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NECESSARY CONDITIONS FOR APPLICABILITY OF POINCARÉ–LIGHTHILL PERTURBATION THEORY*

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Summary. We establish necessary conditions for the applicability of Poincaré–Lighthill (or coordinate stretching) perturbation theory to ordinary differential equations. The criteria are simple consequences of a unique modification of the classical theory of coordinate stretching. The usefulness of the new approach and the role of the criteria of applicability are illustrated by means of simple examples.

1. Introduction. Consider the system of ordinary differential equations

$$dy/dx = F(y, x; \epsilon), \tag{1}$$

for $x \geq 0$ and initial conditions

$$y(1) = Y, \tag{2}$$

where y and F are real vectors, x is a real scalar and ϵ is a parameter. If ϵ is small and F is analytic in ϵ we expand Eq. (1) to obtain

$$dy/dx = f(y, x) + \epsilon g(y, x) + \cdots . \tag{3}$$

Throughout this work we neglect terms $0(\epsilon^2)$ and higher.

Following Poincaré's work [1], Lighthill [2] has introduced an expansion of the independent variable x in addition to an expansion of the dependent variables; thus

$$y = y_0(z) + \epsilon y_1(z) + \cdots , \tag{4}$$

$$x = z + \epsilon x_1(z) + \cdots , \tag{5}$$

where z is a new independent variable. The essence of the Poincaré–Lighthill (or PL) technique is that the function $x_1(z)$ is arbitrary and can be chosen to facilitate the solution of the problem at hand.

By use of Taylor series expansions about the point $(y_0(z), z)$, the Poincaré–Lighthill expansions (4) and (5) give

$$\epsilon^0: (dy_0/dz) = f_0 , \tag{6}$$

$$\epsilon^1: \frac{dy_1}{dz} = y_1^i\left(\frac{\partial f}{\partial y^i}\right)_0 + x_1\left(\frac{\partial f}{\partial z}\right)_0 + f_0 \frac{dx_1}{dz} + g_0 , \tag{7}$$

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[1] See note added in proof at end of paper.

with summation occurring over repeated superscripts. Implicit in the use of Taylor series is the assumption that f_0, g_0, $\cdots$ are nonsingular, where functions with zero subscripts are evaluated at the point $(y_0(z), z)$. Initial conditions can be chosen to be

$$y_0(1) = Y, \qquad y_1(1) = 0, \tag{8}$$

$$z = 1, \qquad x_1(1) = 0. \tag{9}$$

The classical perturbation equations are recovered from Eqs. (4)–(9) by setting

$$x_1 \equiv 0. \tag{10}$$

Then

$$\epsilon^0\colon (dy_0/dx) = f_0\,, \tag{11}$$

$$\epsilon^1\colon dy_1/dx = y_1^i(\partial f/\partial y^i)_0 + g_0\,, \tag{12}$$

and

$$y_0(1) = Y, \tag{13a}$$

$$y_1(1) = 0, \tag{13b}$$

where functions with zero subscripts are now evaluated at the point $(y_0(x), x)$.

Recently we have shown [3] that the PL perturbation equations (6) and (7) may be greatly simplified by means of the identity

$$x_1\left(\frac{\partial f}{\partial z}\right)_0 + f_0\,\frac{dx_1}{dz} = \frac{d}{dz}\,(f_0 x_1) - x_1\left(\frac{\partial f}{\partial y^i}\right)_0 \frac{dy_0^i}{dz}. \tag{14}$$

On substituting Eq. (14) in Eq. (7) and on using Eq. (6) we find

$$d\tilde{y}_1/dz = \tilde{y}_1^i(\partial f/\partial y^i)_0 + g_0\,, \tag{15}$$

where we let

$$\tilde{y}_1 = y_1 - x_1 f_0\,. \tag{16}$$

Eq. (15) now has the same form as the classical first-order perturbation equation (12), yet contains the PL feature through the newly defined dependent variables in Eq. (16). Further, on considering the initial conditions it follows from Eqs. (8), (9) and (16) that

$$\tilde{y}_1(1) = 0, \tag{17}$$

since by assumption f_0 is finite at the boundary.

