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ALMOST-PERIODIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR VOLTERRA SYSTEM*

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1. Introduction. The purpose of this paper is to study the behavior as $t \to \infty$ of solutions of a system of two nonlinear equations of the form

$$\begin{aligned} x_1(t) &= f_1(t) - \int_0^t a_1(t-s)g_1(s, x_1(s))\, ds - \int_0^t a_2(t, s)g_2(s, x_2(s))\, ds, \\ x_2(t) &= f_2(t) - \int_0^t a_2(t-s)g_1(s, x_1(s))\, ds - \int_0^t a_1(t-s)g_2(s, x_2(s))\, ds \end{aligned} \tag{1.1}$$

where $f_1(t)$ and $f_2(t)$ are asymptotically almost periodic and both $g_1(t, x)$ and $g_2(t, x)$ are almost periodic in t uniformly for x on compact sets. We seek conditions which guarantee that the solutions $x_1(t)$ and $x_2(t)$ of (1.1) exist for all $t \geq 0$ and are asymptotically almost periodic.

System (1.1) arises in a natural way from the partial differential equation

$$u_t = u_{xx} \qquad (t > 0, 0 < x < L) \tag{1.2}$$

with initial conditions

$$u(0, x) = F(x) \qquad (0 < x < L) \tag{1.3}$$

and nonlinear boundary conditions of the form

$$u_x(t, 0) = g_1(t, u(t, 0)), \qquad u_x(t, L) = -g_2(t, u(t, 0)), \tag{1.4}$$

for all $t > 0$. Indeed, if $A_1(t) = u_x(t, 0)$ and $A_2(t) = u_x(t, L)$ are assumed to be known functions and if $A_i \in C[0, \infty) \cap C^1(0, \infty)$ with $A_i(t)$ absolutely continuous in a neighborhood of $t = 0$, then well-known elementary methods imply that

$$\begin{aligned} u(t, x) = F_0/2 &+ \sum_{n=0}^{\infty} F_n \exp(-(n\pi/L)^2 t) \cos(n\pi x/L) \\ &- L^{-1} \int_0^t \left\{1 + 2 \sum_{n=1}^{\infty} \exp\{-(n\pi/L)^2(t-s)\} \cos(n\pi x/L)\right\} A_1(s)\, ds \\ &+ L^{-1} \int_0^t \left\{1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp\{-(n\pi/L)^2(t-s)\} \cos(n\pi x/L)\right\} A_2(s)\, ds \end{aligned} \tag{1.5}$$

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where

$$F_n = (2/L) \int_0^L F(x) \cos (n\pi x/L)\, dx \qquad (n = 0, 1, 2, \cdots) \tag{1.6}$$

is the sequence of Fourier cosine coefficients of F. Setting $x = 0$ and then $x = L$ in (1.5) and using (1.4), one obtains the integral equations

$$\begin{aligned} u(t, 0) = {} & F_0/2 + \sum_{n=1}^{\infty} F_n \exp \{-(n\pi/L)^2 t\} \\ & - L^{-1} \int_0^t \Big\{1 + 2 \sum_{n=1}^{\infty} \exp \{-(n\pi/L)^1(t - s)\} g_1(s, u(s, 0))\, ds \\ & - L^{-1} \int_0^t \Big\{1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp \{-(n\pi/L)^2(t - s)\} g_2(s, u(s, L))\, ds, \end{aligned} \tag{1.7a}$$

and

$$\begin{aligned} u(t, L) = {} & F_0/2 + \sum_{n=1}^{\infty} F_n(-1)^n \exp \{-(n\pi/L)^2 t\} \\ & - L^{-1} \int_0^t \Big\{1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp \{-(n\pi/L)^2(t - s)\}\Big\} g_1(s, x_1(s))\, ds \\ & - L^{-1} \int_0^t \Big\{1 + 2 \sum_{n+1}^{\infty} \exp \{-(n\pi/L)^2(t - s)\}\Big\} g_2(s, x_2(s))\, ds. \end{aligned} \tag{1.7b}$$

Eqs. (1.7) clearly have the form (1.1) with $x_1(t) = u(t, 0)$ and $x_2(t) = u(t, L)$. On the other hand, if $u(t, 0)$ and $u(t, L)$ are the known unique solutions of (1.7), then $u(t, x)$ may be obtained using (1.4) and then (1.5). This formal equivalence of (1.2)–(1.4) and (1.7) will be made precise in Sec. II below.

Eqs. (1.2)–(1.4) and also our assumption of almost periodicity may be physically motivated using C. C. Lin's theory of superfluidity of helium, cf. [1]. In three-dimensional space with coordinates (x, y, z) let the planes $x = 0$ and $x = L$ represent two infinite plates. Suppose the region $0 < x < L$ between these plates is filled with liquid helium initially at rest. If the boundary plates $x = 0$ and $x = L$ are both given signosoid oscillations in the y-direction, then a one-dimensional flow will be set up in the liquid. Let $u(t, x)$ be the velocity profile at time $t > 0$ of any point (x, y, z) with first coordinate x. Then $u(t, x)$ satisfies (1.2) and (1.3), $F(x) \equiv 0$. Lin's theory implies boundary conditions of the form (1.4); indeed

$$g_i(t, u) = B(u - C \sin (k_i t))^3 \tag{1.8}$$

for $i = 1, 2$ where B and C are positive constants. For this problem we prove the following:

THEOREM 1. *Suppose* $F \varepsilon C^2[0, L]$. *Let* $g_i(t, u)$ *be given by* (1.8) *where* $B > 0$, $C \neq 0$ *and* $k_i \neq 0$. *Then* (1.7) *has unique continuous solutions* $x_1(t) = u(t, 0)$ *and* $x_2(t) = u(t, L)$ *defined for all* $t \geqq 0$. *Moreover, there exist two almost periodic functions* $X_i(t)$ *with Fourier series of the form*

$$X_i(t) \sim \sum_{m,n=-\infty}^{\infty} X_{mn} \exp (i(mk_1 + nk_2)t) \tag{1.9}$$

such that

$$\lim \{x_i(t) - X_i(t)\} = 0 \quad as \quad t \to \infty.$$

This result follows as a special case of more general theorems which will be proved below. These more general theorems concern a two-dimensional system of the form

$$x(t) = f(t) - \int_0^t A(t - s)G(s, x(s))\, ds, \tag{E}$$

where $A(t)$ is a matrix of the form

$$A(t) = \begin{bmatrix} a_1(t) & a_2(t) \\ a_2(t) & a_1(t) \end{bmatrix}.$$

In Sec. 3 we use the special form of $A(t)$ to show that system (E) may be transformed into an equivalent system of the form

$$y(t) = \int_0^t R_N(t - s)\{y(s) - G_N(s, y(s))\}\, ds \tag{$\mathrm{E_N}$}$$

where $R_N(t)$ is a positive definite, diagonal matrix of class $L^1(0, \infty)$. Subsequent work deals with equations of the form ($\mathrm{E_N}$) rather than directly with (E).

