

DYNAMIC ELASTIC-PLASTIC BUCKLING OF RECTANGULAR PLATES IN SUSTAINED FLOW*

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1. Introduction. A characteristic feature of dynamic buckling of simple structures (rod, plates, shells) is the significant effect of lateral inertia in restraining growth of buckling mode amplitudes during the early stages of the motion. This effect is such that the resulting instabilities may be quite different from those caused by quasi-static loading.

In the case of dynamic buckling of axially compressed bars, rectangular plates and cylindrical shells, the restraint of lateral inertia on the growth of lateral deflections permits large compressive strains to develop before the instabilities can become dominant, even though the instabilities may be initiated at an early stage of loading. For slow loading the inertial effects disappear and in this case the buckling process may be such that instabilities become large for very little increase in axial strain. The wavelengths of the buckled form are then usually much larger than for dynamic buckling. Clearly the transition from the dynamic case to the quasi-static case is a gradual process that depends upon the rate of loading.

For relatively high rates of loading, Florence and Goodier [1] have demonstrated that plastic buckling of cylindrical shells under axial compression is characterized very early in the motion. They obtained a series of high-speed photographs which show that the wavelength of the instability remains fixed and only the amplitude changes as time increases. The nature of the buckling, at least during the early stages, is of the Shanley type; that is, the dominant motion is one of uniform compression which is rapid enough that small perturbations can be introduced without causing any unloading in the plastic sense.

A satisfactory theory for small-amplitude buckling may be obtained assuming Shanley buckling. The mode numbers obtained from this theory will then correspond to large-amplitude buckling in which some unloading does occur, since the results of [1] show that mode numbers do not change with increasing load.

The particular problem considered here is that of a rectangular plate impacted between two converging heavy masses. For sufficiently heavy masses, longitudinal inertia of the plate may be neglected and the compressive strains in the unperturbed state are then uniform along the length of the plate. This problem has been examined both experimentally and theoretically by Goodier [2]. In his analysis, Goodier assumed rigid-plastic material behaviour. Since the final strains involved far exceed strains at the elastic limit it is probable that elastic motion may conveniently be neglected. Cer-

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tainly the extension (to cylindrical shells) given by Vaughan [3] predicted results that were in very good agreement with experiment. However, the shells were of aluminum and final strains were upward of 10%. As mentioned above, even if final strains are large, plastic buckling is characterized very early in the motion when total strains are of the same order as elastic strains. Thus it is possible that elastic behaviour is important in initiating the instability and it remains to be seen what influence elastic strains have on the motion.

In this investigation a theory is developed for a general elastic-plastic material. Thus plates may be considered in which final strains may be entirely elastic or composed of 'elastic' and plastic parts of any relative magnitudes.

The constitutive equations used correspond to a general elastic-plastic continuum which, it is felt, is necessary to describe correctly the flow of metals. A general theory developed by Green and Naghdi [4] has been used as a basis for the particular case of plate buckling. Their theory provides a satisfactory link between the thermodynamic and mechanical considerations and does not limit the size of the strains. Ramsey [5] has previously specialized the general theory to account for instabilities in rectangular plates under tension and compression.

The constitutive postulates envisage small elastic strains in the presence of plastic strains which may be small or large. The constitutive equation for elastic strains is nominally the same as Hooke's law, yet takes account of the result established by Green and Naghdi that during unloading, the strain tensor depends on the current plastic strain, and is not just a function of stress and temperature. This aspect of the constitutive equation for elastic strains is essential in describing elastic-plastic deformations of metals. The plastic strains are related to the von Mises yield condition, suitably generalized for finite deformations.

In this particular analysis the plate deformation is assumed to be uniform up to some instant t_0 at which time a flexural perturbation of the form $w_n(t) \sin(n\pi\theta_1/L)$ is introduced, where θ_1 is a convected coordinate. The lateral inertia effects produced by the flexural motion $w_n(t)$ are important and are included in the equation of motion. At this instant it is assumed that some plastic straining has occurred, although it need not necessarily be large. The response (growth with time) of w_n is examined for a range of values of n . It is possible to find a particular n which shows a strong preference to amplify and this value is taken as the theoretical buckling mode. This method has been used successfully in [3] to predict the response of cylindrical shells under axial impact.

2. Constitutive equation for elastic strains. The general theory of an elastic-plastic continuum in [4] is now followed in deriving constitutive equations for isothermal deformations of metals in a state of plane stress. One of the principal stresses is then always zero, restricting the "elastic" part of the strain, the dilatation in particular, to be small. The notation used in [4] is followed where appropriate. The original coordinates of a particle referred to fixed rectangular cartesian axes are X_K . As the motion proceeds, the coordinates X_K define a convected curvilinear coordinate system. We denote the covariant metric tensor referred to the convected coordinate system X_K by g_{KL} . For continuing motion, the components of g_{KL} depend on time t . The strain tensor may be defined by

$$g_{KL} = \delta_{KL} + 2e_{KL}, \quad (2.1)$$

where e_{KL} is the covariant strain tensor referred to the convected coordinates X_K .