A comparison of the tilde system of Eqs. (15) and (17) with the classical non-PL system of Eqs. (12) and (13b) reveals the interesting fact that they are identical but for the interpretation of the dependent and independent variables. It follows that we may define $\tilde{y}_1(z)$ to be the functional equivalent of the first-order non-PL function $y_1(x)$ when z is replaced by x. Similarly from a comparison of the pairs of equations (6), (8) and (11), (13a), $y_0(z)$ is functionally equivalent to $y_0(x)$. We express these equivalences as follows:

$$y_0(z) \equiv y_0(x), \qquad \tilde{y}_1(z) \equiv y_1(x), \qquad z \equiv x, \tag{18}$$

and from Eq. (16) we have

$$y_1(z) = \tilde{y}_1(z) + x_1(z) f_0\{y_0(z), z\}. \tag{19}$$

The new PL method of coordinate stretching is more easily understood from Eqs. (18) and (19) than from the usual PL formulation in which x_1 occurs in the first-order differential equations (7) themselves. As noted previously [3], the first step is to find (if possible) the classical (non-PL) solutions $y_0(x)$ and $y_1(x)$; this step is taken whether or not it is apparent a priori that PL theory is needed. Then if inspection of $y_1(x)$ indicates the need for coordinate stretching, Eqs. (18) and (19) are employed as a second step. In the following section we derive necessary conditions under which this second step may be applied.

2. Necessary conditions for applicability of PL theory.

A. *Preliminary remarks.* The essential point of this paper is that Eq. (19) also allows us to establish criteria for the applicability of the PL method; for on considering Eq. (19), we want y_1^i to be well behaved, and so we must use the product $x_1 f_0^i$ to subtract out that part of $\tilde{y}_1^i$ that is badly behaved. However, in order to obtain x_1 from one of the k components of $y_1^i = \tilde{y}_1^i + x_1 f_0^i$, we must divide through by f_0^i which may be zero in the domain of interest; it is this possibility that leads to necessary criteria for applicability.

We consider the problem of the nonuniform convergence of k vector components j, $j + 1, \cdots, j + k - 1$ of the series representation of $y(x) = y_0(x) + \epsilon y_1(x)$. In order to avoid obscuring the essential points, we assume that these components of $y_1^i(x)$ are badly behaved at just one point in the domain of interest of x, and that by suitable transformations this point can be brought to the origin $x = 0$. Throughout this section the superscript i denotes any of these k components.

B. *Domain of interest in z.* In problems of nonuniform convergence for which the zeroth order solution $y_0(x)$ is singular at $x = 0$, the singularity becomes progressively more severe as the solution is carried to successively higher orders [2], [4]. The introduction of a stretched coordinate z must be made inter alia in such a way that the singularity in the zeroth and higher order solutions is never reached; i.e., that $z \to \zeta > 0$ as $x \to 0$, where ζ is real. From Eq. (5), this condition can also be expressed as $x_1(z) \to \xi_1 = -\epsilon/\zeta < 0$ as $x \to 0$. Further, since x and z take equal values at the boundary we have that $0 < \zeta \leq z \leq 1$ in the interval $0 \leq x \leq 1$. When the domain of interest $0 \leq x < \infty$ extends beyond the boundary at $x = 1$, we have $0 < \zeta \leq z < \infty$. This condition can be relaxed when the type of misbehavior in $y_0(x)$, $y_1(x)$, $\cdots$ is nonsingular, and by Eqs. (4) and (5) we then require simply that z be finite or zero in the domain of interest. For convenience, in the subsequent discussion we let D denote the domain of interest.

