A BOUND ON THE ERROR IN PLATE THEORY*

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Introduction. A solution to a boundary value problem in the classical two-dimensional theory of plates is generally accepted as an approximate solution to a corresponding boundary value problem in the three-dimensional theory of elasticity provided that the plate is sufficiently thin. This conclusion is supported by several exact solutions for plates in the theory of elasticity [1] and by the fact that the equations of plate theory can be obtained from the equations of elasticity theory as the leading terms in parametric expansions [2], [3]. Further, Morgenstern [4] has shown that the stresses and strains obtained from a solution in plate theory converge in a mean-square sense to a solution in elasticity theory as the plate thickness approaches zero. Related theorems on mean-square convergence of parametric expansions for a problem in beam theory are stated by Babuška and Prager [5].

In the present paper we derive an explicit expression for the mean-square error in the components of stress obtained from a solution in plate theory with respect to the exact solution of a corresponding problem in the theory of elasticity. In addition, a precise bound is given for the relative mean-square error. The derivation employs the hypersphere theorems of Prager and Synge [6] in the theory of elasticity. In the course of the derivation the equations of plate theory are obtained in two ways by minimization of portions of both the potential energy and the complementary energy. The general expression obtained for the error contains only quantities which are available from a solution in plate theory.

Our results and the previous investigations of convergence [4], [5] show that the relative mean-square error in plate theory is proportional to the thickness of the plate in general. This is somewhat surprising since the exact solutions for plates in elasticity theory [1] give a relative error proportional to the square of the thickness. This form for relative error also is indicated by the parametric expansions [2]. The discrepancy in our result can be attributed to the expression obtained for the components of transverse shear stress, which differs from the classical expression. We have been unable to derive the classical expression by the present method. (See note added in proof at the end of this paper.)

1. Function space concepts in elasticity. We consider a three-dimensional elastic body R bounded by a closed surface S. With reference to a system of rectangular Cartesian coordinates x_i ($i = 1, 2, 3$), the field equations of the linear theory of elasticity read as follows [1]:

equilibrium (in the absence of body force)

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$$\sigma_{ij,j} = 0, \qquad \sigma_{ij} = \sigma_{ji}; \tag{1.1}$$ [1]

strain-displacement

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}); \tag{1.2}$$

generalized Hooke's law

$$e_{ij} = A_{ijkl}\sigma_{kl}\ , \qquad \sigma_{ij} = B_{ijkl}e_{kl}\ , \tag{1.3}$$

where σ_{ij} is the stress tensor, e_{ij} is the strain tensor and u_i is the displacement vector. The elastic constants A_{ijkl} and B_{ijkl} satisfy

$$\begin{aligned} A_{ijkl} &= A_{jikl} = A_{ijlk} = A_{klij}\ , \\ B_{ijkl} &= B_{jikl} = B_{ijlk} = B_{klij}\ , \\ A_{ijkl}B_{klmn} &= \tfrac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \end{aligned} \tag{1.4}$$ [2]

and are such that the strain energy density W is positive definite where

$$W = \tfrac{1}{2}A_{ijkl}\sigma_{ij}\sigma_{kl} = \tfrac{1}{2}B_{ijkl}e_{ij}e_{kl}\ . \tag{1.5}$$

We consider boundary value problems where the boundary S is divided into two parts S_u and S_σ on which boundary conditions are

$$u_i = u_i^* \quad \text{on} \quad S_u\ , \tag{1.6}$$

$$\sigma_i = \sigma_{ij}n_j = \sigma_i^* \quad \text{on} \quad S_\sigma\ , \tag{1.7}$$

where u_i^* is the prescribed displacement vector on S_u , σ_i^* is the prescribed stress vector on S_σ and n_i is the outward unit normal vector to S. Somewhat more general linear boundary conditions may be treated without difficulty.