The total strain is split into two parts, a symmetric plastic strain tensor e'_{KL} , and an "elastic" strain tensor e''_{KL} such that

$$e_{KL} = e'_{KL} + e''_{KL}. \quad (2.2)$$

Stress is specified by the symmetric contravariant Kirchhoff stress tensor, which is denoted by S^{KL} when referred to the coordinates X_K .

When the general theory is restricted to isothermal deformation, the part e'_{KL} of the total strain depends on S^{MN} and e''_{MN} . The particular constitutive equation adopted here is

$$Ee'_{KL} = [(1 + \nu)g'_{KM}g'_{LN} - \nu g'_{KL}g'_{MN}]S^{MN}, \quad (2.3)$$

where

$$g'_{KL} = \delta_{KL} + 2e''_{KL}. \quad (2.4)$$

The covariant metric tensor g_{KL} referred to the coordinates X_K may then be written

$$g_{KL} = g'_{KL} + 2e'_{KL}. \quad (2.5)$$

The constitutive relation (2.3) takes account of the result established in [4] that during unloading the strain tensor depends on the existing plastic strain. If the body can be unloaded without residual stresses arising or reverse plastic flow occurring, then g'_{KL} becomes the covariant metric tensor referred to the convected coordinate system X_K in the unloaded state, since the strains e'_{KL} go to zero with the stresses. During reloading from this unloaded state, (2.3) describes elastic strain according to Hooke's law for a linear isotropic material, E and ν being Young's modulus and Poisson's ratio respectively (compare Eq. (5.4.32) in [6]). This constitutive relation implies that these elastic constants are not altered by plastic deformations. In particular, under uniaxial loading as in a standard tension or compression test, the slope of the physical stress (force/current area) versus logarithmic strain curve during unloading is equal to the constant value E , at least for the small reversible strains that occur in metals (Fig. 1).

The constitutive equation for the strain rates \dot{e}'_{KL} , where the superposed dot designates differentiation with respect to time holding the convected coordinates X_K constant,

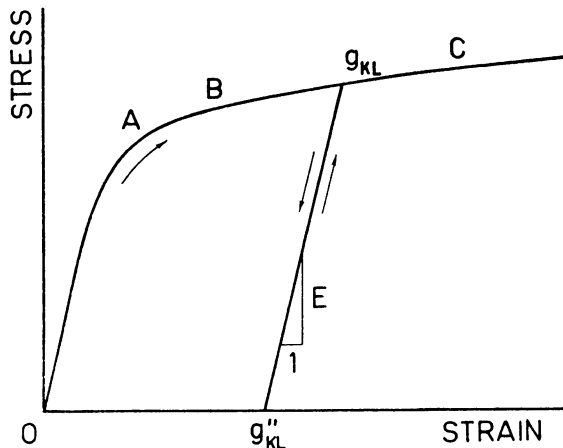


Fig. 1. Physical stress versus logarithmic strain.

is obtained by differentiating (2.3). Since continuing plastic flow is envisaged, the tensor g'_{KL} depends on time through the plastic strains e'_{KL} . Hence, differentiating (2.3) with respect to time yields

$$Ee'_{KL} = [(1 + \nu)(\dot{g}'_{KM}g'_{LN} + g'_{KM}\dot{g}'_{LN}) - \nu(\dot{g}'_{KL}g'_{MN} + g'_{KL}\dot{g}'_{MN})]S^{MN} + [(1 + \nu)g'_{KM}g'_{LN} - \nu g'_{KL}g'_{MN}]\dot{S}^{MN}. \quad (2.6)$$

The strain rates e'_{KL} are seen to depend on the stress rates \dot{S}^{MN} , as to be expected from the usual elementary form of Hooke's law, and in addition on the plastic strain rates e'_{KL} . When the material is in a neutral state or is unloading, that is, when $(\partial f / \partial S^{MN})\dot{S}^{MN} \leq 0$, for isothermal deformations, then $\dot{g}'_{KL} = 0$, and (2.6) reduces to the usual form of Hooke's law provided g'_{KL} is understood to refer to the convected coordinate system X_K in an unloaded state.

3. Constitutive equation for plastic strains. The constitutive equation for the plastic strain rates is based on an approximation to the von Mises yield condition. The approximation is formulated in terms of the tensor invariant function

$$f = \frac{1}{2}(g'_{KM}g'_{LN} - \frac{1}{3}g'_{KL}g'_{MN})S^{MN}S^{KL}, \quad (3.1)$$

which is taken as the yield function. In view of (2.4), f depends on the plastic strains e'_{KL} as well as S^{KL} . For a rigid-plastic material, g'_{KL} is the current covariant metric tensor referred to the convected coordinate system X_K . A second convected coordinate system can be found for which the metric tensor is currently the Kronecker delta. If, in addition, the material is incompressible, the stress components S^{KL} when referred to this second coordinate system coincide with the usual cartesian components of stress. The correspondence between (3.1) and the usual statement of the von Mises yield condition is apparent, f coinciding with J'_2 , the second invariant of the deviatoric stress tensor. For an elastic-plastic material, a yield condition based on (3.1) is a close approximation to the von Mises yield condition, provided the part e'_{KL} of the total strain is small.