C. *x_1 and the auxiliary vector ϕ_1^i.* Let us sketch the procedure involved in choosing x_1. When coordinate stretching is needed, the components $y_1^i(x)$, or equivalently (by expression 18) $\tilde{y}_1^i(z)$, are more badly behaved than $y_0^i(x)$ or $y_0^i(z)$ at their respective origins. We let $\tilde{y}_1^i$ equal the sum of the badly behaved part $\tilde{y}_{1,s}^i$ and a well-behaved part $\tilde{y}_{1,w}^i$

$$\tilde{y}_1^i = \tilde{y}_{1,w}^i + \tilde{y}_{1,s}^i . \tag{21}$$

Breaking up $\tilde{y}_1^i$ in this fashion is possible since we can always choose $\tilde{y}_{1,w}^i \equiv 0$. From Eq. (19) we use $x_1 f_0^i$ to subtract out $\tilde{y}_{1,s}^i$

$$y_1^i = \tilde{y}_{1,w}^i + \tilde{y}_{1,s}^i + x_1 f_0^i , \tag{22}$$

and we let

$$\phi_1^i = \tilde{y}_{1,s}^i + x_1 f_0^i . \tag{23}$$

Hence

$$y_1^i = \tilde{y}_{1,w}^i + \phi_1^i \,. \tag{24}$$

Since $\tilde{y}_{1,w}^i$ is by definition no more badly behaved than y_0^i at $z = 0$, and since we want y_1^i to be similarly behaved in order to ensure uniform convergence, it follows that ϕ_1^i must be so chosen that it, too, is no more badly behaved than y_0^i at $z = 0$. (This and subsequent conditions on ϕ_1^i and x_1 are summarized in Sec. 2E.)

In order to determine x_1 it is necessary first to decide upon the form of just one component of ϕ_1^i, say ϕ_1^j; then

$$x_1 = (\phi_1^j - \tilde{y}_{1,s}^j)/f_0^j \,, \tag{25}$$

and the remaining components $\phi_1^{i>j}$ can be found from Eq. (23) whence all y_1^i are found from Eq. (24). The derived components $\phi_1^{i>j}$ and $y_1^{i>j}$ must, of course, still satisfy the conditions that they be well behaved for all values of z in D and, near $z = 0$, that they be no more badly behaved than $y_0^{i>j}$. Moreover, by Eq. (24), ϕ_1^i must be such that the initial conditions on x_1 and y_1^i be satisfied.

To ensure that the initial condition on x_1 is satisfied, it is necessary to let ϕ_1^j be parametric in a constant C^j; for on applying the initial conditions (8), (9) for x_1 and y_1^j, to Eqs. (23), (24), and since f_0 is finite at the boundary, we have

$$\phi_1^j(1; C^j) = \tilde{y}_{1,s}^j(1), \tag{26a}$$

$$\phi_1^j(1; C^j) = -\tilde{y}_{1,w}^j(1). \tag{26b}$$

Either one of Eqs. (26a, b) serves to determine C^j; these equations are not independent since their difference always satisfies the initial condition on $\tilde{y}_1^j$ by Eqs. (17) and (21). While the constant C^j arises through the necessity of satisfying the initial condition on x_1, it may also be regarded as an integration constant, because according to conventional PL theory x_1 must be determined by solving a suitably chosen scalar first-order differential equation (cf. Eq. (7)).

The above procedure ensures that the initial conditions on y_1^j and x_1 are satisfied. The initial conditions (8) on the remaining $k - 1$ components of $y_1^{i>j}$ at $x = z = 1$ are then always satisfied through Eqs. (9), (17), (21), and (22).

D. *Conditions on* x_1. From the expression (25) for x_1 is it necessary that

$$(\phi_1^j(z) - \tilde{y}_{1,s}^j(z))/f_0^j(z) \neq \infty \tag{27}$$

throughout D in order for the unified perturbation analysis to be applicable.