Sec. 4 contains results concerning the global existence and boundedness of solutions of ($\mathrm{E_N}$). In Sec. 5 we study the existence and uniqueness of almost periodic solutions of a related equation of the form

$$Y(t) = \int_{-\infty}^t R_N(t - s)\{Y(s) - G_N(s, Y(s))\}\, ds. \tag{1.10}$$

In the last section we show that the solutions $y(t)$ and $Y(t)$ of ($\mathrm{E_N}$) and (1.10) are asymptotic, that is

$$\lim \{y(t) - Y(t)\} = 0 \quad \text{as} \quad t \to \infty.$$

Transforming ($\mathrm{E_N}$) back to (E) then yields Theorem 1 as a corollary. Sec. 6 also contains results concerning the mean values of the solution $x(t)$ of (E). This information on mean values is important in any study of the behavior of the nonlinear problem (1.2)–(1.4).

If $L = +\infty$ and if the second boundary condition in (1.4) is dropped, then (1.2)–(1.4) and (1.8) model the limiting case of a one-dimensional flow in a half space. This problem has been studied by Levinson [2]. Some of Levinson's results have been generalized in papers of Friedman [3], [4] and Miller [5]. A similar problem involving heat flow has been extensively studied by Mann and Wolf [6] and others [7], [8], [9]. The methods used in this paper are extensions of the methods used in [5]. The main tools in our analysis will be the "variation of constants" equation ($\mathrm{E_N}$) and invariance results similar to those used in [5, Sec. V].

2. Equivalence of the problems. Let R^2 denote real Euclidean two-dimensional space of column vectors $x = \text{col}\,(x_1, x_2)$. Throughout the remainder of this paper the norm $|x|$ in R^2 will always mean $|x| = \max\{|x_1|, |x_2|\}$. Many of our results are explicitly dependent on the use of this norm rather than some other equivalent norm.

Define

$$f_1(t) = F_0/2 + \sum_{n=1}^{\infty} F_n \exp\{-(n\pi/L)^2 t\}, \tag{2.1a}$$

and

$$f_2(t) = F_0/2 + \sum_{n=1}^{\infty} F_n(-1)^n \exp\{-(n\pi/L)^2 t\} \tag{2.1b}$$

where F_n is defined by (1.6). Define

$$a_1(t) = \pi\left(1 + 2\sum_{n=1}^{\infty} \exp\{-(\pi n/L)^2 t\}\right)\Big/ L \tag{2.2a}$$

$$a_2(t) = \pi\left(1 + 2\sum_{n=1}^{\infty} (-1)^n \exp\{-(n\pi/L)^2 t\}\right)\Big/ L \tag{2.2b}$$

and

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{2.3}$$

Let $A(t)$ be the matrix

$$A(t) = \begin{bmatrix} a_1(t) & a_2(t) \\ a_2(t) & a_1(t) \end{bmatrix}. \tag{2.4}$$

Then Eq. (1.7) has the form (E) where

$$G(t, x) = \begin{bmatrix} g_1(t, x_1) \\ g_2(t, x_2) \end{bmatrix}. \tag{2.5}$$

THEOREM 2. *Suppose $u(t, x)$ is a function which satisfies the following conditions:*

(i) *$u(t, x)$ is continuous on $\{0 \leqq t < \infty, 0 \leqq x \leqq L\}$.*

(ii) *u_t and u_{xx} exist and are continuous for all (t, x) in the set $\{0 < t < \infty, 0 < x < L\}$.*

(iii) *$u(t, x)$ satisfies (1.2), (1.3) and also (1.4) in the sense that*

$$\lim_{x\to 0^+} u_x(t, x) = g_1(t, u(t, 0))$$

and

$$\lim_{x\to L^-} u_x(t, x) = -g_2(t, u(t, L)), \qquad (t > 0).$$

(iv) *The functions $A_1(t) = g(t, u(t, 0))$ and $A_2(t) = -g_2(t, u(t, L))$ are of class $C[0, \infty) \cap C^1(0, \infty)$ and are absolutely continuous in a neighborhood of $t = 0$.*

If $F \in C^2[0, L]$ and if $g_1, g_2 \in C^1$ for all (t, ∞), then the functions

$$x_1(t) = u(t, 0), \qquad x_2(t) = u(t, L)$$

satisfy (1.1) for all $t \geqq 0$.

Proof. Define functions $\alpha(x) = x^2/2L$, $K(t, x) = \alpha(x)A_2(t) - \alpha(L - x)A_1(t)$ and $v(t, x) = u(t, x) - K(t, x)$. Then

$$v_t - v_{xx} = Q(t, x) \equiv \alpha(x)A_2'(t) - \alpha(L - x)A_1'(t) + \{A_2(t) - A_1(t)\}/L,$$

$$v_x(t, 0) = v_x(t, L) = 0,$$

and

$$v(0, x) = H(x) \equiv F(x) - \alpha(x)A_2(0) + \alpha(L - x)A_1(0).$$

The functions H and Q are sufficiently smooth in order to solve uniquely for $v(t, x)$ in the usual way (cf. [10, Theorems 1 and 2]). Therefore

$$\begin{aligned} u(t, x) &= K(t, x) + v(t, x) \\ &= K(t, x) + H_0/2 + \sum_{n=1}^{\infty} H_n \exp\{-(n\pi/L)^2 t\} \cos(n\pi x/L) \\ &\quad + \int_0^t \int_0^L \Big\{L^{-1} + \sum_{n=1}^{\infty} \exp\{-(n\pi/L)^2(t-s)\} \cos(n\pi y/L) \\ &\quad \cdot \cos(n\pi x/L)\Big\} Q(s, y)\, dy\, ds. \end{aligned} \tag{2.6}$$

Here H_n is the sequence of Fourier cosine coefficients of H. By the definition of $\alpha(x)$ it follows that

$$\alpha(x) = L/6 + 2 \sum_{n=1}^{\infty} (-1)^n L^2 (n\pi)^{-2} \cos(n\pi x/L)$$

and

$$\alpha(L - x) = L/6 + 2 \sum_{n=1}^{\infty} L^2 (n\pi)^{-2} \cos(n\pi x/L)$$

when $0 < x < L$. Therefore, the definitions of K, Q and H together with integration by parts suffice to put the above expression for $u(t, x)$ into the form (1.5). Since $u(t, x)$ is continuous, then setting $x = 0$ and $x = L$ in (1.5) yields (1.7). Q.E.D.

THEOREM 3. *Suppose* (2.1)–(2.5) *are true,* $F \in C^2[0, 1]$ *and the functions* $g_1(t, u)$ *and* $g_2(t, u)$ *are of class* C^1 *for all* (t, u). *If the solution* $x(t)$ *of Eq.* (E) *exists for all* $t \geqq 0$, *then* $u(t, 0) = x_1(t)$ *and* $u(t, L) = x_2(t)$ *are the boundary values of a function* $u(t, x)$ *which satisfies conditions* (i)–(iv) *of Theorem* 2.

Proof. The conditions $F \in C^2[0, L]$ and (2.1) are sufficient to insure that $f \in C[0, \infty) \cap C^1(0, \infty)$ and that f' is locally of class L^1 on $0 \leqq t < \infty$. Since g_1 and $g_2 \in C^1$, then it follows from results in [11] that $x_1(t)$ and $x_2(t)$ have these same smoothness properties, that is $x(t) \in C[0, \infty) \cap C^1(0, \infty)$ and $x'(t) \in L^1$ near $t = 0$.

