n(\mu_0)\begin{cases} I_{n+1/2}(\frac{1}{2}r)K_{n+1/2}(\frac{1}{2}r_0) & (r < r_0), \\ K_{n+1/2}(\frac{1}{2}r)I_{n+1/2}(\frac{1}{2}r_0) & (r > r_0), \end{cases} \tag{5.3}$$

satisfying the equation

$$r^2\mathcal{G}_{rr} + (1 - \mu^2)\mathcal{G}_{\mu\mu} + 2r\mathcal{G}_r - 2\mu\mathcal{G}_\mu - \mu r^2\mathcal{G}_r - r(1 - \mu^2)\mathcal{G}_\mu = -\frac{\delta(r - r_0)\,\delta(\mu - \mu_0)}{2\pi r^2} \tag{5.4}$$

we determine particular integrals corresponding to f_1, f_2 and f_3, respectively, to be

$$\text{(i)} \quad T^{(1)} = -\frac{1}{16\pi^2}e^{-\frac{1}{2}r(1-\mu)}\cdot\frac{\log r}{r},$$

$$\text{(ii)} \quad T^{(2)} = \frac{1}{16\pi^2}\left[\frac{1}{2}e^{r\mu}Ei(-r) + \frac{e^{-r(1-\mu)}}{2r}\right], \tag{5.5}$$

$$\text{(iii)} \quad T^{(3)} = \frac{1}{16\pi^2}\frac{e^{\frac{1}{2}r\mu}}{r^{1/2}}\cdot\int_r^b ds(2s^{-1} + s^{-2})$$

$$\cdot \sum_0^\infty \pi^{1/2}(2n+1)e^{-s}I_{n+1/2}(\tfrac{1}{2}s)P_n(\mu)$$

$$\cdot [I_{n+1/2}(\tfrac{1}{2}s)K_{n+1/2}(\tfrac{1}{2}r) - I_{n+1/2}(\tfrac{1}{2}r)K_{n+1/2}(\tfrac{1}{2}s)].$$

The upper limit b of the integral in (5.5, iii) is an as yet undetermined constant greater than zero which is determined by the boundary conditions (2.18), (2.19). The integral is in fact well-behaved even for b tending to infinity. The series are absolutely and uniformly convergent for all $r > 0$ and $|\mu| \leq 1$. This follows by comparison of each of the series with the series

$$\sum_0^\infty (2n+1)e^{-2z}I_{n+1/2}(z)I_{n+1/2}(Z)K_{n+1/2}(\lambda Z)$$

$$= \frac{(\lambda z)^{1/2}}{(2\pi)^{1/2}} e^{-2z} \int_{-1}^1 e^{z\alpha} \cdot \frac{\exp(-Z\sqrt{1+\lambda^2-2\lambda\alpha})}{\sqrt{1+\lambda^2-2\lambda\alpha}}\, d\alpha \qquad (\lambda \geq 1)$$

which may be obtained by multiplication of series given in Watson [8, pp. 365, 369], and integration with respect to α over the interval of orthogonality of the Legendre polynomials. The expressions for $T^{(3)}$ and the partial derivative of $T^{(3)}$ with respect to r are therefore both continuous for all $r > 0$ and $|\mu| \leq 1$.

The general solution of (5.1) can be expressed as the sum of the expressions (5.5) together with (5.2). Since T_1 must remain finite as $r \to \infty$, the constants A_n for all n must be chosen equal to zero in (5.2). Furthermore, the temperature is finite along the axis of symmetry and hence the constants D_n must all be zero. The remaining constants are determined by requiring that the integral boundary condition (2.18) and the condition (2.19) for the gradient of T_1 on the axis be satisfied. This is accomplished by considering separately the contributions of the individual terms of the integrand to the value of the integral.