We note that, in terms of S^{KL} , J'_2 may be written¹

$$J'_2 = \frac{1}{2}(\rho/\rho_0)^2(g_{KM}g_{LN} - \frac{1}{3}g_{KL}g_{MN})S^{KL}S^{MN}, \quad (3.2)$$

where ρ is the current density of the material and ρ_0 the initial density. J'_2 does not satisfy the general requirements of form postulated in Sec. 5 of [4] for the yield function f , inasmuch as g_{KL} incorporates dependence on e'_{KL} , from (2.1) and (2.2). Hence J'_2 could not be used for f in the constitutive Eqs. (3.3) and (3.4) to follow, which relate the plastic strain rates e'_{KL} to e'_{MN} , S^{MN} , and \dot{S}^{MN} .

The constitutive equation for the plastic strain rates given in [4] is, for isothermal deformations,

$$e'_{KL} = \lambda \beta_{KL} \frac{\partial f}{\partial S^{MN}} \dot{S}^{MN}, \quad \left(\frac{\partial f}{\partial S^{MN}} \dot{S}^{MN} > 0 \right). \quad (3.3)$$

For f defined as in (3.1), the normality condition

¹ In the infinitesimal theory, c_{KL} , c'_{KL} , and e'_{KL} are all of $O(\epsilon)$, where $O(\epsilon)$ refers to an infinitesimal of first order. Then it is apparent that $g'_{KL} = \delta_{KL} + O(\epsilon)$, $g_{KL} = \delta_{KL} + O(\epsilon)$ and $\rho/\rho_0 = 1 + O(\epsilon)$, and the right-hand sides of (3.1) and (3.2) both reduce to the usual expression for the von Mises yield function when terms of $O(\epsilon)$ are neglected.

$$\beta_{KL} = \partial f / \partial S^{KL} = (g'_{KM} g'_{LN} - \frac{1}{3} g'_{KL} g'_{MN}) S^{MN} \quad (3.4)$$

is a generalization of the Reuss flow rule to arbitrary curvilinear coordinate systems. For an incompressible rigid-plastic material, when (3.4) is referred to rectangular cartesian coordinates, g'_{KL} is replaced by the Kronecker delta and S^{MN} by the cartesian stress tensor. Then β_{KL} is recognizable as the deviatoric stress tensor. Provided the strains e'_{KL} are small in an elastic-plastic material, (3.4) affords a close approximation to the Reuss flow rule.

In the general theory [4], λ is a scalar function of S^{KL} and e'_{KL} when the temperature is constant. A result due to Hill [7, Eq. (30), p. 39] suggests an appropriate expression for λ which relates λ to the slope H (the prime in Hill's notation is dropped) of the equivalent stress/plastic-strain curve. With respect to current rectangular cartesian coordinates, ϵ'_{ij} is the plastic strain rate tensor and σ'_{ij} is the deviatoric stress tensor. When the equivalent stress $\bar{\sigma}$ is replaced by $\sqrt{(3J_2)}$, Hill's relationship for the plastic strain rates may be written

$$\epsilon'_{ij} = \frac{3}{4} \frac{1}{H} \frac{J'_2}{J_2} \sigma'_{ij}. \quad (3.5)$$

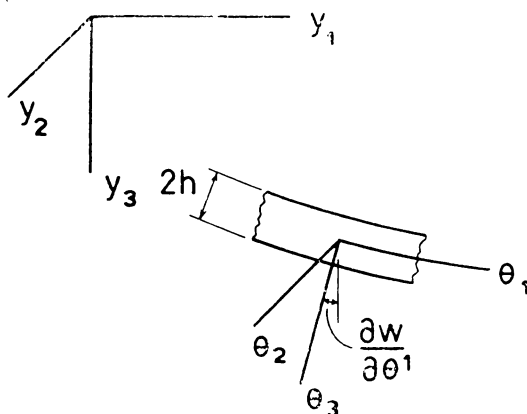
We identify the scalar factor $3/4HJ'_2$ in (3.5) with λ in (3.3). Hence, in (3.3) we now put

$$\lambda = 3/4Hf. \quad (3.6)$$

H can be deduced from the uniaxial stress-strain curve, and is a function of stress. For multiaxial states of stress, since the constitutive theory for the plastic strain rates is based on J'_2 , it would be appropriate to regard H as a function of f . It would not be appropriate in the context of the general theory in [4] for H to depend on a functional of the entire history of the deformation.