Define $A_1(t) = g_1(t, x_1(t))$, $A_2(t) = -g_2(t, x_2(t))$ and define $u(t, x)$ by line (1.5). Condition (i) of Theorem 2 can easily be verified directly using (1.5). Since $A_1(t)$ and $A_2(t)$ are smooth, the steps in the proof of Theorem 2 can be reversed to obtain (2.6). Therefore, the results in [10] imply (ii), (1.2), (1.3) and the boundary conditions

$$\lim_{x \to 0} u_x(t, x) = A_1(t), \qquad \lim_{x \to L} u_x(t, x) = A_2(t).$$

Setting $x = 0$ in (1.5) and using the present definitions of A_1 and A_2 it follows that

$$u(t, 0) = f_1(t) - \int_0^t a_1(t - s) g_1(s, x_1(s))\, ds - \int_0^t a_2(t - s) g_2(s, x_2(s))\, ds.$$

(There is a similar formula for $u(t, L)$.) Comparing this with (1.1a) one sees that $u(t, 0) = x_1(t)$ for all $t \geqq 0$. Similarly $u(t, L) = x_2(t)$. Q.E.D.

3. Preliminary transformations. Given any matrix $A(t)$ the *resolvent* $R(t)$ of $A(t)$ is defined to be the solution of the linear equation

$$R(t) = A(t) - \int_0^t A(t-s)R(s)\,ds. \tag{3.1}$$

If the entries of $A(t)$ are locally of class L^1 on $0 \leqq t < \infty$ then it is known [12, Chapter IV] that $R(t)$ exists a.e., is locally L^1 on $0 \leqq t < \infty$, and $R(t)$ also satisfies the equation

$$R(t) = A(t) - \int_0^t R(t-s)A(s)\,ds \tag{3.1$'$}$$

a.e. on $0 < t < \infty$.

Let Q denote the symmetric, unitary matrix

$$Q = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.2}$$

Then clearly Q diagonalizes any matrix of the form (2.4), that is $QA(t)Q$ is diagonal.

Lemma 1. *Suppose $A(t)$ is any matrix of the form* (2.4) *where $a_1(t)$ and $a_2(t)$ are locally L^1 on $0 \leqq t < \infty$. For any $N > 0$ define*

$$A_N(t) = NQA(t)Q$$

and let $R_N(t)$ be the resolvent of $A_N(t)$. Then the following statements are true:

(i) $A_N(t) = N$ *diagonal* $(a_1(t) + a_2(t), a_1(t) - a_2(t))$.

(ii) $R_N(t) =$ *diagonal* $(\lambda_{1N}(t), \lambda_{2N}(t))$.

(iii) *If $a_1(t)$ and $a_2(t)$ are defined by* (2.2) *then $\lambda_{1N}(t)$ and $\lambda_{2N}(t)$ are positive and continuous on $0 < t < \infty$ and*

$$\int_0^\infty \lambda_{1N}(t)\,dt = 1, \qquad \int_0^\infty \lambda_{2N}(t)\,dt < 1.$$

Proof. The first two parts follow immediately from (3.2) and Eq. (3.1) for the resolvent. Indeed, $\lambda_{1N}(t)$ is the resolvent of the scalar function $W_1(t) = N\{a_1(t) + a_2(t)\}$ and $\lambda_{2N}(t)$ is the resolvent of the function $W_2(t) = N\{a_1(t) - a_2(t)\}$.

If (2.2) is true, then

$$W_1(t) = N\{2 + 4 \sum_{n \text{ even}} \exp\{-(n\pi/L)^2 t\}\},$$

$$W_2(t) = N\{4 \sum_{n \text{ odd}} \exp\{-(n\pi/L)^2 t\}\}.$$

These formulas show that W_1 and W_2 are nonconstant, locally integrable, and completely monic on $0 < t < \infty$, that is $(-1)^j (W_k)^{(j)}(t) > 0$ for $0 < t < \infty$, $j = 0, 1, 2, \cdots$ and $k = 1, 2$. It follows from a theorem of Reuter [13] that $\lambda_{kN}(t)$ is completely monic on $0 < t < \infty$. The results in [5, Sec. II] immediately give the two integral estimates in (iii). Q.E.D.

Lemma 2. *Suppose* (E) *satisfies* (2.3)–(2.4), *Q is defined by* (3.2) *and both $a_1(t)$ and $a_2(t)$ are functions which are locally L^1 on $0 \leqq t < \infty$. For any fixed $N > 0$ let R_N be the resolvent of the matrix valued function $A_N(t) = NQA(t)Q$. Then the transformation*

$$y = Q\{x - f(t)\} \qquad (\text{or } x = Qy + f(t))$$

may be used to transform (E) *into the equivalent system*

$$y(t) = \int_0^t R_N(t-s)\{y(s) - G_N(s, y(s))\}\,ds \tag{E$_N$}$$

where

$$G_N(t, y) = QG(t, Qy + f(t))/N. \tag{3.3}$$

Proof. Define $\delta(t) = \text{diagonal }(\delta_d(t), \delta_d(t))$ where $\delta_d(t)$ is the Dirac delta function. Let $*$ denote the convolution operation. Then the resolvent equation

$$R_N(t) = A_N(t) - \int_0^t A_N(t-s)R_N(s)\,ds$$

may be written in the symbolic form

$$R_N = A_N - A_N * R_N\,,$$

or equivalently,

$$(\delta - R_N) * (\delta + A_N) = \delta. \tag{3.4}$$

Eq. (E) has the form $x = f - A * G(x)$. If $y = Q(x - f)$ then (E) becomes

$$y = -(QA) * G(Qy + f) = -N(QAQ) * (QG(Qy + f)/N) = -A_N * G_N(y).$$

Adding $A_N * y$ to both sides of this equation yields

$$y + A_N * y = (\delta + A_N) * y = A_N * \{y - G_N(y)\}.$$

Applying $\delta - R_N$ to both sides and using (3.4) one obtains

$$y = \delta * y = (\delta - R_N) * (\delta + A_N) * y = (\delta - R_N) * A_N * \{y - G_N(y)\}$$

or $y = R_N * \{y - G_N(y)\}$. This is Eq. (E_N). The calculation is completely reversible so that (E_N) also implies (E). Q.E.D.

4. Existence of bounded solutions. Assume the functions f, G and A of Eq. (E) satisfy the following conditions:

(A1) f, A and G satisfy (2.2)–(2.4).
(A2) $f \in C[0, \infty)$ and $f(t)$ is bounded on $[0, \infty)$.
(A3) $G(t, x) \in C(R^3)$ and $G(t, 0) \equiv 0$ for all $t \geqq 0$.
(A4) There exist positive numbers N and K such that if $|y| \leqq K$ then $|y - G_N(t, y)| \leqq K$ uniformly in $t \in R^1$.

Note that more generally one could assume the existence of a vector-valued function $r(t)$ such that $G(t, r(t)) \equiv 0$ for all $t \geqq 0$. (This is the situation in Theorem 1 above.) However, the transformation $X = x - r(t)$ puts (E) in the form

$$X(t) = \{f(t) - r(t)\} - \int_0^t A(t-s)G(s, r(s) + X(s))\,ds.$$

If $r(t)$ is continuous and if $|f(t) - r(t)|$ is bounded, then the new equation satisfies (A3).

THEOREM 4. *Suppose* (A1)–(A4), (3.2) *and* (3.3) *are all true. Then there exists a solution $x(t)$ of* (E) *such that $|x(t)| \leqq K$ for all $t \geqq 0$.*

Proof. Let $C = C([0, \infty), R^2)$ be the space of all continuous functions $\varphi : [0, \infty) \to R^2$.