Firstly, with T_0 given by (2.14) and ψ_1 given by (4.4), the contribution of the first term in the integrand of (2.18) is

$$\int_{-1}^1 T_0 \frac{\partial \psi_1}{\partial \mu}\, d\mu = -\frac{1}{16\pi^2}\left(1 - \frac{3}{2r} + \frac{2}{r}e^{-r} - \frac{1}{2r}e^{-2r}\right). \tag{5.6}$$

A consideration of each of the quantities in (5.5) separately gives

$$\begin{aligned}
&\text{(i)} \quad \int_{-1}^1 \left[T_1^{(1)} \frac{\partial \psi_0}{\partial \mu} + r^2 \frac{\partial T_1^{(1)}}{\partial r}\right] d\mu = -\frac{1}{16\pi^2}\left(\frac{2}{r} - 2\log r - \frac{2}{r}e^{-r}\right),\\
&\text{(ii)} \quad \int_{-1}^1 \left[T_1^{(2)} \frac{\partial \psi_0}{\partial \mu} + r^2 \frac{\partial T_1^{(2)}}{\partial r}\right] d\mu = -\frac{1}{16\pi^2}\left(\frac{1}{2r} - \frac{1}{2r}e^{-2r}\right), \qquad (5.7)\\
&\text{(iii)} \quad \int_{-1}^1 \left[T_1^{(3)} \frac{\partial \psi_0}{\partial \mu} + r^2 \frac{\partial T_1^{(3)}}{\partial r}\right] d\mu\\
&\qquad = -\frac{1}{16\pi^2}\left(\frac{-1}{r} + 2\log r + \frac{e^{-2r}}{r} + \frac{1}{b} - 2\log b - \frac{e^{-2b}}{b}\right).
\end{aligned}$$

The evaluation of the first two integrals in (5.7) is straightforward, but considerably more effort is required to evaluate the third. Since $T_1^{(3)}$ and $\partial T_1^{(3)}/\partial r$ are continuous for all $r > 0$ and $|\mu| \leq 1$ the left-hand side of the boundary condition can be expressed in the alternative form

$$\int_{-1}^{1} T_1^{(3)} \frac{\partial \psi_0}{\partial \mu}\, d\mu + r^2 \frac{d}{dr} \int_{-1}^{1} T_1^{(3)}\, d\mu.$$

The series in $T_1^{(3)}$, being uniformly convergent for $|\mu| \leq 1$, permit the order of the integration with respect to μ and summation to be interchanged. If the integrals

$$\int_{-1}^{1} e^{\frac{1}{2}r\mu} P_n(\mu)\, d\mu = (2\pi)^{1/2} \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}}, \qquad \int_{-1}^{1} e^{\frac{1}{2}r\mu} P_n(\mu)\mu\, d\mu = (2\pi)^{1/2} \frac{d}{d(\frac{1}{2}r)} \left(\frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}}\right)$$

are noted, the resultant integration with respect to μ gives

$$\begin{aligned}
&\int_{-1}^{1} T_1^{(3)} \frac{\partial \psi_0}{\partial \mu}\, d\mu + r^2 \frac{d}{dr} \int_{-1}^{1} T_1^{(3)}\, d\mu \\
&= \frac{1}{8\pi} r^2 \Bigg[\sum_0^\infty (2n+1) \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \frac{d}{dr} \left(\frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}}\right) \cdot \int_r^b e^{-s}(2s^{-1} + s^{-2}) K_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s)\, ds \\
&\quad - \sum_0^\infty (2n+1) \frac{K_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \frac{d}{dr} \left(\frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}}\right) \cdot \int_r^b e^{-s}(2s^{-1} + s^{-2}) I_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s)\, ds \Bigg] \\
&\quad - \frac{1}{16\pi^2} r^2 \frac{d}{dr} \Bigg[\sum_0^\infty (2n+1) \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \int_r^b e^{-s}(2s^{-1} + s^{-2}) K_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s)\, ds \\
&\quad - \sum_0^\infty (2n+1) \frac{K_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \frac{I_{n+1/2}(\frac{1}{2}r)}{(\frac{1}{2}r)^{1/2}} \cdot \int_r^b e^{-s}(2s^{-1} + s^{-2}) I_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s)\, ds \Bigg].
\end{aligned}$$