4. Motion of the plate. A thin rectangular homogeneous plate is oriented with respect to fixed cartesian axes y_m so that the middle surface of the plate lies in the plane $y_3 = 0$. The edges of the plate are parallel to the y_1 and y_2 axes. The plate is deformed by uniform finite extensions due to compressive stress acting parallel to the y_1 axis. This compressive stress increases continuously with time, and at time t_0 the plate thickness is $2h$ and the length parallel to the y_1 axis is L . The width parallel to the y_2 axis is unspecified but is very much greater than the thickness $2h$. At time t_0 the plate is tested for instability by imposing on it a transverse velocity which varies only with y_1 . The plate bends, the middle surface forming a cylindrical surface whose generator moves parallel to the y_2 axis. During this motion, the bending of the plate superposes a plane strain perturbation on the uniform finite extension of the plate. The growth of the perturbation is followed for a small time interval $t - t_0$.

The method developed in [6, Chap. IV] is now followed in formulating the mathematical description of the motion of the plate. A system of convected coordinates θ_i is chosen by imagining that the plate, deformed by uniform finite extensions, is unloaded at time t_0 . This convected coordinate system is taken to coincide, in the hypothetical unloaded state, with the fixed rectangular cartesian coordinates y_m . This specification of convected coordinates differs somewhat from that used in [6] in that the convected coordinates used there coincide with fixed rectangular cartesian coordinates when the body is in the loaded state. During the motion subsequent to time t_0 , the middle surface of the plate lies in the coordinate surface $\theta_3 = 0$, and the coordinate curves θ_2 are straight parallel to the y_2 axis (Fig. 2).

FIG. 2. Convected coordinate system θ_i .

The transformation relating y_m and θ_i at time $t \geq t_0$ is

$$y_1 = [1 + \epsilon_1(t)]\theta_1 - \theta_3 \partial w / \partial \theta^1, \quad y_2 = [1 + \epsilon_2(t)]\theta_2, \quad y_3 = [1 + \epsilon_3(t)]\theta_3 + w, \quad (4.1)$$

where $w = w(\theta_1, t)$ and $w = 0$ at $t = t_0$. The function w in (4.1) describes the transverse displacement of the middle surface of the plate from the plane $y_3 = 0$. If w remains zero, (4.1) describes continuing uniform extension of the plate, and hence w describes the superposed perturbation on the uniform deformation. Consequently, $\partial w / \partial \theta^1$ is restricted to be small compared to unity. The functions $\epsilon_1(t)$, $\epsilon_2(t)$, $\epsilon_3(t)$ at time t_0 are the uniform elastic strains. For a sufficiently small time interval subsequent to time t_0 , these three functions remain small compared to unity, independent of the magnitude of total strain at time t_0 .

The covariant metric tensor referred to the convected coordinate system θ_i is denoted by G_{ij} , and the covariant strain tensor by γ_{ij} . The strain rates $\dot{\gamma}_{11}$ and $\dot{\gamma}_{22}$ are required later in the constitutive equations and are now derived from (4.1). Since the coordinate curves θ_1 lie in planes $y_2 = \text{constant}$, the differential of arc length ds_1 along these curves is given by

$$ds_1^2 = dy_1^2 + dy_3^2 \quad (4.2)$$

where, from (4.1), for $t \geq t_0$,

$$dy_1 = \left(1 + \epsilon_1 - \theta_3 \frac{\partial^2 w}{\partial \theta^1 \partial \theta^1}\right) d\theta^1 \quad \text{and} \quad dy_3 = \frac{\partial w}{\partial \theta^1} d\theta^1. \quad (4.3)$$

When squares and products of ϵ_1 and $\partial^2 w / \partial \theta^1 \partial \theta^1$ are neglected compared to unity we have

$$G_{11} = \left(\frac{ds_1}{d\theta^1}\right)^2 = 1 + 2\epsilon_1 - 2\theta_3 \frac{\partial^2 w}{\partial \theta^1 \partial \theta^1} \quad (4.4)$$

and

$$\dot{\gamma}_{11} = \frac{1}{2} \dot{G}_{11} = \dot{\epsilon}_1 - \theta_3 \frac{\partial^2 \dot{w}}{\partial \theta^1 \partial \theta^1}. \quad (4.5)$$

Along the coordinate curves θ_2 ,

$$ds_2 = dy_2 = (1 + \epsilon_2) d\theta^2, \quad (4.6)$$

giving

$$G_{22} = (ds_2/d\theta^2)^2 = 1 + 2\epsilon_2 \quad (4.7)$$

and

$$\dot{\gamma}_{22} = \frac{1}{2}\dot{G}_{22} = \dot{\epsilon}_2. \quad (4.8)$$

To the same order of approximation as in (4.4) and (4.7), we find from (4.3) that, along the coordinate curves θ_1 ,

$$dy_3/dy_1 = \partial w/\partial\theta^1, \quad (4.9)$$

and similarly along the coordinate curves θ_3 , that

$$dy_1/dy_3 = -\partial w/\partial\theta^1. \quad (4.10)$$

Hence the coordinate curves θ_1 and θ_3 are orthogonal. The coordinate transformation (4.1) describing the motion of the plate incorporates the usual hypothesis of plate bending; that normals to the middle surface remain normal.