Let C have the topology of uniform convergence on compact subsets of the interval $0 \leqq t < \infty$. Define

$$S = \{\varphi \in C: |\varphi(t)| \leqq K \quad \text{for all} \quad t \geqq 0\}.$$

For any $\varphi \in S$ define

$$(M\varphi)(t) = \int_0^t R_N(t-s)\{\varphi(s) - G_N(s, \varphi(s))\}\, ds.$$

Clearly, $M: S \to C$ and M is completely continuous. Since the norm $|z| = |(z_1, z_2)|$ is defined by $|z| = \max\{|z_1|, |z_2|\}$, then (A4), Lemma 1 parts (ii) and (iii) and the definitions of S and M easily imply that $|(M\varphi)(t)| \leqq K$ for all $t \geqq 0$. This means that $M\varphi \in S$ if $\varphi \in S$. By the Schauder fixed point theorem the operator M has at least one fixed point $x(t)$. This fixed point solves (E_N) on $0 \leqq t < \infty$ and thus also solves (E). Q.E.D.

It can be shown that if G is defined by (1.8) then (A4) is true. More generally assume:

(A4′) $G(t, x_1, x_2) = \text{col}\,(g(t, x_1), g(t, x_2))$ for all $(t, x_1, x_2) \in R^3$. Moreover, $g(t, z)$ is an odd, nondecreasing function of z and is bounded in $t \in R^1$ uniformly for z on compact subsets of R^1.

LEMMA 3. *Suppose G satisfies* (A3) *and* (A4′). *Let* $B = \sup\{|f(t)| : t \geqq 0\}$. *Then for any $M > \sqrt{2}B$ and for any ε in the range $0 < \varepsilon < B$ there exists $N > 0$ such that* (A4) *is true with $K = M + \varepsilon$.*

Proof. Fix any such values of M and ε. Pick $N > 0$ so large that $2\,|g(t, z)| < N\varepsilon$ uniformly in $t \geqq 0$ and $|z| \leqq 5M$. The map $w = u - G_N(t, u)$ may be written in the form

$$w_1 - u_1 = -\{g(t, (u_1 + u_2)/\sqrt{2} + f_1(t)) + g(t, (u_1 - u_2)/\sqrt{2} + f_2(t))\}/(\sqrt{2}N) \quad (4.1)$$

and

$$w_2 - u_2 = -\{g(t, (u_1 + u_2)/\sqrt{2} + f_1(t)) - g(t, (u_1 - u_2)/\sqrt{2} + f_2(t))\}/(\sqrt{2}N) \quad (4.2)$$

for all $t \geqq 0$.

If $|u| = \max\{|u_1|, |u_2|\} \leqq M$ then for any t one has

$$|(u_1 + u_2)/\sqrt{2} + F_1(t)|, \qquad |(u_1 - u_2)/\sqrt{2} + F_2(t)| \leqq \sqrt{2}M + M \leqq 5M.$$

Thus (4.1) and the choice of N imply $|w_1| < |u_1| + \{\varepsilon/\sqrt{2} + \varepsilon/\sqrt{2}\}/\sqrt{2} \leqq M + \varepsilon$. Similarly, $|w_2| < M + \varepsilon$.

Now consider the region $D = \{u \in R^2: M < |u| \leqq M + \varepsilon\}$. We must show that if $u \in D$ then $|w_1|, |w_2| < M + \varepsilon$. For $j = 1, 2, 3, \cdots, 8$ define

$$S_j = \{(u_1, u_2) \in R^2: u_1 + iu_2 = re^{i\theta} \text{ for some } r > 0$$
$$\text{and some } \theta \text{ in } (j-1)(\pi/2) \leqq \theta \leqq j\pi/2\}$$

and define $D_j = D \cap S_j$. Since $g(t, x)$ is an odd, nondecreasing function of x with $g(t, 0) = 0$, then the map $G(t, z_1, z_2) = \text{col}\,(g(t, x_1), g(t, x_2))$ maps each region S_j into itself. Also recall that $G_N(t, z) = QG(t, Qz + f(t))/N$.

If $u \in D_1$, then (3.2) implies that $v = Qu \in D_1$. Since $B < M/\sqrt{2}$ and $|f(t)| \leqq B$, then $y = v + f(t)$ lies in S_1, S_2 or S_8. Again Q maps $S_1 \to S_1$, $S_2 \to S_8$ and $S_8 \to S_2$ so that

$$Z = Qx = QG(t, Qu + f(t))/N = G_N(t, u)$$

is in S_1, S_2 or S_8. Finally, (4.1)–(4.2) show that $Z = G_N(t, u) = w - u$.

If $Z = G_N(t, u)$ is in S_1 or S_2, then the right-hand sides of (4.1) and (4.2) both lie in the range $(-\varepsilon, 0)$. Since $u \in D_1$, then $M < u_1 \leqq M + \varepsilon$ and $0 \leqq u_2 \leqq M + \varepsilon$. Therefore, $w_1 = u_1 + z_1$ lies in the range $0 < M - \varepsilon < w_1 < M + \varepsilon$ and $w_2 = u_1 + z_2$ lies in the range $-\varepsilon < w_2 < M + \varepsilon$. Therefore, $|w| \leqq M + \varepsilon$.

Now suppose $u \in D_1$ and $Z = G_N(t, u) \in S_8$. Since $|f(t)| \leqq B < M/\sqrt{2}$ one must have $M < u_1 \leqq M + \varepsilon$ and $0 \leqq u_2 \leqq \sqrt{2}B < M$ in order that $Z \in S_8$. Therefore, the right-hand side of (4.1) is in the range $(-\varepsilon, 0)$ and the right-hand side of (4.2) in the range $(0, \varepsilon)$. This and $u_1 \in D_1$ mean that $M - \varepsilon < w_1 < M + \varepsilon$ and $\varepsilon < w_2 \leqq \sqrt{2}B + \varepsilon < M + \varepsilon$.

The analysis of the other seven regions $S_2, S_3, \cdots, S_8$ follows in a similar manner. The various maps involved in the analysis are illustrated in Fig. 1. Q.E.D.

Corollary 1. *Suppose* (A1–3), (A4′), (3.2) *and* (3.3) *are true. If G is sufficiently smooth to insure the uniqueness of the solution $x(t)$ of* (E) *then $x(t)$ exists for all $t \geqq 0$*