Let us examine the conditions under which the expression (27) holds. By assumption (Sec. 2A above), $\tilde{y}_{1,s}^j(z)$ is misbehaved only at $z = 0$ which lies outside D, and from Eq. (24) and the discussion following it, $\phi_1^j(z)$ can be misbehaved at $z = 0$ which also lies outside D. Since $f_0^j(z)$ is never singular in D, it follows that $\phi_1^j(z)$ must be so chosen as to be nonsingular in D. But it is possible for one or both of the numerator and denominator to have *zeros* in the domain of interest. We distinguish three cases:

Case 1.

$$f_0^j(z) \neq 0, \tag{28}$$

for all z in D. Hence x_1 is zero or finite in D and PL theory is applicable.

Case 2.

$$f_0^j(\bar{z}) = 0, \qquad \phi_1^j(\bar{z}) - \tilde{y}_{1,s}^j(\bar{z}) \neq 0, \tag{29}$$

where $\bar{z}$ lies in D. In this case $x_1(\bar{z})$ is singular leading to an infinite distortion of the x-coordinate, and PL theory is not applicable in the vicinity of $\bar{z}$.

Case 3.

$$f_0^i(\bar{z}) = 0, \qquad \phi_1^i(\bar{z}) - \tilde{y}_{1,s}^i(\bar{z}) = 0. \tag{30}$$

In this case PL theory is applicable only if the order of the zero of $(\phi_1^i - \tilde{y}_{1,s}^i)$ is greater than or equal to that of f_0^i, i.e., if

$$\lim_{z \to \bar{z}} (\phi_1^i(z) - \tilde{y}_{1,s}^i(z))/f_0^i(z) \neq \infty. \tag{31}$$

E. *Summary and additional remarks.* Let us summarize the conditions on ϕ_1^i and x_1 which were deduced in sections C and D above:

(1) $\phi_1^i(z)$ must be chosen so that it is no more badly behaved than $y_0^i(z)$ near $z = 0$. (2Ea)

(2) $\phi_1^i(z)$ must be chosen to be nonsingular throughout the domain of interest in z. (2Eb)

(3) When, $y_0(z)$ is singular at $z = 0$, then $\phi_1^i(z)$ must be chosen so that $x_1(z) \to \xi_1 < 0$, $z \to \zeta > 0$ as $x \to 0$, where ξ_1 and ζ are real. (2Ec)

(4) $\phi_1^i(z)$ must be parametric in a constant C^i determined from one of Eqs. (26a, b) in order to satisfy the initial conditions on x_1 and y_1^i. (2Ed)

(5) $\phi_1^{j>i}(z)$ derived from Eq. (24) must satisfy the same conditions on $\phi_1^i(z)$ as stated in (2Ea, b). (2Ee)

When (2Ea–e) are satisfied, then PL theory is applicable provided the condition (27) holds. These conditions are necessary for the applicability of Poincaré–Lighthill theory to problems of nonuniform convergence.

We note that (2Ee) is redundant in the scalar case and also in the vector case when the number of misbehaved components $k = 1$. The conditions under which (2Ee) will be satisfied when $k > 1$ depend ultimately on the nature of the original problem through Eqs. (3), (6), (15), (21) and (23); this question is not investigated here.

3. Wasow's criterion. Wasow [5] has considered the case

$$(x + \epsilon y)(dy/dx) + q(x)y = r(x), \qquad y(1) = b, \qquad (0 \leq x \leq 1), \tag{32}$$

with the functions q and r regular near the origin. By use of the classical PL perturbation theory he finds a necessary condition for convergence

$$q(z)y_0 - r(z) \neq 0, \qquad (0 < z \leq 1). \tag{33}$$

If we consider Eq. (32) in zeroth order we see that in this case

$$f_0(y_0, z) = \frac{1}{z}[r(z) - y_0 q(z)],$$

so that criterion (33) is a special case of Eq. (28).

4. Illustrative examples. To illustrate Cases 1–3 of Sec. 2D, we apply the new formulation to three simple problems of nonuniform convergence of series solutions.