5. Constitutive equations for the perturbed motion. We denote the metric tensor referred to the convected coordinates θ_i in the original unloaded state at time $t = 0$ by G_{ij}^0 . Then

$$G_{ij}'' = G_{ij}^0 + 2\gamma_{ij}''. \quad (5.1)$$

G_{ij}'' and g_{KL}'' in (2.4) are components of the same covariant tensor referred to coordinates θ_i and X_K respectively. From (2.5) it follows that

$$G_{,i} = G_{ij}' + 2\gamma_{ij}'. \quad (5.2)$$

We recall that the convected coordinates θ_i were chosen to coincide with fixed rectangular cartesian coordinates when the uniformly deformed plate is unloaded, hypothetically, at time t_0 . Hence,

$$G_{ij}'' = \delta_{ij} \quad (t = t_0). \quad (5.3)$$

It then follows, from (5.2), (5.3), (4.4), (4.7) and the corresponding equation for G_{33} , and from (2.3), that

$$\gamma_{ij}'(t_0) = \begin{bmatrix} \epsilon_1(t_0) & 0 & 0 \\ 0 & \epsilon_2(t_0) & 0 \\ 0 & 0 & \epsilon_3(t_0) \end{bmatrix} = \frac{P}{E} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

where P is the Kirchhoff compressive stress in the y_1 direction, that is, the compression force acting over unit area of cross-section in the unloaded state obtained by unloading the uniformly deformed plate at time t_0 . Thus it has been confirmed that $\epsilon_1(t_0)$, $\epsilon_2(t_0)$, $\epsilon_3(t_0)$ are the uniform "elastic" strains at time t_0 , referred to the coordinate system θ_i .

A plastic strain increment tensor $\Delta\gamma_{ij}''$ is now introduced, and defined by

$$\Delta\gamma_{ij}''(t) = \gamma_{ij}''(t) - \gamma_{ij}''(t_0), \quad (t \geq t_0). \quad (5.5)$$

Then

$$G_{ij}'' = \delta_{ij} + 2\Delta\gamma_{ij}'', \quad (t \geq t_0). \quad (5.6)$$

At time t_0 , $\Delta\gamma'_{ij} = 0$. Since we are following the motion for a small time interval $t - t_0$, the components of $\Delta\gamma'_{ij}$ are always small compared to unity. It is to be emphasized, however, that the components of the plastic strain tensor γ'_{ij} are not restricted to be small.

The symmetric contravariant Kirchhoff stress tensor referred to the convected coordinates θ_i at time t ($\geq t_0$) is denoted by

$$\tau^{ij} = \begin{bmatrix} (-P + s^{11}) & 0 & 0 \\ 0 & s^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.7)$$

where P is the aforementioned compressive stress in the y_1 direction at time t_0 and s^{11} and s^{22} , which depend on t , are the small changes in the stresses occurring over the small time interval $t - t_0$. This representation admits the usual assumption of thin plate theory: that stresses normal to the middle surface are negligible. Also, for the deformation described by (4.1), the absence of τ^{12} and τ^{13} is a consequence of material isotropy.

The elastic strain rates $\dot{\gamma}'_{11}$ and $\dot{\gamma}'_{22}$ at time t are now found by substituting (5.7) into an equivalent form of (2.6) with g'_{KM} replaced by G'_{ij} as given by (5.6). These substitutions give

$$\begin{aligned} E\dot{\gamma}'_{11} &= [-4P(1 + 2\Delta\gamma'_{11}) + 4s^{11} - 2\nu s^{22}]\dot{\gamma}'_{11} - 2\nu s^{22}\dot{\gamma}'_{22} \\ &\quad + (1 + 4\Delta\gamma'_{11})\dot{s}^{11} - \nu(1 + 2\Delta\gamma'_{11} + 2\Delta\gamma'_{22})\dot{s}^{22}, \\ E\dot{\gamma}'_{22} &= [2\nu P(1 + 2\Delta\gamma'_{22}) - 2\nu s^{11}]\dot{\gamma}'_{11} + [4s^{22} + 2\nu P(1 + 2\Delta\gamma'_{11}) - 2\nu s^{11}]\dot{\gamma}'_{22} \\ &\quad - \nu(1 + 2\Delta\gamma'_{11} + 2\Delta\gamma'_{22})\dot{s}^{11} + (1 + 4\Delta\gamma'_{22})\dot{s}^{22}. \end{aligned} \quad (5.8)$$

In the derivation of (5.8), squares and products of s^{11} and s^{22} , and also of $\Delta\gamma'_{11}$ and $\Delta\gamma'_{22}$, are neglected. Similarly, in all following expressions, these squares and products are neglected, as are terms of order P/E compared to unity.