The preceding expression is considerably simplified by noting that $(fg)' = 2f'g + W(f, g)$ for any differentiable functions f and g and applying this result to the functions $f = (\frac{1}{2}r)^{-1/2} I_{n+1/2}(\frac{1}{2}r)$ and $g = (\frac{1}{2}r)^{-1/2} K_{n+1/2}(\frac{1}{2}r)$ with Wronskian $-2r^{-2}$. The cancellation of series which then becomes possible provides the simpler result

$$\begin{aligned}
&\int_{-1}^{1} T_1^{(3)} \frac{\partial \psi_0}{\partial \mu}\, d\mu + r^2 \frac{d}{dr} \int_{-1}^{1} T_1^{(3)}\, d\mu \\
&\qquad = \frac{1}{8\pi} \sum_0^\infty (2n+1) \int_r^b e^{-s}(2s^{-1} + s^{-2}) I_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s)\, ds.
\end{aligned}$$

The series is uniformly convergent for $0 \leq s < \infty$ with sum (see e.g. [8, p. 365 (6)])

$$\sum_0^\infty (2n+1) I_{n+1/2}(\tfrac{1}{2}s) I_{n+1/2}(\tfrac{1}{2}s) = (2\pi)^{-1/2} s^{1/2} I_{1/2}(s).$$

The integration with respect to s is now performed by noting that $I_{1/2}(s) = (2/\pi)^{1/2} s^{-1/2} \sinh s$, and the final result is Eq. (5.7, iii).

The sum of the integrals in the expressions (5.6)–(5.7) reduces to the constant value $-[16\pi^2 b]^{-1}[b + 1 - 2b \log b - e^{-2b}]$. The nonzero value is due to the quantity $T_1^{(3)}$ which possesses a singularity of $O(r^{-1})$ as $r \to 0$ corresponding to the $n = 0$ term in the complementary function (5.2). By choosing $b = 1$, $B_0 = (2\pi)^{-1/2} \cdot (32\pi^2)^{-1} \cdot (e^{-2} - 2)$ and $B_n = 0 (n \geq 1)$, the boundary condition (2.18) is completely satisfied. The perturbation temperature is given therefore by

$$T_1(r,\mu) = \frac{1}{16\pi^2}\left[\tfrac{1}{2}e^{r\mu}Ei(-r) + \frac{e^{-r(1-\mu)}}{2r} - e^{-\frac{1}{2}r(1-\mu)}\frac{\log r}{r} - \left(1 - \frac{1}{2e^2}\right)\frac{e^{-\frac{1}{2}r(1-\mu)}}{r}\right. \tag{5.8}$$
$$- \frac{e^{\frac{1}{2}r\mu}}{r^{1/2}}\int_r^1 ds(2s^{-1} + s^{-2})\sum_0^\infty \pi^{1/2}(2n+1)e^{-s}P_n(\mu)I_{n+1/2}(\tfrac{1}{2}s)$$
$$\left.\cdot\{I_{n+1/2}(\tfrac{1}{2}s)K_{n+1/2}(\tfrac{1}{2}r) - I_{n+1/2}(\tfrac{1}{2}r)K_{n+1/2}(\tfrac{1}{2}s)\}\right].$$

It can be verified directly that the condition (2.19) on the gradient of T_1 normal to the axis is satisfied uniformly in r. The logarithmic singularity of $\partial T_1^{(1)}/\partial\mu$ as r tends to zero is cancelled by the behaviour of $\partial T_1^{(3)}/\partial\mu$ as $r \to 0$, which is determined by noting the behaviour

$$I_{n+1/2}(\tfrac{1}{2}r) \sim \frac{r^{n+1/2}}{2^{2n+1}\Gamma(n+\frac{3}{2})}, \qquad K_{n+1/2}(\tfrac{1}{2}r) \sim \pi^{1/2}\frac{(2n)!}{n!}r^{-(n+1/2)}$$

of the modified Bessel functions for small r.