A. We illustrate Case 1 above by means of the scalar problem $(x + \epsilon y)(dy/dx) + y = 0$, $y(1) = 1$ for $\epsilon > 0$, $x \geq 0$. The zeroth and first order solutions are [4] $y_0 = 1/x$, $y_1 = (1/2x) - (1/2x^3)$. Since the order of the singularity in y_1 at $x = 0$ is greater than that of y_0, the classical perturbation solution is nonuniformly convergent at the origin. According to the new formulation, we let $\tilde{y}_{1,s} = -1/2z^3$ and $\tilde{y}_{1,w} = 1/2z$. From Eq. (24) we choose $\phi_1 = -1/2z + C^1/z$ to be parametric in a constant C^1. Applying the initial conditions (26a, b) we find $C^1 = 0$. Hence by Eq. (23) $x_1 = (1/2z)(1/z^{-2} - 1)/f_0(z)$ where

$$f_0(z) = -1/z^2. \tag{34}$$

Thus $y = 1/z$ and $x = z + \frac{1}{2}\epsilon z(1 - (1/z^{-2})$. As $x \to 0+$, $z \to +[\epsilon/(2+\epsilon)]^{1/2} = \zeta$, and $\xi_1 = -[\epsilon(2+\epsilon)]^{-1/2}$. From Eq. (34) it follows that $f_0(z)$ is never zero in the domain of interest $0 < \zeta \leq z < \infty$. The PL method is necessary for small x and by Eq. (28) it is applicable for all x.

B. We illustrate Cases 2 and 3 by means of the scalar problem

$$(x + \epsilon y)(dy/dx) + y = 2x, \qquad y(1) = b > 1 \tag{35}$$

for $0 \leq x < \infty$, $\epsilon > 0$. (This is a special case of the group of equations in [6].) The zeroth and first order solutions are

$$y_0 = x + (b - 1)/x, \tag{36}$$

$$y_1 = (1/2x)[b^2 - 2b + 2 - (b-1)^2/x^2 - x^2]. \tag{37}$$

The non-PL method gives nonuniform convergence for small values of x since from Eqs. (36) and (37) $y_0 \sim x^{-1}$, $y_1 \sim x^{-3}$.

We proceed in accordance with Sec. 2 to eliminate the singular portion of $\tilde{y}_1$ by letting $\tilde{y}_{1,s} = -(b-1)^2/2z^3$ and $\tilde{y}_{1,w} = (b^2 - 2b + 2 - z^2)/2z$, where

$$f_0 = 1 - (b-1)/z^2. \tag{38}$$

We have by Eq. (23)

$$\phi_1 = -(b-1)^2/2z^3 + x_1(1 - (b-1)/z^2). \tag{39}$$

According to 2Ea, ϕ_1 must be no more singular than $y_0(z) \sim z^{-1}$ as $z \to 0$. Thus as $z \to 0$, we must have the asymptotic dependence $x_1 \sim -(b-1)/2z$ in order to eliminate the third order singularity in Eq. (39). By 2Ed, x_1 must be parametric in a constant C^1 in order that $x_1(1) = 0$. Multiplication of the asymptotic form by C^1 does not suffice, and neither does the addition of C^1, since then there will be a term in z^{-2} in ϕ_1 which violates 2Ea. Hence we add a term $-(1/2)(b-1)C^1z$ to the asymptotic form and the initial condition on $x_1(1)$ gives $C^1 = -1$. Thus

$$x_1 = -((b-1)/2)(1/z - z). \tag{40}$$

Eq. (39) then gives

$$\phi_1 = -((b-1)/2)(b/z - z), \tag{41}$$

which is nonsingular in the domain of interest, thereby satisfying 2Eb. From Eq. (24) we find

$$y_1 = ((2-b)/2))(1/z - z),$$

which satisfies the initial condition $y_1(1) = 0$. The solution for $y(x)$ to first order in ϵ is thus given by

$$y = z + (b-1)/z + (\epsilon/2)(2-b)(1/z - z), \qquad x = z + (\epsilon/2)(1-b)(1/z - z),$$