Following Prager and Synge [6] and Synge [7], for states of stress σ_{ij} such that

$$\int_R A_{ijkl}\sigma_{ij}\sigma_{kl}\, dV < \infty \tag{1.8}$$

we consider the vector space with componentwise addition and inner product defined by

$$\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'' = \int_R A_{ijkl}\sigma'_{ij}\sigma''_{kl}\, dV, \tag{1.9}$$

where $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ denote two states of stress with components σ'_{ij} and σ''_{ij} which satisfy (1.8). The norm of $\boldsymbol{\sigma}$ is defined as

$$\|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{1/2}. \tag{1.10}$$

It can be verified that the foregoing definitions satisfy the basic postulates for a linear vector space [8].

In what follows unprimed quantities denote the actual solution to the boundary value problem (1.1) to (1.7). For this same boundary value problem, primed quantities satisfy (1.1), (1.3) and (1.7), and double-primed quantities satisfy (1.2), (1.3) and (1.6).

[1] Commas denote partial differentiation and repeated indices imply summation.

[2] The Kronecker delta δ_{ij} takes the value 1 if $i = j$ and 0 if $i \neq j$.

Then, with use of the divergence theorem, it follows that

$$\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'' = \int_{S_u} u_i^* \sigma_i' \, dS + \int_{S_\sigma} u_i'' \sigma_i^* \, dS. \tag{1.11}$$

Since $\boldsymbol{\sigma}'$ or $\boldsymbol{\sigma}''$ in (1.11) can be replaced by $\boldsymbol{\sigma}$, we have

$$\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'' + \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'' = 0,$$

which is equivalent to

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_A|| = E, \tag{1.12}$$

where

$$\boldsymbol{\sigma}_A = \tfrac{1}{2}(\boldsymbol{\sigma}' + \boldsymbol{\sigma}''), \qquad E = ||\tfrac{1}{2}(\boldsymbol{\sigma}' - \boldsymbol{\sigma}'')|| \, .$$

Thus, if $\boldsymbol{\sigma}_A$ is considered an approximation to $\boldsymbol{\sigma}$, (1.12) gives the error E of this approximation in the integral-square norm (1.10). In vector space geometry (1.12) implies that $\boldsymbol{\sigma}$ lies on a hypersphere with center at $\boldsymbol{\sigma}_A$ and radius E.

In order to investigate relative error we recall the inequalities [7]

$$||\boldsymbol{\sigma}_A|| - E \leq ||\boldsymbol{\sigma}|| \leq ||\sigma_A|| + E, \tag{1.13}$$

which follow from (1.12) and the triangle inequality. By (1.13), assuming that $E < ||\boldsymbol{\sigma}_A||$, we have the following bounds on the relative error $E/||\boldsymbol{\sigma}||$:

$$\frac{E}{||\boldsymbol{\sigma}_A|| + E} \leq \frac{E}{||\boldsymbol{\sigma}||} \leq \frac{E}{||\sigma_A|| - E}. \tag{1.14}$$

Since, by (1.12),

$$||\boldsymbol{\sigma}_A||^2 = \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'' + E^2, \tag{1.15}$$

(1.14) can be written as

$$\eta(\sqrt{(1 + \eta^2)} - \eta) \leq E/||\boldsymbol{\sigma}|| \leq \eta(\sqrt{(1 + \eta^2)} + \eta) \tag{1.16}$$

where

$$\eta = E/(\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'')^{1/2}. \tag{1.17}$$

From (1.16) we see that η is the leading term in an expansion of the relative error in powers of η.

A convenient method of obtaining $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ follows from the easily verified relation [7]

$$E^2 = \tfrac{1}{2}(V_c + V_p) \tag{1.18}$$

where

$$V_c = \int_R \tfrac{1}{2} A_{ijkl} \sigma_{ij}' \sigma_{kl}' \, dV - \int_{S_u} u_i^* \sigma_i' \, dS$$

$$V_p = \int_R \tfrac{1}{2} A_{ijkl} \sigma_{ij}'' \sigma_{kl}'' \, dV - \int_{S_\sigma} u_i'' \sigma_i^* \, dS$$

are the complementary energy and the potential energy, respectively. Thus, the usual methods of energy minimization can be interpreted as minimizations of the error E.

