COMPARISON THEOREMS FOR EIGENVALUES*

By A. M. FINK (*Iowa State University*)

1. Introduction. It is the purpose of this note to announce two comparison theorems for the lowest eigenvalue of a nonhomogeneous string. Comparison theorems of the sort that we prove have two advantages over many of the results in the literature. Firstly, the results of the literature, e.g. Banks [1], give upper and lower bounds for classes of functions. Comparison theorems give upper and lower bounds for a particular function which can be improved by astute choices of comparison functions. Secondly, our theorems remove nonnegative conditions on p , replacing them by nonnegative conditions on integrals of p .

2. The comparison theorems. The main tool in proving our comparison theorem is an integral mean value theorem due to Banks [2], which we state for reference.

LEMMA 1. *Let f be integrable on A and ϕ a bounded real-valued measurable function. For y a real number, let $A(y) = \{x \mid \phi(x) \geq y\}$, Then*

$$\int_A f\phi = m \int_A f + \int_m^M \left(\int_{A(y)} f \right) dy$$

where $m \leq \phi(x) \leq M$ on A .

We shall be considering the eigenvalue problem

$$(1) \quad (ry')' + \lambda py = 0, \quad y(-a) = y(a) = 0$$

where $r > 0$ is continuous and p is continuous. We use two facts about the smallest positive eigenvalue of the problem (1). It is known that the associated eigenfunction is nonzero on $(-a, a)$ and that

$$(2) \quad \lambda = \min \frac{\int_{-a}^a r(y')^2}{\int_{-a}^a py^2}$$

where the minimum is taken over continuous functions with piecewise continuous derivatives that satisfy the boundary conditions and for which the denominator is positive. Furthermore, no function other than the eigenfunction yields the minimum. We write $\lambda(p)$ to designate the relationship between λ and p .

It is clear that if one requires $p > 0$ and $r = 1$, then the first eigenfunction is concave. This nonnegative hypothesis of most of the results in the literature may be removed by the following observation.

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LEMMA 2. (St. Mary [3]) Let y be the solution to $(ry')' + qy = 0$, $y(-a) = y'(b) = 0$ and $y > 0$ on $(-a, b)$. If $\int_t^b q \geq 0$ for $t \in (-a, b)$ then $y'(t) \geq 0$ on $(-a, b)$. Here $r(t) > 0$.

Proof. From the differential equation we obtain

$$\begin{aligned} r(t)y'(t) &= \int_t^b q(s)y(s) ds \\ &= y(t) \int_t^b q(\tau) d\tau + \int_t^b \left(\int_t^b q(\tau) d\tau \right) y'(s) ds \\ &\geq \int_t^b \left(\int_t^b q(\tau) d\tau \right) y'(s) ds. \end{aligned}$$

Hence Gronwall's inequality applied to

$$-r(t)y'(t) \leq \int_t^b \left(\frac{1}{r(s)} \int_s^b q(\tau) d\tau \right) (-r(s)y'(s)) ds$$

yields $r(t)y'(t) \geq 0$ and the desired result.

Of course a similar result holds on (b, a) , namely $y(a) = y'(b) = 0$, $y(t) > 0$ on (b, a) and $\int_t^a q \geq 0$ for $t \in (b, a)$ implies $y'(t) \leq 0$ on (b, a) . We apply the above results to the smallest eigenfunction with q replaced by λp to get that if $y'(b) = 0$ and $(b-t) \int_t^b p \geq 0$ for $t \in (-a, a)$ then $y' \geq 0$ on $(-a, b)$ and $y' \leq 0$ on (b, a) . We are now ready to prove the first comparison theorem.

THEOREM 1. Let Z be the first eigenfunction for p in the problem (1). If $Z'(b) = 0$ and $0 \leq (b-t) \int_t^b p \leq (b-t) \int_t^b q$ for all $t \in (-a, a)$ then $\lambda(p) \leq \lambda(q)$. Strict inequality holds unless $\int_t^b p = \int_t^b q$ for all t .

Proof. Notice that if one considers the eigenvalue problem $(ry')' + \lambda py = 0$, $y(-a) = y'(b) = 0$, then Z is the first eigenfunction since it is nonzero on $(-a, b]$. Consequently,

$$\lambda(p) = \frac{\int_{-a}^b r(Z')^2}{\int_{-a}^b pZ^2}.$$

Now we apply Lemma 1 to $\int_{-a}^b pZ^2$. We have $A(y) = \{x \mid Z^2(x) \geq y\} = [t(y), b]$ for an increasing function $t(y)$ with $t(0) = -a$ and $t(M) = b$, $M = \max Z^2$. Here we use Lemma 2. Thus

$$\int_{-a}^b pZ^2 = \int_0^M \left(\int_{t(y)}^b p \right) dy \leq \int_0^M \left(\int_{t(y)}^b q \right) dy = \int_{-a}^b qZ^2,$$

so that $\lambda(p) \geq \int_{-a}^b r(Z')^2 / \int_{-a}^b qZ^2$. Similarly $\lambda(p) \geq \int_a^b r(Z')^2 / \int_a^b qZ^2$. Equality holds only if $\int_t^b q = \int_t^b p$ for all t . Now if $0 < a/b \leq c/d$ for positive a, b, c, d then $(a+c)/(b+d) \leq c/d$. Thus

$$\lambda(q) \leq \frac{\int_{-a}^a r(Z')^2}{\int_{-a}^a qZ^2} = \frac{\int_{-a}^b r(Z')^2 + \int_b^a r(Z')^2}{\int_{-a}^b qZ^2 + \int_b^a qZ^2} \leq \lambda(p).$$

In order to apply the comparison Theorem 1, it is necessary to know the zero of Z' . One case which is of particular interest is when both r and p are even. Then it is easy to verify that the eigenfunctions of (1) are also even so that $y'(0) = 0$. Now the hypothesis

of Theorem 1 yields

$$0 \leq \int_{-t}^t p \leq \int_{-t}^t q$$

for all $t \in (0, a)$, but not conversely. We get a new comparison theorem for this situation.