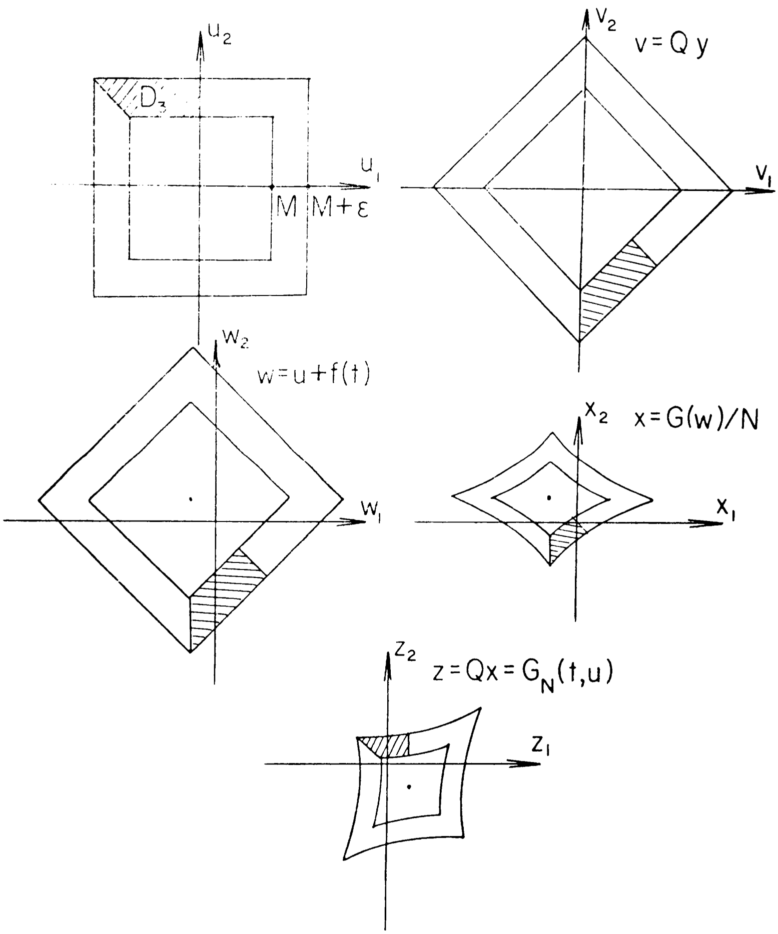

Fig. 1

and satisfies

$$|Q\{x(t) - f(t)\}| \leqq \sqrt{2}B \qquad (0 \leqq t < \infty)$$

where $B = \sup \{|f(t)| : 0 \leqq t < \infty\}$.

Proof. By Lemma 3 and Theorem 4 above the solution $x(t)$ satisfies

$$\sup \{|Q\{x(t) - f(t)\}| : t \geqq 0\} \leqq M + \varepsilon$$

for each $\varepsilon > 0$ and each $M > \sqrt{2}B$. Q.E.D.

5. Almost periodic solutions. The purpose of this section is to study the existence and uniqueness of almost periodic solutions of equations of the form (1.10). First, we give appropriate definitions and background information concerning almost periodic functions. The first result in this section (Theorem 5) asserts that if $Y(t)$ is an almost periodic solution of (1.10) for some fixed $N_0 > 0$ then it is also a solution of (1.10) for all other $N > 0$. This result will be important since one value of N will be needed to prove existence of almost periodic solutions of (1.10) and a second value of N will be needed to obtain uniqueness and prove the asymptotic relationships between solutions of (E_N) and (1.10).

The rest of the section is devoted to the existence and uniqueness of almost periodic solutions of (1.10). Lemma 4 is an invariance theorem for bounded solutions of (1.10). Lemma 5 asserts the uniqueness of bounded solutions of (1.10). The last result of the section asserts that the unique bounded solution of (1.10) is almost periodic.

DEFINITION. *A continuous function $S(t, x)$ defined for all $(t, x) \in R^{n+1}$ is called almost periodic in t (uniformly for x on compact sets) if and only if given any sequence $\{t_n\}$ of real numbers there exists a subsequence $\{t_{nk}\}$ and a function $S^*(t, x)$ such that*

$$\lim_{k \to \infty} S(t + t_{nk}, x) = S^*(t, x)$$

with convergence uniform in (t, x) for all $t \in R^1$ and x on compact subsets of R^n. In this case we write $S \in AP$.

The set of all functions S^* which may be obtained in this way is called the closed hull of S, written $CH(S)$.

As general references on almost periodic functions see the books of Favard [14] and Besicovitch [15] or the original papers of Bohr [16]. The results listed below are well-known results in this field.

Given a function $S(t, x)$ which is almost periodic in t uniformly for x on compact sets define $FM(S)$ to be the set of all almost periodic functions $f(t)$ with range in the same space as S and satisfying the following condition:

> If $\{t_n\}$ is any real sequence such that $\{S(t + t_n, x)\}$ is a Cauchy sequence uniformly in $t \in R^1$ and x on compact subsets of R^n, then $\{f(t + t_n)\}$ is a Cauchy sequence uniformly in $t \in R^1$.

The set $FM(S)$ is called the *function module of* S.

Given $S \in AP$ there exists a countable set of Fourier exponents $\{\lambda_n\} \subset R^1$ and a set $\{S_n(x)\}$ of continuous nontrival functions such that S has Fourier series

$$S(t, x) \sim \sum_{n=1}^{\infty} S_n(x) \exp (i\lambda_n t).$$

If S is independent of x, then so are the functions S_n. The *module of* S, written $M(S)$, is the additive group of real numbers generated by the sequence $\{\lambda_n\}$ of Fourier exponents. In other words $M(S)$ is the smallest additive subgroup of R^1 containing the set $\{\lambda_n\}$. An almost periodic function f is in the function module $FM(S)$ if and only if the Fourier exponents of f are contained in the module $M(S)$.

Let the functions f, A and G satisfy (A1)–(A4) and in addition some or all of the following conditions:

(A5) There exist almost periodic functions $p(t)$ and $h_i(t, x)$ such that

$$\lim_{t\to\infty} \{f(t) - p(t)\} = 0, \qquad \lim_{t\to\infty} \{g_i(t, x) - h_i(t, x)\} = 0$$

with the last limit uniform in x on compact sets of R^1.

(A6) For each $t \in R^1$ and for $j = 1, 2$ the function $h_j(t, x)$ is nondecreasing in x.

(A7) The functions $h_1(t, x)$ and $h_2(t, x)$ are locally Lipschitz continuous in x with Lipschitz constants independent of $t \in R^1$.

In Theorem 1 above $g_1(t, x) = g_2(t, x) = h_1(t, x) = h_2(t, x) = Bx^3$. Moreover, (2.1) implies that

$$f_i(t) + C \sin (k_i t) \to F_0/2 + C \sin (k_i t)$$

as $t \to \infty$. Thus (A5)–(A7) are all true for this special case. Note that (A5) implies that $G(t, x_1, x_2)$ has the special form (2.5).

Under the above assumptions the invariance theorem in [17, Theorem 1] implies that the equation

$$y(t) = \int_0^t R_N(t - s)\{y(s) - G_N(s, y(s))\}\, ds \tag{E_N}$$

has the limiting form

$$Y(t) = \int_{-\infty}^t R_N(t - s)\{Y(s) - H_N(s, Y(s))\}\, ds \tag{5.1}$$

where $H(t, y) = H(t, y_1, y_2) = \text{col}\,(h_1(t, y_1), h_2(t, y_2))$ and

$$H_N(t, y) = QH(t, Qy + p(t))/N. \tag{5.2}$$

THEOREM 5. *Suppose* (A1)–(A5), (3.2) *and* (5.2) *are true. Suppose* $Y(t)$ *is any almost periodic solution of* (5.1) *for some fixed* N. *If* $Y \in FM(H, p)$ *then* Y *is also a solution of* (5.1) *for all other values of* $N > 0$.

Proof. Pick any $M > 0$ with $M \neq N$. Let S_N be the resolvent of $NA(t)$ and S_M the resolvent of $MA(t)$. Write (5.1) in the form

$$Y(t) = h(t) + \int_0^t R_N(t - s)\{Y(s) - H_N(s, Y(s))\}\, ds \tag{5.3}$$

where

$$\begin{aligned} h(t) &= \int_{-\infty}^0 R_N(t - s)\{Y(s) - H_N(s, Y(s))\}\, ds \\ &= \int_t^\infty R_N(s)\{Y(t - s) - H_N(t - s, Y(t - s))\}\, ds \to 0 \end{aligned}$$

as $t \to \infty$. Let $\delta_d(t)$ be the Dirac delta function and let $\delta(t) = \text{diagonal}\,(\delta_d(t), \delta_d(t))$.