2. Boundary value problems for plates. We consider an elastic body in the form of a plate bounded by the faces $x_3 = \pm h$ and the edge surface $\tilde{S}$ which is generated by normals to the middle plane S through the edge curve C. Attention will be confined to plates of constant thickness $2h$ although there is no difficulty in extending the results to variable thickness.

Stress boundary conditions are imposed on the faces of the plate. For simplicity we consider the faces to be subject only to prescribed normal components of stress, i.e.

$$\begin{gathered} \sigma_{13} = \sigma_{23} = 0 \quad \text{on} \quad x_3 = \pm h, \\ \sigma_{33}(x_1, x_2, h) = \sigma_3^+(x_1, x_2), \qquad \sigma_{33}(x_1, x_2, -h) = -\sigma_3^-(x_1, x_2). \end{gathered} \tag{2.1}$$

The edge surface of the plate is divided into two parts $\tilde{S}_u$ and $\tilde{S}_\sigma$ which have displacement and stress boundary conditions of the form (1.6) and (1.7), namely

$$u_i = u_i^* \quad \text{on} \quad \tilde{S}_u, \tag{2.2}$$

$$\sigma_{i\alpha} n_\alpha = \sigma_i^* \quad \text{on} \quad \tilde{S}_\sigma, \tag{2.3}$$

where n_α is the normal to C in the middle plane and Greek indices take values 1 and 2. Here, we restrict u_i^* and σ_i^* to the forms

$$u_\alpha^* = \beta_\alpha^* x_3, \tag{2.4}$$

$$u_3^* = w^* + g^* x_3^2, \tag{2.5}$$

$$\sigma_\alpha^* = \frac{3}{2h^3} M_\alpha^* x_3, \tag{2.6}$$

$$\sigma_3^* = \frac{3}{4h}\left(1 - \frac{x_3^2}{h^2}\right) Q^*, \tag{2.7}$$

where β_α^*, w^*, g^*, M_α^* and Q^* are independent of x_3. The constants in (2.6) and (2.7) have been chosen so that M_α^* and Q^* represent the stress couple and the shear stress resultant, respectively, i.e.,

$$\int_{-h}^{h} \sigma_\alpha^* x_3 \, dx_3 = M_\alpha^*, \qquad \int_{-h}^{h} \sigma_3^* \, dx_3 = Q^*. \tag{2.8}$$

We now have a boundary value problem in the theory of elasticity for a plate with boundary conditions (2.1) to (2.7). The aim of the theory of plates is to reduce this three-dimensional boundary value problem to a two-dimensional problem involving quantities which are independent of x_3. Here we will derive the equations of the classical theory of plates by consideration of both the complementary energy and the potential energy (1.18). Then, (1.12) will yield an expression for the error in an approximation for the stresses in the plate. Since the well-known edge conditions for the classical theory of plates [1] are not as general as (2.4) to (2.7), we expect to obtain restrictions on the quantities on the right-hand side of these equations.

For convenience we restrict attention to anisotropic plates for which the middle plane is a plane of elastic symmetry. Then (1.3) takes the form [1]

$$e_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + A_{\alpha\beta33}\sigma_{33}, \quad e_{33} = A_{33\gamma\delta}\sigma_{\gamma\delta} + A_{3333}\sigma_{33}, \quad e_{\alpha3} = 2A_{\alpha3\beta3}\sigma_{\beta3}, \tag{2.9}$$

and

$$\sigma_{\alpha\beta} = B_{\alpha\beta\gamma\delta}e_{\gamma\delta} + B_{\alpha\beta33}e_{33}, \quad \sigma_{33} = B_{33\gamma\delta}e_{\gamma\delta} + B_{3333}e_{33}, \quad \sigma_{\alpha3} = 2B_{\alpha3\beta3}e_{\beta3}. \tag{2.10}$$