from which it follows that as $x \to 0+$ from the boundary at $x = z = 1$, then $\zeta \approx +[(\epsilon/2)(b-1)]^{1/2} > 0$, and $\xi_1 \approx -[(\epsilon/2)(b-1)]^{-1/2} < 0$ thereby satisfying 2Ec. Also $y \approx +[(2/\epsilon)(b-1)]^{1/2}$ which agrees with the asymptotic value of y derived from the exact solution to Eq. (35), viz

$$xy + \tfrac{1}{2}\epsilon y^2 = x^2 + b - 1 + \tfrac{1}{2}\epsilon b^2.$$

Let us now examine the role of criterion (27) in the light of Cases 2 and 3 of section 2D. From Eq. (38) we see that $f_0 = 0$ at $\bar{z}^2 = b - 1$, and we might expect therefore according to Case 2 (Eq. (29)) that PL theory is inapplicable in the vicinity of $\bar{z} = +(b-1)^{1/2} > \zeta$. It is clear from Eq. (40), however, that x_1 is nonsingular everywhere in D, so that PL theory is everywhere applicable. The apparent contradiction is resolved by the fact that both $f_0(z)$ and $(\phi_1 - y_{1,s})$ have cancelling zeros in $\bar{z}$ in accordance with Case 3 (Eqs. (30) and (31)), as can be verified by substituting expression (41) in Eq. (39), and it follows, therefore, that PL theory is applicable throughout D.

Another solution to this problem provides an instructive illustration of Case 2 (Eq. (29)). It can be shown that a PL solution is also

$$y = z + (b-1)/z + (\epsilon/2z)(z + b^2 - 2b + 2)(1 - z), \quad x = z + (\epsilon/2)(b-1)^2(1-z^3)/z^3 f_0(z),$$

which as required is accurate in the vicinity of the origin $x = 0$ at which $z = \zeta \approx + [(\epsilon/2)(b-1)]^{1/2}$, but breaks down at $\bar{z} = +(b-1)^{1/2}$ by virtue of Eqs. (29) and (38), where $\bar{z} > \zeta$ lies in D.

C. We illustrate Cases 2 and 3 by means of the vector system

$$dy^1/dx = -(1+2\epsilon)^{1/2} y^2, \qquad y^1(0) = 1, \tag{42a}$$

$$dy^2/dx = (1+2\epsilon)^{1/2} y^1, \qquad y^2(0) = 0, \tag{42b}$$

for $-\infty < x < \infty$. The exact solution is

$$y^1 = \cos\,(1+2\epsilon)^{1/2} x, \tag{43a}$$

$$y^2 = \sin\,(1+2\epsilon)^{1/2} x. \tag{43b}$$

The applicability of PL perturbation theory to this type of problem is classic, since the effect of the perturbation is purely a coordinate stretching.

We rewrite Eqs. (42a, b) in the form of Eq. (3)

$$dy^1/dx = -y^2 - \epsilon y^2 - \cdots, \qquad dy^2/dx = y^1 + \epsilon y^1 + \cdots$$

and by the method of Sec. 1 we find

$$y_0^1 = \cos z, \tag{44a}$$

and

$$y_0^2 = \sin z. \tag{44b}$$

Hence

$$f_0^1 = -\sin z, \tag{45a}$$

and

$$f_0^2 = \cos z. \tag{45b}$$

In first order we have

$$\tilde{y}_1^1 = -z \sin z, \tag{46a}$$

and

$$\tilde{y}_1^2 = z \cos z, \tag{46b}$$

which satisfy the initial conditions $\tilde{y}_1^i(0) = 0$, $(i = 1, 2)$. In terms of classical perturbation theory $(z \equiv x)$, the secular-periodic terms in the first order solution (46a, b) will adequately represent the true solution only when $|\epsilon x|$ is small, as can be seen by expanding the true solution (43a, b) and comparing it with equations (44a, b), and (46a, b). It is well known that the difficulty with the classical approach arises through attempting to use a finite number of terms in an infinite series to represent the true solution, as noted toward the end of the last century by Lindstedt, Poincaré and others [7].