6. Discussion. With the zero- and first-order perturbation solutions now determined completely for a Prandtl number of one we investigate the consequences of assuming a regular perturbational procedure. If L, M, N and K denote the linear partial differential operators

$$\begin{aligned} L &\equiv r^2\frac{\partial^2}{\partial r^2} + (1-\mu^2)\frac{\partial^2}{\partial\mu^2} + 4r\frac{\partial}{\partial r} - 4\mu\frac{\partial}{\partial\mu}, \\ M &\equiv r^2\frac{\partial^2}{\partial r^2} + (1-\mu^2)\frac{\partial^2}{\partial\mu^2} + 2r\frac{\partial}{\partial r} - 2\mu\frac{\partial}{\partial\mu}, \\ N &\equiv r^2\frac{\partial^2}{\partial r^2} + (1-\mu^2)\frac{\partial^2}{\partial\mu^2}, \\ K &\equiv r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial\mu}, \end{aligned} \tag{6.1}$$

then Eqs. (2.6)–(2.8), which have been presumed to possess series solutions for small ϵ of the form (2.13), become

$$\begin{aligned} &L(l_0) + \epsilon L(l_1) + \epsilon^2 L(l_2) + \cdots - \epsilon K(T_0) - \epsilon^2 K(T_1) - \cdots \\ &\qquad = \frac{\partial(\psi_0, l_0)}{\partial(r,\mu)} + \epsilon\left[\frac{\partial(\psi_1, l_0)}{\partial(r,\mu)} + \frac{\partial(\psi_0, l_1)}{\partial(r,\mu)}\right] \\ &\qquad\quad + \epsilon^2\left[\frac{\partial(\psi_2, l_0)}{\partial(r,\mu)} + \frac{\partial(\psi_1, l_1)}{\partial(r,\mu)} + \frac{\partial(\psi_0, l_2)}{\partial(r,\mu)}\right] + \cdots, \\ &M(T_0) + \epsilon M(T_1) + \epsilon^2 M(T_2) + \cdots \\ &\qquad = \sigma\frac{\partial(\psi_0, T_0)}{\partial(r,\mu)} + \epsilon\sigma\left[\frac{\partial(\psi_1, T_0)}{\partial(r,\mu)} + \frac{\partial(\psi_0, T_1)}{\partial(r,\mu)}\right] \\ &\qquad\quad + \epsilon^2\sigma\left[\frac{\partial(\psi_2, T_0)}{\partial(r,\mu)} + \frac{\partial(\psi_1, T_1)}{\partial(r,\mu)} + \frac{\partial(\psi_0, T_2)}{\partial(r,\mu)}\right] + \cdots, \\ &N(\psi_0) + \epsilon N(\psi_1) + \epsilon^2 N(\psi_2) + \cdots = -r^4(1-\mu^2)[l_0 + \epsilon l_1 + \epsilon^2 l_2 + \cdots]. \end{aligned} \tag{6.2}$$

The analytical solutions for l_0, ψ_0, T_0 and for l_1, ψ_1, T_1 given by (2.14), (3.5), (4.4), and (5.8), respectively, can be used to calculate the order of the terms neglected in each of the first two stages of the perturbational process. The ratio of neglected self-convection terms to forced convection terms outside the wake region is calculated readily to be $O(\epsilon)$ for large r.

The order in r, when r tends to zero, of the diffusion, buoyancy and convection terms is calculated from solutions for l_0, ψ_0, T_0 and l_1, ψ_1, T_1 to be $O(r^{-1})$, $O(r^{-1})$ and $O(1)$ respectively, and hence in the neighborhood of the source the order of neglected terms is at most $O(\epsilon)$.