The plastic strain rates $\dot{\gamma}'_{11}$ and $\dot{\gamma}'_{22}$ are obtained from equivalent forms of (3.3), (3.4), and also from (3.6). In particular, from (3.4), (5.6) and (5.7), we have

$$\begin{aligned} \beta_{11} &= -\frac{2P}{3} \left[1 + 4\Delta\gamma'_{11} - \frac{s^{11}}{P} + \frac{s^{22}}{2P} \right], \\ \beta_{22} &= \frac{P}{3} \left[1 + 2\Delta\gamma'_{11} + 2\Delta\gamma'_{22} - \frac{s^{11}}{P} + \frac{2s^{22}}{P} \right]. \end{aligned} \quad (5.9)$$

Substituting from (5.6) and (5.7) into (3.1) yields

$$f = \frac{P^2}{3} \left[1 + 4\Delta\gamma'_{11} - \frac{2s^{11}}{P} + \frac{s^{22}}{P} \right]. \quad (5.10)$$

Then, for f as given in (5.10), (3.6) becomes

$$\lambda = \frac{9}{4HP^2} \left[1 - 4\Delta\gamma'_{11} + \frac{2s^{11}}{P} - \frac{s^{22}}{P} \right]. \quad (5.11)$$

Finally, substitution from (5.7), (5.9), and (5.11) into (3.3) gives

$$\begin{bmatrix} \dot{\gamma}'_{11} \\ \dot{\gamma}'_{22} \end{bmatrix} = \begin{bmatrix} A'' & B'' \\ B'' & C'' \end{bmatrix} \begin{bmatrix} \dot{s}^{11} \\ \dot{s}^{22} \end{bmatrix} \quad (5.12)$$

where

$$\begin{aligned} A'' &= (1 + 4\Delta\gamma'_{11})/H, \\ B'' &= -(1 + 2\Delta\gamma'_{11} + 2\Delta\gamma'_{22} + 3s^{22}/2P)/2H, \\ C'' &= (1 + 4\Delta\gamma'_{22} + 3s^{22}/P)/4H. \end{aligned}$$

We note that the square matrix comprising the elements A'' , B'' , C'' in (5.12) is singular. Combining (5.8) and (5.12) in (2.2), we have

$$\begin{bmatrix} \dot{\gamma}_{11} \\ \dot{\gamma}_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \dot{s}^{11} \\ \dot{s}^{22} \end{bmatrix} \quad (5.13)$$

where

$$\begin{aligned} A &= A'' + (1 + 4\Delta\gamma'_{11})/E, \\ B &= B'' - \nu(1 + 2\Delta\gamma'_{11} + 2\Delta\gamma'_{22})/E, \\ C &= C'' + (1 + 4\Delta\gamma'_{22})/E. \end{aligned}$$

The terms in (5.8) in $\dot{\gamma}'_{11}$ and $\dot{\gamma}'_{22}$ are neglected in (5.13) since they add contributions to the total strain rates at most of order P/E compared to unity.

6. Differential equations governing material flow. Having established general expressions for the strain rates, we now introduce simplifying assumptions common to thin plate bending theories, namely, that stress and strain variations through the plate thickness are linear. The coordinate transformation (4.1) introduces the condition that total strain varies linearly through the plate thickness. In addition it is now assumed that stress and "plastic" strain vary linearly through the thickness. Accordingly we make the following expansions:

$$\begin{aligned} s^{11} &= a_0 + \theta_3 a_1, & s^{22} &= \theta_3 b_1, \\ \Delta\gamma'_{11} &= c_0 + \theta_3 c_1, & \Delta\gamma'_{22} &= d_0 + \theta_3 d_1. \end{aligned} \quad (6.1)$$

The stress s^{22} does not have a uniform part because there is no resultant in-plane force in the y_2 direction. The functions $a_0(t)$, $c_0(t)$ and $d_0(t)$ describe the conditions in the plate due to the continuing uniform compression and the functions $a_1(t, \theta_1)$, $b_1(t, \theta_1)$, $c_1(t, \theta_1)$ and $d_1(t, \theta_1)$ account for the superimposed perturbation. Hence $|ha_1| \ll |a_0|$, $|hc_1| \ll |c_0|$ and $|hd_1| \ll |d_0|$ in order that there be no unloading; that is, everywhere $(\partial/\partial S^{MN})\dot{S}^{MN} > 0$.

In the present theory the amplitudes of the instabilities that arise from the perturbations are expected to be much less than the plate thickness during the small time interval $t - t_0$. Since the plate is thin and the dominant motion is sustained axial compressive flow, the variation through the plate in the strain-hardening resulting from the instabilities will be negligible. Consequently H is taken as constant through the plate thickness. With this assumption the expansions (6.1) show, through (5.11), that the variation in λ through the plate thickness is also linear. Neglecting the variation in H through the plate thickness still admits a dependence of H upon time; that is, H may change as we proceed along the stress-strain curve. However, for the sake of simplicity, H is now taken as an absolute constant over the small time interval $t - t_0$; different values are later assigned to this constant, corresponding to different values of t_0 when the stability of

the plate is tested. Substituting Eqs. (6.1) into (5.13) and taking a constant value for H leads to two equations that hold for all θ_3 . Partitioning the equations into θ_3 independent and θ_3 dependent sets gives