To discuss this problem in accordance with the development of Sec. 2, we should apply a transformation of the kind $x = a(1 - v)$ in order to bring the boundary of the new coordinate to $v = 1$; this corresponds merely to a shift of origin with a scale factor $|\epsilon a| \gtrsim 1$ chosen to bring the point of misbehavior to the origin. However, unlike the previous problems the misbehavior here occurs for all finite $|\epsilon x| \gtrsim 1$, and is nonsingular, so that no special significance need be attached to the origin $v = 0$ (cf. Sec. 2b). Consequently we retain the Eqs. (42a, b) in their original form, and condition 2Ec reduces in this case simply to the requirement that x and z be finite or zero in the domain of interest.

From Eqs. (46a, b) we let $\tilde{y}_{1,w}^i \equiv 0$, $\tilde{y}_{1,s}^i = \tilde{y}_1^i$ $(i = 1, 2)$ and Eqs. (21)–(24) give

$$\phi_1^1 \equiv y_1^1 = -z \sin z + x_1 f_0^1 , \tag{48}$$

$$\phi_1^2 \equiv y_1^2 = z \cos z + x_1 f_0^2 , \tag{49}$$

where f_0^i is given by Eqs. (45a, b). One or the other of these equations serves to determine x_1 once y_1 for that equation has been chosen subject to the initial conditions

$$y_1^i(0) = 0 \tag{50a}$$

and

$$x_1(0) = 0. \tag{50b}$$

Moreover, y_1 must be chosen to be as well behaved as the zeroth order solutions (44a, b) which are purely periodic, and also so that x_1 remains finite in D.

To illustrate the role of criterion (27) let us first choose for y_1^i the simple periodic function $\sin z$ which satisfies Eq. (50a). Then from Eq. (48) we have $x_1 = -(1 + z)$ which does not satisfy the initial condition (50b) on x_1, while from Eq. (49) we have $x_1 = \tan z - z$ which satisfies Eq. (50b) but is singular at $z = \pm(2n + 1)\pi/2$, $(n = 0, 1, \cdots)$. This illustrates Case 2.

Next we try

$$y_1^j \equiv 0, \qquad (j = 1 \text{ or } 2) \tag{51}$$

which satisfies Eq. (50a) and is well behaved, and both Eqs. (48), (49) give

$$x_1 = -z \tag{52}$$

while the other component of y_1 is also identically zero. Eq. (52) satisfies Eq. (50b), and is nonsingular for all finite z and conditions (2Ea–e) on $\phi_1 \equiv y_1$ are all satisfied. Thus, for this choice and with reference to Eq. (31) (Case 3) and the expressions (45a, b) for f_0' we see that both the numerator and denominator of x_1 derived from Eqs. (48) and (49) have zeros at $z = \pm n\pi$ and $\pm(n + \frac{1}{2})\pi$, $(n = 0, 1, \cdots)$ respectively but these zeros cancel and PL theory is applicable throughout the domain of interest. This illustrates Case 3. Finally, using Eqs. (44a, b) and (51) we have $x = z(1 - \epsilon)$, $y^1 = \cos z$, $y^2 = \sin z$, and y^1, y^2 agree with the exact solution (43a, b) up to terms of first order in ϵ.

5. Concluding remarks. We have discussed a method which represents a great saving of effort in problems which are of the Poincaré–Lighthill type. The method enables one to obtain the PL expansion and an ordinary expansion simultaneously. A criterion for the applicability of the PL method is established which is an elementary consequence of our method and some new insights on the theory of coordinate stretching are given. In particular, the violation of the condition (27) shows up by our method as an extreme distortion of the coordinate system and thus appears to be a fundamental limitation of the PL method.

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Note added in proof: This section summarizes some results of reference [3] which were arrived at independently by M. F. Pritulo, J. Appl. Math. Mech. **26**, 661 (1962) (Prikladnaia Matematika i Mekhanika **26**, 444 (1962)).

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