THEOREM 2. *Let r and p be even and $0 \leq \int_{-t}^t p \leq \int_{-t}^t q$ for $t \in (0, a)$. Then $\lambda(p) \leq \lambda(q)$ and strict inequality holds if $\int_{-t_0}^t p < \int_{-t_0}^t q$ for some t_0 .*

Proof. We use the fact that $A(y) = \{x \mid Z^2(x) \geq y\} = [-t(y), t(y)]$ when Z is the eigenfunction associated with p . Then for $M = \max Z^2$

$$\int_{-a}^a p Z^2 = \int_0^M \left(\int_{-t(v)}^{t(v)} p \right) dy \leq \int_0^M \left(\int_{-t(v)}^{t(v)} q \right) dy = \int_{-a}^a q Z^2$$

so that

$$\lambda(p) = \frac{\int_{-a}^a r(Z')^2}{\int_{-a}^a p Z^2} \geq \frac{\int_{-a}^a r(Z')^2}{\int_{-a}^a q Z^2} \geq \lambda(q).$$

COROLLARY 1. (Beesack [4]). *If there exists $b \in (0, a)$ such that p is even and $p(x) \leq q(x)$ on $(-b, b)$ and $p(x) \geq q(x)$ on $(-b, -a) \cup (a, b)$ with $\int_{-a}^a p \leq \int_{-a}^a q$ then $\lambda(p) \leq \lambda(q)$ provided that $\int_{-t}^t p \geq 0$ for $t \in (0, a)$ and r is even.*

Proof. From the geometry of p and q and $\int_{-a}^a p \leq \int_{-a}^a q$ one gets $\int_{-t}^t p \leq \int_{-t}^t q$ on $(0, a)$ so that Theorem 2 applies.

We remark that Beesack requires that q be even, that $\int_{-a}^a p = \int_{-a}^a q$, and $p \geq 0$. It is seen that the evenness of q is not required and that $p \geq 0$ may be replaced by $\int_{-t}^t p \geq 0$. In fact, this change of the condition of positivity from $p \geq 0$ to $\int_{-t}^t p \geq 0$ leads to generalizations of many of the results in the literature. One is

COROLLARY 2. *Suppose q is convex, even, $\int_{-a}^a q > 0$ and let $r = 1$. Then $\lambda(q) \geq \pi^2(2a \int_{-a}^a q)^{-1}$.*

Proof. Let $p = (1/2a) \int_{-a}^a q > 0$. Then p and q satisfy the hypothesis of Corollary 1 with $\lambda(p) = \pi^2(2a \int_{-a}^a q)^{-1}$.

3. Applications. We give only a few typical applications to illustrate the use of the above. If $\lambda(p)$ is the smallest eigenvalue, then $\lambda((p(x) + p(-x))/2) \geq \lambda(p(x))$ (Krein [5]). Let $p^*(x) = (p(x) + p(-x))/2$.

Let p be a concave function; then $\lambda(p)2a \int_{-a}^a p \leq \lambda(p^*)2a \int_{-a}^a p$. If $q = (1/2a) \int_{-a}^a p > 0$, then $\int_{-t}^t q \leq \int_{-t}^t p^*$ so that $\lambda(q) \geq \lambda(p^*)$ and consequently $\lambda(p)2a \int_{-a}^a p \leq \pi^2$. This generalizes a result of Galbraith [6] who requires $p \geq 0$.

As a second example, let p be convex and increasing. Let $Z'(b) = 0$, where Z is the eigenfunction corresponding to p . Consider the convex functions q such that

- (i) q is continuous;
- (ii) q is linear on $[c, a]$ where $-a \leq c \leq b$;
- (iii) $q = \min_{-a \leq x \leq a} p$ on $(-a, c)$ if $c \neq -a$; and
- (iv) $\int_{-a}^a q = \int_{-a}^a p$.

It is easy to see that for $q(b)$ in $(0, (1/2a) \int_{-a}^a p)$ such a function exists. It is also clear that for $q(b)$ near 0, $\int_{-a}^a q < \int_{-a}^a p$ and for $q(b)$ near $(1/2a) \int_{-a}^a p$ the reverse holds. Hence there exists a q_0 satisfying (i)–(iv) such that $\int_{-a}^a q_0 = \int_{-a}^a p$ and $\int_b^a q_0 = \int_b^a p$. In fact

$q_0(b) > p(b)$. Hence $(b - t) \int_t^b q_0 \geq (b - t) \int_t^b p$ so that $\lambda(q_0) \leq \lambda(p)$. $\lambda(q_0)$ may be computed, or lower bounds for $\lambda(q_0)$ may be computed, by using Bessel functions.

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