3. Potential energy. Here, (1.2), (1.3) and (1.6) must be satisfied and V_p in (1.18) is to be minimized. Guided by the three-dimensional solution for pure bending of a plate [1], we assume displacements of the form

$$u_\alpha'' = -w_{,\alpha}x_3 , \qquad u_3'' = w + gx_3^2 , \tag{3.1}$$

where w and g are independent of x_3. The first of (3.1) represents the classical Kirchhoff assumption that normals to the undeformed middle plane remain normal to the deformed middle surface.

By (1.2) and (3.1), the strains are

$$e_{\alpha\beta}'' = -w_{,\alpha\beta}x_3 , \qquad e_{\alpha 3}'' = \tfrac{1}{2}g_{,\alpha}x_3^2, \qquad e_{33}'' = 2gx_3 . \tag{3.2}$$

If we take

$$g = \frac{B_{33\alpha\beta}}{2B_{3333}} w_{,\alpha\beta} , \tag{3.3}$$

then, by (2.10) and (3.2)

$$\sigma_{33}'' = 0 \tag{3.4}$$

as in the solution for pure bending. The discrepancy between (2.1) and (3.4) is admissible since σ_{ij}'' need not satisfy stress boundary conditions on S_σ. By (2.10), (3.2) and (3.3), the remaining stresses are given by

$$\sigma_{\alpha\beta}'' = -\tilde{B}_{\alpha\beta\gamma\delta}w_{,\gamma\delta}x_3 , \qquad \sigma_{\alpha 3}'' = B_{\alpha 3\beta 3}g_{,\beta}x_3^2 , \tag{3.5}$$

where

$$\tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - B_{\alpha\beta 33}B_{\gamma\delta 33}/B_{3333} .$$

The displacements (3.1) must satisfy (2.2) with (2.4) and (2.5), i.e.,

$$\left.\begin{aligned} w &= w^*(s), \\ \frac{\partial w}{\partial n} &= -\beta_\alpha^*(s)n_\alpha , \\ g &= g^*(s), \\ \epsilon_{\alpha\beta}n_\beta\beta_\alpha^* &= \frac{\partial w^*}{\partial s} \end{aligned}\right\} \text{ on } C. \tag{3.6}$$ [3]

where s is the arc length on C and $\partial w/\partial n$ denotes the normal derivative of w. Since g is determined by (3.3), the function $g^*(s)$ cannot be specified arbitrarily but must be compatible with (3.3). Thus, $g^*(s)$ is not known until w has been obtained for a particular problem.[4] The last of (3.6) is a compatability requirement on w^* and β_α^* as a consequence of the Kirchhoff hypothesis.

By (2.1), (2.3), (2.6), (2.7), (3.2) and (3.3), the potential energy (1.18) can be written

[3] The components of the ϵ-symbol $\epsilon_{\alpha\beta}$ have the values $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = 1$ and $\epsilon_{21} = -1$.

[4] It is possible to avoid this unpleasantness by not invoking (3.3) and leaving g as a basic variable akin to w. However, classical plate theory does not result.

as

$$V_p = V_p^{(0)} + V_p^{(2)} h^2 \tag{3.7}$$

where

$$V_p^{(0)} = \int_S [\tfrac{1}{3}h^3 \bar{B}_{\alpha\beta\gamma\delta} w_{,\alpha\beta} w_{,\gamma\delta} - pw]\, dx_1\, dx_2 + \int_{C_e} [M_\alpha^* w_{,\alpha} - Q^* w]\, ds,$$

$$V_p^{(2)} = \int_S [\tfrac{1}{5}h^3 B_{\alpha 3\beta 3} g_{,\alpha} g_{,\beta} - pg]\, dx_1\, dx_2 - \frac{1}{5}\int_{C_e} gQ^*\, ds,$$