$$\begin{aligned}\dot{\epsilon}_1 &= \dot{c}_0 + (1 + 4c_0)\dot{a}_0/E, \\ \dot{\epsilon}_2 &= \dot{d}_0 - \nu(1 + 2c_0 + 2d_0)\dot{a}_0/E, \\ -\frac{\partial^2 \dot{w}}{\partial \theta^1 \partial \theta^1} &= \dot{c}_1 + (1 + 4c_0)\dot{a}_1/E + 4c_1\dot{a}_0/E - \nu(1 + 2c_0 + 2d_0)\dot{b}_1/E, \\ 0 &= \dot{d}_1 - \nu(1 + 2c_0 + 2d_0)\dot{a}_1/E - \nu(2c_1 + 2d_1)\dot{a}_0/E + (1 + 4d_0)\dot{b}_1/E,\end{aligned}\tag{6.2a, b, c, d}$$

where $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ are the uniform parts of the total strain rates given by (4.5) and (4.8).

The above four equations are supplemented by two constitutive equations for the plastic strain rates and are obtained from (5.12) and (6.1). Partitioning for θ_3 yields the following four equations:

$$\begin{aligned}\dot{c}_0 &= (1 + 4c_0)\dot{a}_0/H, \\ \dot{d}_0 &= -(1 + 2c_0 + 2d_0)\dot{a}_0/2H, \\ \dot{c}_1 &= (1 + 4c_0)\dot{a}_1/H + 4c_1\dot{a}_0/H - (1 + 2c_0 + 2d_0)\dot{b}_1/2H, \\ \dot{d}_1 &= -(1 + 2c_0 + 2d_0)\dot{a}_1/2H - (2c_1 + 2d_1 + 3b_1/2P)\dot{a}_0/2H \\ &\quad + (1 + 4d_0)\dot{b}_1/4H.\end{aligned}\tag{6.3a, b, c, d}$$

Note that $c_0 = 0$ at $t = t_0$, so that (6.3a) gives $\dot{a}_0 = H\dot{c}_0$, confirming that H is the slope of the uniaxial stress-plastic strain curve.

The solutions of Eqs. (6.2 a, b) and (6.3 a, b) describe the continuing uniform motion of the plate. For example, if $\dot{\epsilon}_1$ is given, we easily obtain from (6.2a) and (6.3a)

$$\dot{c}_0 = \dot{\epsilon}_1 H / (H + E), \quad \dot{a}_0 = H\dot{c}_0 / (1 + 4c_0).\tag{6.4a, b}$$

Eliminating \dot{a}_0 from (6.2a) and (6.2b) and using the binomial theorem gives

$$\dot{c}_0(1 - 2c_0) = 2d_0(1 - 2d_0),$$

which, upon integration, gives

$$c_0(1 - c_0) = -2d_0(1 - d_0).\tag{6.5a}$$

Hence d_0 can be found in terms of ϵ_1 through (6.4a). Finally, using (6.2b) we get

$$\dot{\epsilon}_2 = \dot{d}_0(1 + 2\nu H/E).\tag{6.5b}$$

The perturbation is examined by solving Eqs. (6.2c), (6.2d), (6.3c) and (6.3d). These four equations are not sufficient for finding the five unknowns a_1 , b_1 , c_1 , d_1 and w . One more equation is provided by the equation of motion of the plate. To the order of accuracy of linear plate bending theory, no distinction need be made between the components of the Kirchhoff stress tensor and the physical components of stress, since the dilatation is small, and the convected coordinates are orthogonal and differ only slightly from rectangular cartesian coordinates. Accordingly, the bending moment M per unit length is

$$M = \int_{-h}^h s^{11} \theta_3 \, d\theta_3 = \frac{2h^3}{3} a_1.\tag{6.6}$$

The appropriate equation of motion for a compressed plate is obtained from [8] and is

$$\frac{\partial^2 M}{\partial \theta^1 \partial \theta^1} + 2h(-P + a_0) \frac{\partial^2 w}{\partial \theta^1 \partial \theta^1} - 2h\rho_0 \ddot{w} = 0. \quad (6.7)$$

Eqs. (6.4), (6.5) and (6.7) are sufficient to describe completely the development of the perturbation as time increase from t_0 .

7. Solution for constant axial stress rate. We now let

$$\dot{a}_0 = H\alpha \quad (7.1)$$

where α is a negative constant, since the axial compressive stress is increasing. Integrating (7.1) with respect to time gives $a_0 = H\alpha(t - t_0)$. Letting $\tau = -\alpha(t - t_0)$ it follows that

$$a_0 = -H\tau \quad (7.2)$$

where τ is a positively increasing nondimensional time variable. From (6.3a) we have $\dot{c}_0 = (1 + 4c_0)\alpha$. Since $c_0 \ll 1$, α is approximately the axial strain rate. Upon integration we have $\log(1 + 4c_0) = -4\tau$. Now c_0 is the increase in the axial strain during the time interval $(t - t_0)$ and will at most be a few percent. Hence we may take $\log(1 + 4c_0) = 4c_0 - 8c_0^2$ so that $c_0(1 - 2c_0) = -\tau$; that is,

$$c_0 = -\tau + O(\tau^2). \quad (7.3)$$

From (6.3b) we now get

$$d_0 = \tau/2 + O(\tau^2). \quad (7.4)$$

In the following equations τ^2 and higher powers are neglected in comparison with unity.