If $*$ denotes convolution then (5.3) may be written in the form

$$Y = h + R_N * \{Y - QH(QY + p)/N\}.$$

Since $QR_NQ = S_N =$ resolvent of $NA(t)$, then the transformation $Z = QY$ puts the equation in the form $Z = Qh + S_N * \{Z - H(Z + p)/N\}$, or $(\delta - S_N) * Z = Qh - S_N * H(Z + p)/N$. Applying $(\delta + NA)$ to both sides, one obtains

$$Z = (\delta + NA) * Qh - A * H(Z + p) = (\delta + NA) * Qh - (MA) * H(Z + p)/M.$$

Add $(MA) * Z$ to both sides and apply $(\delta - S_M)$:

$$Z = (\delta - S_M) * (\delta + NA) * Qh + S_M * \{Z - H(Z + p)/M\}.$$

Letting $Y = QZ$, one obtains

$$Y = Q(\delta - S_M) * (\delta + NA) * Qh + R_M * \{Y - QH(Qy + p)/M\}.$$

Note that

$$Q(\delta - S_M) * (\delta + NA)Q = Q\{\delta - S_M + NA - (N/M)(MA - S_M)\}Q$$
$$= \delta + (1 - N/M)QS_MQ = \delta + (1 - N/M)R_M .$$

Therefore $Y = h + (1 - N/M)R_M * h + R_M * \{Y - H_M(Y)\}$. Writing this equation in the usual form, one has

$$Y(t) = h(t) + \int_0^t (1 - N/M)R_M(t - s)h(s)\, ds + \int_0^t R_M(t - s)\{Y(s) - H_M(s, Y(s))\}\, ds \tag{5.4}$$

for $t \geqq 0$.

Let $t_n \to \infty$ be an increasing sequence such that $p(t + t_n) \to p(t)$ and $H(t + t_n, y) \to H(t, y)$ as $n \to \infty$. Since $Y \in FM(H, p)$, then $Y(t + t_n) \to Y(t)$ as $n \to \infty$. Note that $h(t) \to 0$ as $t \to \infty$ and $R_N \in L^1(0, \infty)$ implies that $h(t) + (1 - N/M)R_M * h(t) \to 0$ as $t \to \infty$. Replacing t by $t + t_n$ in (5.4) yields

$$Y(t + t_n) = h(t + t_n) + (1 - N/M)R_M * h(t + t_n)$$
$$+ \int_{-t_n}^t R_M(t - s)\{Y(s + t_n) - H_M(s + t_m , Y(s + t_n))\}\, ds.$$

Taking the limit as $n \to \infty$ gives (5.1) with N replaced by M. Q.E.D.

We now turn to the existence-uniqueness problem. The following lemma will be needed.

LEMMA 4. *Suppose* (5.2), (A1)–(A3) (*and* A5) *are true. Suppose* (5.1) *has a bounded solution* $y(t)$ *on* $-\infty < t < \infty$. *Then given any sequence* $\{t_n\}$ *of real numbers there exists a subsequence* $\{t_{nk}\}$, *a function* $(H_N)^* \in CH(H_N)$ *and a function* $y^*(t)$ *such that*

$$y(t + t_{nk}) \to y^*(t), \qquad H_N(t + t_{nk} , y) \to H_N^*(t, y)$$

and

$$y^*(t) = \int_{-\infty}^t R_N(t - s)\{y^*(s) - H_N^*(s, y^*(s))\}\, ds \qquad (-\infty < t < \infty). \tag{5.5}$$

Proof. If $\{t_n\}$ contains a subsequence which tends to a finite limit point τ, then the result is trivial with $y^*(t) = y(t + \tau)$ and $H_N^*(t, y) = H_N(t + \tau, y)$. Therefore, assume $t_n \to \pm\infty$. Since $p(t)$ and $H(t, y)$ are almost periodic in t, then there is a subsequence (which we shall also index by n) and functions $p^* \varepsilon CH(p)$, $H^* \varepsilon CH(H)$ such that $p(t + t_n) \to p^*(t)$ and $H(t + t_n, y) \to H^*(t, y)$. Then

$$H_N(t + t_n, y) = QH(t + t_n, Qy + p(t + t_n))/N \to QH^*(t, Qy + p^*(t)/N = H_N^*(t, y).$$

Since $y(t)$ is bounded and H is almost periodic, then $|y(t) - H_N(t, y(t))|$ is bounded on $-\infty < t < \infty$. The convolution of a function in $L^1(-\infty, \infty)$ and a function of class $L^\infty(-\infty, \infty)$ results in a bounded uniformly continuous function. Since $y(t)$ solves (5.1), then $y(t)$ must be uniformly continuous. This in turn means that the sequence $\{y(t + t_n)\}$ is a uniformly bounded, equicontinuous family of functions on each finite subinterval of R^1. By possibly taking a subsequence we may assume that $y(t + t_n) \to y^*(t)$ as $n \to \infty$ for some function y^*. Replacing t by $t + t_n$ in (5.1) one obtains

$$y(t + t_n) = \int_{-\infty}^{t} R_N(t - s)\{y(s + t_n) - H_N(s + t_n, y(s + t_n))\}\, ds$$

taking the limit as $n \to \infty$ gives (5.5). Q.E.D.

LEMMA 5. *Assume the hypotheses of Lemma 4. Assume* (A6)–(A7) *are also true. If $N > 0$ is sufficiently large, depending only on a bound for $y(t)$ and the Lipschitz constant of H, then $y(t)$ is the unique bounded solution of* (5.1).

Proof. Suppose there exist two distinct solutions $y(t)$ and $z(t)$ of (5.1). Pick a sequence t_n such that

$$|y(t_n) - z(t_n)| \to L = \sup\{|y(t) - z(t)| : -\infty < t < \infty\}.$$

By possibly taking a subsequence we may assume that $y(t_n) - z(t_n) \to u_0$ as $n \to \infty$ where u_0 is some point on the boundary of the square $\{u: |u| \leqq L\}$. By possibly taking another subsequence Lemma 8 insures that $H_N(t + t_n, y) \to H_N^*(t, y)$, $y(t + t_n) \to y^*(t)$ and $z(t + t_n) \to z^*(t)$ where $H_N^* \varepsilon CH(H_N)$ and y^* and z^* solve (5.5). Clearly H_N^* satisfies the same hypotheses as H_N. Moreover, $u_0 = y^*(0) - z^*(0) = \lim\{y(t_n) - z(t_n)\}$ as $n \to \infty$. Thus we have reduced the problem to the case where $|y(0) - z(0)| = L = \sup\{|y(t) - z(t)| : -\infty < t < \infty\}$.