$$p = \sigma_3^+ + \sigma_3^- .$$

On minimization of $V_p^{(0)}$ by standard techniques of the calculus of variations we obtain the Euler equation

$$\tfrac{2}{3}h^3 \bar{B}_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta} = p \quad \text{in} \quad S \tag{3.8}$$

and the natural boundary conditions

$$\left.\begin{aligned} M_N &= M_N^* \\ Q_\alpha n_\alpha - \frac{\partial M_T}{\partial s} &= Q^* - \frac{\partial M_T^*}{\partial s} \end{aligned}\right\} \quad \text{on } C_e \tag{3.9}$$

where

$$M_N^* = M_\alpha^* n_\alpha , \qquad M_T^* = \epsilon_{\alpha\beta} M_\alpha^* n_\beta ,$$
$$M_N = M_{\alpha\beta} n_\alpha n_\beta , \qquad M_T = \epsilon_{\beta\gamma} M_{\alpha\beta} n_\alpha n_\gamma . \tag{3.10}$$

$$M_{\alpha\beta} = -\tfrac{2}{3}h^3 \bar{B}_{\alpha\beta\gamma\delta} w_{,\gamma\delta} , \qquad Q_\alpha = -\tfrac{2}{3}h^3 \bar{B}_{\alpha\beta\gamma\delta} w_{,\beta\gamma\delta} . \tag{3.11}$$

The quantities $M_{\alpha\beta}$ and Q_α may be interpreted as stress couple and transverse shear force resultant, respectively. The subscripts N and T denote normal and tangential components of stress couple on C. The derivatives in (3.9) should be understood in the symbolic sense of the theory of generalized functions, i.e., a jump discontinuity in M_T gives a Dirac delta symbol for $\partial M_T/\partial s$ which represents a concentrated force.

4. Complementary energy. In this approach V_c is to be minimized for admissible stresses which meet (1.1) and (1.7). If we assume that the x_3 variation of stresses is of the form

$$\sigma'_{\alpha\beta} = \frac{3x_3}{2h^3} M'_{\alpha\beta} ,$$

$$\sigma'_{\alpha 3} = \frac{3}{4h}\left(1 - \frac{x_3^2}{h^2}\right) Q'_\alpha , \tag{4.1}$$

$$\sigma'_{33} = \frac{3}{4h}\left(1 - \frac{x_3^2}{3h^2}\right) x_3 p + q,$$

where

$$P = \sigma_3^+ + \sigma_3^- , \qquad q = \tfrac{1}{2}(\sigma_3^+ - \sigma_3^-),$$

then (1.1) are satisfied provided that

$$M'_{\alpha\beta,\beta} - Q'_\alpha = 0, \qquad -Q'_{\alpha,\alpha} = p. \tag{4.2}$$

Further, (4.1) satisfies (2.1) identically and guided by (3.9) we impose the boundary conditions

$$M'_N = M^*_N\,, \qquad Q'_\alpha n_\alpha - \partial M'_T/\partial s = Q^* - \partial M^*_T/\partial s \quad \text{on } C_\sigma \tag{4.3}$$

where

$$M'_N = M'_{\alpha\beta} n_\alpha n_\beta\,, \qquad M'_T = \epsilon_{\beta\gamma} M'_{\alpha\beta} n_\alpha n_\gamma$$

As mentioned in Sec. 2, classical plate theory requires a restriction on M^*_α and Q^* in (2.6) and (2.7). Namely, we can specify only the two quantities M^*_N and $Q^* - \partial M^*_T/\partial s$ on C_σ and we must accept the results of the plate theory solution for M^*_α and Q^*. With this restriction on the elasticity problem, the stress (4.1) under (4.2) and (4.3) are admissible as σ'_{ij} in the formulas of Sec. 1.