The ring circulation density flux vectors for diffusion, forced convection and self-convection of l, and the corresponding vectors for heat flux can be calculated using the zero-order and first-order solutions obtained for ψ, l and T, and the effect of the perturbation solutions compared with predictions made physically. For example, diffusion effects would be expected to predominate near the source. The various forms of flux of l are:

$$\begin{aligned}
\bar{J}_D &= -\operatorname{grad} l = \frac{\epsilon}{8\pi}\left(e^{-\frac{1}{2}r(1-\mu)}\left[\frac{1-\mu}{2r}+\frac{1}{r^2}\right], e^{-\frac{1}{2}r(1-\mu)}.\frac{(1-\mu^2)^{1/2}}{2r}\right), \\
\bar{J}_F &= \bar{1}l = \frac{\epsilon}{8\pi}\left(e^{-\frac{1}{2}r(1-\mu)}.\frac{\mu}{r}, -e^{-\frac{1}{2}r(1-\mu)}.\frac{(1-\mu^2)^{1/2}}{r}\right), \qquad (6.3) \\
\bar{J}_S &= \bar{u}_1 l = \frac{\epsilon^2}{32\pi^2}\left(\frac{e^{-r(1-\mu)}-e^{-\frac{1}{2}r(1-\mu)}}{r^2}+e^{-r(1-\mu)}.\frac{1+\mu}{2r},\right. \\
&\qquad \left.\frac{e^{-r(1-\mu)}-e^{-\frac{1}{2}r(1-\mu)}}{r^2}\left(\frac{1+\mu}{1-\mu}\right)^{1/2}-e^{-r(1-\mu)}\frac{(1-\mu^2)^{1/2}}{2r}\right).
\end{aligned}$$

Outside the wake the distribution of l throughout the fluid is mainly by diffusion, and convection by the uniform stream. The ratio of self-convected flux of l to the other forms is small in this region, being $O(\epsilon r^{-1})$ for $r \to \infty$.

Inside the wake, for r tending to zero,

$$\bar{J}_D = O(\epsilon r^{-2}), \qquad \bar{J}_F = O(\epsilon r^{-1}), \qquad \bar{J}_S = O(\epsilon^2 r^{-1}).$$

In this region diffusion of l dominates forced convection which in turn dominates self-convection of l. Similar conclusions hold for the distribution of heat flux in each of the regions above.

The flux vectors of l in (6.3) can be expressed in terms of a component in the direction of the uniform stream and another normal to it. It is then apparent that the downstream component of diffusion reinforces the corresponding component of convection near the axis of symmetry which implies there is concentration of heat and momentum in the neighborhood of the center of the wake.

References

[1] G. K. Batchelor, *Heat conduction and buoyancy effects in fluids,* Quart. J. Roy. Met. Soc. **80**, 339–358 (1954)

[2] D. R. Breach, *Axisymmetric viscous flow past finite bodies,* Ph.D. dissertation, University of Toronto, Toronto, Canada, 1964

[3] H. S. Carslaw and J. C. Jaegar, *Conduction of heat in solids,* 2nd ed., Oxford Univ. Press, London, 1959

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of integral transforms*, Vol. 1, 2, edited by A. Erdelyi, Bateman Manuscript Project, McGraw-Hill, New York, 1954

[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. 1, 2, 3, edited by A. Erdelyi, Bateman Manuscript Project, McGraw-Hill, New York, 1953

[6] L. Howarth, *Modern developments in fluid dynamics, high speed flow. I*, Oxford Univ. Press, London, 1953

[7] J. J. Mahony, *Heat transfer at small Grashof numbers*, Proc. Roy. Soc. **A238**, 412–423 (1956/57).

[8] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, New York, 1958

[9] C. S. Yih, *Free convection due to a point source of heat*, Proc. First U. S. Natl. Congress Appl. Mech., pp. 941–947