The two components of $b(t, y) = y - H_N(t, y)$ have the form

$$\begin{aligned} b_1(t, y) = y_1 - \{&h_1(t, (y_1 + y_2)/\sqrt{2} + p_1(t)) \\ &+ h_2(t, (y_1 - y_2)/\sqrt{2} + p_2(t))\}/(\sqrt{2}N) \end{aligned} \tag{5.7a}$$

and

$$\begin{aligned} b_1(t, y) = y_2 - \{&h_1(t, (y_1 + y_2)/\sqrt{2} + p_1(t)) \\ &- h_2(t, (y_1 - y_2)/\sqrt{2} + p_2(t))\}/(\sqrt{2}N). \end{aligned} \tag{5.7b}$$

Set $u(t) = y(t) - z(t)$ and define

$$\begin{aligned} m_1(t) = \{h_1(t, (y_1(t) + y_2(t))/\sqrt{2} + p_1(t)) - h_1(t, (z_1(t) + z_2(t))/\sqrt{2} \\ + p_1(t))\eta\}\{(u_1(t) + u_2(t))/\sqrt{2}\}^{-1} \end{aligned} \tag{5.8a}$$

if $u_1(t) + u_2(t) \neq 0$ and $m_1(t) = 0$ if $u_1(t) + u_2(t) = 0$. Since $u(t)$ is bounded and (A7) is true, then $m_1(t) \varepsilon L^\infty(-\infty, \infty)$. Moreover, $m_1(t) \geqq 0$ by (A6). Similarly define

$$m_2(t) = \{h_2(t, (y_1(t) - y_2(t))/\sqrt{2} + p_2(t)) - h_2(t, (z_1(t) - z_2(t))/\sqrt{2} + p_2(t))\eta\}\{(u_1(t) - u_2(t))/\sqrt{2}\}^{-1} \tag{5.8b}$$

if $u_1(t) \neq u_2(t)$ and $m_2(t) = 0$ otherwise. Since $u(t) = y(t) - z(t)$, then (5.1), (5.7) and (5.8) imply that

$$u_1(t) = \int_{-\infty}^{t} \lambda_{1N}(t - s)(\{1 - (m_1(s) + m_2(s))/N\}u_1(s) + \{(m_2(s) - m_1(s))/N\}u_2(s))\, ds \tag{5.9a}$$

and

$$u_2(t) = \int_{-\infty}^{t} \lambda_{2N}(t - s)(\{(m_2(s) - m_1(s))/N\}u_1(s) + \{1 - (m_1(s) + m_2(s))/N\}u_2(s))\, ds \tag{5.9b}$$

for all t in R^1. In system form (5.9) becomes

$$u(t) = \int_{-\infty}^{t} R_N(t - s)(I - M(s)/N)u(s)\, ds \tag{5.9$'$}$$

where $M(s)$ is the appropriate matrix.

Pick $N > 0$ so large that $0 \leqq m_1(t), m_2(t) \leqq N/3$ a.e. on $-\infty < t < \infty$. For any fixed s the map $\hat{u} = (I - M(s)/N)u$ maps the square $S = \{u: |u_1|, |u_2| \leqq L\}$ into the region

$$S' = \{u: |u_1|, |u_2| \leqq L \max\{1 - 2m_1(s)/N, 1 - 2m_2(s)\}\}.$$

Since $0 \leqq 2m_i(s)/N \leqq 2/3$ (by the choice of N), then $S' \subset S$. Using the properties of λ_{2N} obtained in Lemma 2, (5.6) and (5.9b) it follows that

$$|u_2(t)| \leqq \int_{-\infty}^{t} \lambda_{2N}(t - s)L\, ds = L \int_0^\infty \lambda_{2N}(s)\, ds = L_0 < L.$$

Letting $t = 0$ we see that $|u_2(0)| < L$. Therefore, $u(t)$ is in the set

$$S_0 = \{u: |u_1| \leqq L, |u_2| \leqq L_0\}.$$

Since $\hat{u} = (I - M(s)/N)\, u$ maps S_0 strictly inside of the square S, say $|\hat{u}| \leqq \beta L$ $(0 < \beta < 1)$, then for any t line (5.9a) implies that

$$|u_1(t)| \leqq \int_{-\infty}^{t} \lambda_{1N}(t - s)\beta L\, d_s = \left(\int_0^\infty \lambda_{1N}(s)\, ds\right)(\beta L) = \beta L < L.$$

Therefore, $|u_1(0)| < L$ and $|u_2(0)| < L$ which contradicts $|u(0)| = \max\{|u_1(0)|, |u_2(0)|\} = L$.
Q.E.D.

THEOREM 6. *Suppose the hypotheses of Lemma 4 are true. Then $y(t) \varepsilon FM(H, p)$ so that in particular $y \varepsilon AP$. Moreover, $y(t)$ solves each equation (5.1) for all $N > 0$ and for N sufficiently large is the unique bounded solution of (5.1).*

Proof. Fix $N > 0$ and large. Let $\{t_n\}$ be any real sequence such that $\{p(t + t_n)\}$ and $\{H(t + t_n, y)\}$ are Cauchy sequences, $p(t + t_n) \to p^*(t) \varepsilon CH(p)$ and $H(t + t_n, y) \to$

$H^*(t, y) \in CH(H)$. It must be shown that $\{y(t + t_n)\}$ is a Cauchy sequence uniformly in $t \in R^1$. Suppose this is not true. Then there exists $\epsilon > 0$, subsequences n_k and m_k, and a sequence τ_k such that $n_k > m_k \geqq k$ and $|y(\tau_k + t_{nk}) - y(\tau_k + t_{nk})| \geqq \epsilon$. Define

$$T_k = \tau_k + t_{nk}, \; S_k = \tau_k + t_{mk} \qquad (k = 1, 2, 3, \cdots).$$

By possibly taking a subsequence of the k's it follows that

$$H^*(t + \tau_k, y) \to H_0(t, y), \qquad p^*(t + \tau_k) \to p_0(t)$$

for some functions $H_0 \in CH(H^*)$ and $p_0 \in CH(p^*)$. Then $p(t + T_k) = p(t + \tau_k + t_{nk}) \to p_0(t)$ and

$$H(t + T_k, y) = H(t + \tau_k + t_{nk}) \to H_0(t, y)$$

as $k \to \infty$. Similarly $p(t + S_k) \to p_0(t)$ and $H(t + S_k, y) \to H_0(t, y)$.

By Lemma 4 there exists a subsequence (which will again be indexed by k) and functions $y_1(t)$ and $y_2(t)$ such that $y(t + S_k) \to y_1(t)$, $y(t + T_k) \to y_2(t)$ and

$$y_j(t) = \int_{-\infty}^{t} R_N(t - s)\{y_j(s) - (H_0)_N(s, y_j(s))\}\, ds$$

for $j = 1, 2$ and $t \in R^1$. Since

$$|y_1(0) - y_2(0)| = \lim_{k\to\infty} |y(S_k) - y(T_k)| \geqq \varepsilon,$$

then $y_1(t) \not\equiv y_2(t)$. But this violates the uniqueness asserted in Lemma 5. This contradiction shows that $y \in FM(H, p)$. Theorem 5 shows that y solves (5.1) for all $N > 0$. The uniqueness of y is Lemma 5. Q.E.D.

6. Proof of Theorem 1 and generalizations.

LEMMA 6. *Suppose* (A1)–(A7) *and* (3.2) *are true. Then for any $N > 0$ Eq.* (5.1) *has at least one bounded solution. In particular, then* (5.1) *has a solution $Y \in FM(H, p)$.*

Proof. By Theorem 4 Eq. (E) has a bounded solution $x(t)$. Since (E) is equivalent to (E_N) for all $N > 0$, then each (E_N) has the same bounded solution. The results in [17] imply the existence of at least one bounded solution of (5.1) for any $N > 0$. Now apply Theorem 6. Q.E.D.