On substitution of (4.1) into (1.18), V_c can be written as

$$V_c = V_c^{(0)} + V_c^{(2)} h^2 \tag{4.4}$$

where

$$V_c^{(0)} = \int_S \frac{3}{4h^3} A_{\alpha\beta\gamma\delta} M'_{\alpha\beta} M'_{\gamma\delta}\, dx_1\, dx_2 - \int_{c_u} [\beta^*_\alpha M'_{\alpha\beta} n_\beta + w^* Q'_\alpha n_\alpha]\, ds,$$

$$V_c^{(2)} = \int_S \left[\frac{3}{5h^3} A_{\alpha\beta33} M'_{\alpha\beta} p + A_{3333}\left(\frac{17p^2}{140h} + \frac{q^2}{h}\right) + \frac{6}{5h^3} A_{\alpha3\beta3} Q'_\alpha Q'_\beta\right] dx_1\, dx_2 - \frac{1}{5}\int_{c_u} g^* Q'_\alpha n_\alpha\, ds.$$

In order to minimize $V_c^{(0)}$ subject to (4.2), we introduce the Lagrange multiplier w' and minimize

$$V_c^{(0)} - \int_S w'(M'_{\alpha\beta,\alpha\beta} + p)\, dx_1\, dx_2\,. \tag{4.5}$$

By the calculus of variations we obtain the Euler equation

$$\frac{3}{2h^3} A_{\alpha\beta\gamma\delta} M'_{\gamma\delta} + w'_{,\alpha\beta} = 0 \quad \text{in } S, \tag{4.6}$$

which, by (1.4), is equivalent to

$$M'_{\alpha\beta} = -\frac{2h^3}{3} \tilde{B}_{\alpha\beta\gamma\delta} w'_{,\gamma\delta}\,, \tag{4.7}$$

where $\tilde{B}_{\alpha\beta\gamma\delta}$ is defined at (3.5). We also obtain the natural boundary conditions

$$w' = w^*(s), \qquad \frac{\partial w'}{\partial n} = -\beta^*_\alpha(s) n_\alpha \qquad \text{on } C_u\,, \tag{4.8}$$

which agrees with (3.6). Substitution of (4.7) into (4.2) results in

$$Q'_\alpha = -\frac{2h^3}{3} \tilde{B}_{\alpha\beta\gamma\delta} w'_{,\beta\gamma\delta} \tag{4.9}$$

and

$$\frac{2h^3}{3}\tilde{B}_{\alpha\beta\gamma\delta}w'_{,\alpha\beta\gamma\delta} = p, \tag{4.10}$$

which is identical with (3.8).

5. Approximate solution and error. Since the field equations and boundary conditions of Secs. 3 and 4 are identical we make no distinction between primed and unprimed quantities w, $M_{\alpha\beta}$ and Q_α for the plate. Further, the equations and boundary conditions for w, $M_{\alpha\beta}$, Q_α are easily recognized as those of classical plate theory [1].

In a particular boundary value problem of plate theory, after a solution has been obtained for w, then g^*, M^*_α and Q^* can be determined from (3.3), (4.3), and (4.7) and (4.9). These values of g^*, M^*_α and Q^* together with the prescribed values $\sigma_3^{\pm}(x_1, x_2)$, w^* and $\beta^*_\alpha n_\alpha$ define a three-dimensional boundary value problem for an elastic plate as discussed in Sec. 2. For this problem we determine the error in the approximate solution of plate theory by the method of Sec. 1.