THEOREM 7. *Suppose* (A1)–(A7) *and* (3.2) *are true. Then there exists a unique function $X \in FM(H, p)$ such that if $x(t)$ is any bounded solution of* (E) *then $x(t) - X(t) \to 0$ as $t \to \infty$.*

Proof. Let $Y \in FM(H, p)$ be the function given by Lemma 6. Define $X(t) = QY(t) + p(t)$ and $y(t) = Q(x(t) - f(t))$. We must show that $x(t) - X(t) \to 0$ as $t \to \infty$ or equivalently that $y(t) - Y(t) \to 0$. If this is not true, then there exists an $\epsilon > 0$ and a sequence $t_n \to \infty$ such that $|y(t_n) - Y(t_n)| \geqq \epsilon$. By Lemma 3 we may assume that $p(t + t_n) \to p^*(t)$, $H(t + t_n, y) \to H^*(t, y)$ and $Y(t + t_n) \to Y^*(t)$ as $n \to \infty$ where $p^* \in CH(p)$, $H^* \in CH(H)$ and Y^* solves (5.5). Write (E_N) in the form

$$y(t) = E(t) + \int_0^t R_N(t - s)\{y(s) - H_N(s, y(s)\}\, ds,$$

where

$$E(t) = \int_0^t R_N(t - s)\{H_N(s, y(s)) - G_N(s, y(s))\}\, ds.$$

Assumption (A5), (3.3) and (5.2) insure that $H_N(t, y(t)) - G_N(t, y(t)) \to 0$ as $t \to \infty$. Since $R_N \in L^1(0, \infty)$, then $E(t) \to 0$ as $t \to \infty$. Therefore, Theorem 1 of [17] implies that by possibly taking a subsequence of t_n one has $y(t + t_n) \to y^*(t)$ where $y^*(t)$ solves (5.5) for the same value of N. By uniqueness of solutions of (5.5) for large N, $y^*(t) \equiv Y^*(t)$. On the other hand

$$|y^*(0) - Y^*(0)| = \lim_n |y(t_n) - Y(t_n)| \geqq \epsilon > 0.$$

This contradiction proves the theorem. Q.E.D.

THEOREM 8. *Under the hypotheses of Theorem 7 the function $H(t, X(t))$ has mean value zero. Moreover, if the mean values of $p_1(t)$ and $p_2(t)$ are equal then the two components of $X(t)$ have equal mean values.*

Proof. Recall that for any $\varphi(t) \in AP$ the mean value of φ is defined to be

$$m(\varphi) = \lim_{T \to \infty} T^{-1} \int_0^T \varphi(s)\, ds.$$

Since $Y(t) = Q(X(t) - p(t))$, then Y solves (5.1) on $-\infty < t < \infty$. Taking mean values of both sides of (5.1) one obtains

$$m(Y) = R_N^*(0)(m(Y) - m(H_N)), \tag{6.1}$$

where $R_N^*(\omega)$ is the Fourier transform of R_N. By Lemma 1 above

$$R_N^*(0) = \text{diagonal } (1, N\beta(1 + N\beta)^{-1})$$

where

$$\beta = \int_0^\infty \{a_1(t) - a_2(t)\}\, dt.$$

Write $Y(t) = \text{col}\,(Y_1(t), Y_2(t))$. Then the equation in the first component of (6.1) is $m(Y_1) = m(Y_1) - m(H_{N1})$. Therefore, $m(H_{N1}) = 0$, that is the first component of $QH(t, QY(t) + p(t))/N = QH(t, X(t))/N$ has mean value zero. The second component of (6.1) is

$$m(Y_2) = N\beta(1 + N\beta)^{-1} m(Y_2) - m(H_{N2}),$$

or

$$m(H_{N2}) = -N(1 + N\beta)^{-1} m(Y_2) \qquad (0 < N < \infty). \tag{6.2}$$

Since the left-hand side of (6.2) is independent of N, then (6.2) can be true for all $N > 0$ only if $m(Y_2) = m(H_{N2}) = 0$. We have shown that the mean value of

$$H_N(t, Y(t)) = QH(t, QY(t) + p(t))/N = QH(t, X(t))/N$$

is the zero vector. Since Q is not singular and $N \neq 0$, then $H(t, X(t))$ also have mean value zero. We have also shown that $m(Y_2) = m(X_1 - X_2 - p_1 + p_2)/\sqrt{2} = 0$, that is $m(X_1 - X_2) = m(p_1 - p_2)$. Since $m(p_1 - p_2) = 0$, then $m(X_1) = m(X_2)$. Q.E.D.

Theorem 1 follows as a special case of the results in this section. In this special case $g(x) = Bx^3$ is smooth so that the solution of (E) is unique. Assumptions (A1)–(A7) are easily verified with $p(t) = \text{col}\,(p_1(t), p_2(t))$ having components $p_i(t) = F_0/2 + c \sin (k_i t)$.

If k_1 and k_2 are linearly independent over the integers, then $p(t)$ is quasiperiodic with fundamental frequencies $(k_1\ ,\ k_2)$. If k_1 and k_2 are linearly dependent then there exist integers M_1 and M_2 such that $k_3 = M_1k_1 + M_2k_2$ and $k_3/2\pi$ is the least common period of $p(t)$. In this case the functions X_i in (1.9) will have the form

$$X_i(t) = \sum_{n=-\infty}^{\infty} X_{in} \exp \{\text{in } (M_1k_1 + M_2k_2)t\}.$$

If $u(t, x)$ is the function defined by (1.5) and X_i the functions defined by (1.9) then

$$u(t, 0) = X_1(t) + E_1(t) \tag{6.3a}$$

and

$$u(t, L) = X_2(t) + E_2(t), \tag{6.3b}$$

where X_1 and X_2 both have the same mean value, $E_i(t) \varepsilon C[0, \infty \cap C^1(0, \infty)$, $d/dt\ E_i(t)$ is L^1 near $t = 0$ and $E_i(t) \to 0$ as $t \to \infty$. Therefore, one would conjecture that $\lim_{t\to\infty} u(t, x) = U(t, x)$ where U solves the problem

$$\begin{aligned} &U_t = U_{xx} \qquad (-\infty < t < \infty, 0 < x < L) \\ &U(t, 0) = X_1(t), \qquad U(t, L) = X_2(t) \qquad (-\infty < t < \infty). \end{aligned} \tag{6.4}$$

Similarly $u(t, x)$ satisfies boundary conditions of the form

$$\partial u/\partial x(t, 0) = B\{X_1(t) - C \sin (k_1t)\}^3 + E_1(t) \tag{6.5a}$$

and

$$\partial u/\partial x(t, L) = -B\{X_2(t) - C \sin (k_2t)\}^3 + E_2(t), \tag{6.5b}$$

for all $t \geqq 0$. Here E_1 and E_2 have the same properties as the corresponding terms in (6.3) and the two functions $B\{X_i(t) - C \sin (k_i(t)\}^3$ have mean value zero. If it is true that $u(t, x)$ tends to a solution U of (6.4) then $U(t, x)$ should also satisfy the boundary conditions

$$\partial U/\partial x(t, 0) = B\{X_1(t) - C \sin (k, t)\}^3$$

and

$$\partial U/\partial x(t, L) = -B\{X_2(t) - C \sin (k_2(t)\}^3,$$

for $-\infty < t < \infty$.

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