For use in the error formula (1.12), by (3.4), (3.5), (4.1), (4.7) and (4.9), the approximate plate stresses are

$$\begin{aligned}
\sigma^A_{\alpha\beta} &= -\tilde{B}_{\alpha\beta\gamma\delta}w_{,\gamma\delta}x_3 , \\
\sigma^A_{\alpha 3} &= \tfrac{1}{4}\left[\frac{B_{\alpha3\beta3}B_{33\gamma\delta}}{B_{3333}}x_3^2 - \tilde{B}_{\alpha\beta\gamma\delta}(h^2 - x_3^2)\right]w_{,\beta\gamma\delta} , \\
\sigma^A_{33} &= \frac{3}{8h}(1 - x_3^2/3h^2)x_3 p + \tfrac{1}{2}q,
\end{aligned} \tag{5.1}$$

where p and q are defined by (4.1). By (1.12), (3.4), (3.5), (4.1), (4.7) and (4.9) we find that

$$\begin{aligned}
E^2 = \frac{1}{2}\int_S \Big[\tfrac{1}{5}h^5 B_{\alpha3\beta3}g_{,\alpha}g_{,\beta} - \tfrac{1}{5}h^2 g_{,\alpha}Q_\alpha + A_{3333}h\left(\frac{17}{140}p^2 + q^2\right) \\
+ \frac{6}{5h}A_{\alpha3\beta3}Q_\alpha Q_\beta\Big]\,dx_1\,dx_2 ,
\end{aligned} \tag{5.2}$$

$$\mathbf{d}'\cdot\mathbf{d}'' = \int_S \left[\tfrac{2}{3}h^3 B_{\alpha\beta\gamma\delta}w_{,\alpha\beta}w_{,\gamma\delta} + \tfrac{1}{5}h^2(g_{,\alpha}Q_\alpha + 4gp)\right]dx_1\,dx_2 .$$

Thus, the error in the stresses (5.1) is given by (1.12) and (5.2). Bounds on the relative error are given by (1.16) and (5.2).

By (3.8) and (3.11), (5.2) can be written as

$$E^2 = C_1h^5 + C_2h^7, \qquad \mathbf{d}'\cdot\mathbf{d}'' = C_3h^3 + C_4h^5, \tag{5.3}$$

where C_1, C_2, C_3, C_4 depend only on w and the elastic constants. Thus, by (5.3), we have

$$\eta = E/(\mathbf{d}'\cdot\mathbf{d}'')^{1/2} = Ch + O(h^3), \tag{5.4}$$

where

$$\begin{aligned}
C = \left(\frac{C_1}{C_3}\right)^{1/2} = \Bigg\{\int_S \Big[\frac{1}{10}B_{\alpha3\beta3}g_{,\alpha}g_{,\beta} - \frac{1}{15}g_{,\alpha}\tilde{B}_{\alpha\beta\gamma\delta}w_{,\beta\gamma\delta} \\
+ \frac{4}{15}A_{\alpha3\beta3}B_{\alpha\gamma\delta\eta}w_{,\gamma\delta\eta}B_{\beta\mu\zeta\xi}w_{,\mu\zeta\xi}\Big]\,dx_1\,dx_2 \Big/ \int_S \frac{2}{3}B_{\alpha\beta\gamma\delta}w_{,\alpha\beta}w_{,\gamma\delta}\,dx_1\,dx_2\Bigg\}^{1/2}.
\end{aligned}$$

Equations (1.16) and (5.4) show that the relative error in the approximate stresses (5.1) is $O(h)$. This result is somewhat surprising in view of the elasticity solutions for plates [1] where corrections to classical plate theory are $O(h^2)$. A study of these solutions shows that the relative error of $O(h)$ in our formulas arises from the shear stress $\sigma^A_{\alpha\beta}$ in (5.1) whereas $\sigma^A_{\alpha\beta}$ and σ^A_{33} contribute $O(h^2)$ to the error. Since $\sigma_{\alpha\beta}$ are usually the largest stresses in plate bending, (5.4) may be unduly pessimistic from a practical viewpoint.

Note added in proof: James G. Simmonds (in a paper accepted for publication in this journal) has improved the bound on the relative error to $O(h^2)$ for isotropic plates by use of a more elaborate displacement field than (3.1). This result appears to hold for anisotropic plates as well.

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