An inspection of Eqs. (6.2c), (6.2d), (6.3c), (6.3d), (6.6), and (6.7) shows that a solution exists of the form

$$[a_1, b_1, c_1, d_1, w] = [\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, C] \sin(n\pi\theta_1/L) \quad (7.5)$$

and upon making the substitutions (7.1)–(7.4) they become

$$\begin{aligned} \frac{n^2\pi^2}{L^2} \dot{C} - (1 + 2\nu H/E)\dot{c}_1 + (1 - 2\nu)(4H/E)c_1 - (1 - 2\nu)(1 - 4\tau)\dot{a}_1/E &= 0, \\ d_1 + 2\nu H d_1/E + 2\nu H c_1/E + (1 + 2\tau)\dot{b}_1/E - \nu(1 - \tau)\dot{a}_1/E &= 0, \\ \dot{c}_1 + 4c_1 + (1 - \tau)\dot{b}_1/2H - (1 - 4\tau)\dot{a}_1/H &= 0, \\ (1 + 2\tau)\dot{b}_1/E + 2\nu H c_1/E + \dot{d}_1 + 2\nu H d_1/E - \nu(1 - \tau)\dot{a}_1/E &= 0, \\ -\frac{n^2\pi^2}{L^2} \frac{h^2}{3} a_1 + \frac{n^2\pi^2}{L^2} H(P/H + \tau)C - \rho_0\alpha^2 \ddot{C} &= 0, \end{aligned} \quad (7.6)$$

where dots now denote differentiation with respect to τ and where the bars have been omitted for convenience.

Eqs. (7.6) are best solved numerically. There are five unknowns $a_1(\tau)$, $b_1(\tau)$, $c_1(\tau)$, $d_1(\tau)$ and $C(\tau)$, and Eqs. (7.6) provide five ordinary linear differential equations. Since initial conditions are known or can be specified, direct numerical integration is possible. As previously mentioned, Eqs. (7.6) are to be integrated over a range of τ corresponding to a small time interval $t - t_0$ for a range of values of n . Since a transverse velocity is

given to the uniformly deformed plate at time t_0 , the initial conditions for (7.6) at $\tau = 0$ are

$$\begin{aligned} C(0) &= 0 & \dot{C}(0) &= C_1 \\ a_1(0) &= 0 & b_1(0) &= 0 \\ c_1(0) &= 0 & d_1(0) &= 0. \end{aligned} \tag{7.7}$$

The procedure for finding buckling mode numbers is similar to that used in [2]. The equations are integrated with respect to τ for a range of values of n (C_1 independent of n). As time increases a narrow band of harmonics can be found such that the response of the corresponding amplitudes grows exponentially. In particular, a most responsive mode can be found which is independent of time and this mode is taken as the theoretical buckling mode. The time interval over which the equations are integrated is chosen such that the amplitude of the most responsive mode grows to about one-quarter of the plate thickness.

The first numerical results reported here refer to 10-inch aluminum plates of thickness 0.5 inches and 0.7 inches, for which Young's modulus was taken as 10^7 psi. Three sets of values of P and H were used which correspond to the three points A , B and C on the stress-strain curve, (Fig. 1). These values are

$$\begin{aligned} A: \quad P &= 42,000 \text{ psi}, & H &= 500,000 \text{ psi} \\ B: \quad P &= 45,000 \text{ psi}, & H &= 100,000 \text{ psi} \\ C: \quad P &= 50,000 \text{ psi}, & H &= 50,000 \text{ psi}. \end{aligned}$$

C_1 , the initial value of \dot{C} , was taken as $-1/\alpha$, corresponding to a transverse velocity dC/dt of 1 inch/sec at $t = t_0$ (we recall that the superposed dots in (7.6) and (7.7) denote differentiation with respect to τ , where $\tau = -\alpha(t - t_0)$). The initial conditions (7.7) are somewhat arbitrary. In addition to a transverse velocity, the plate could be given an initial transverse displacement in which case all the initial values in (7.7) could be nonzero. One such set of initial conditions that has been considered is

$$\begin{aligned} C(0) &= 0.01 \text{ inch} & \dot{C}(0) &= 0.01 \text{ inch} \\ a_1(0) &= 5 \text{ psi/inch} & b_1(0) &= 10 \text{ psi/inch} \\ c_1(0) &= 0.0001/\text{inch} & d_1(0) &= 0. \end{aligned} \tag{7.8}$$

For given values of P , H and α , no change was obtained in the preferred harmonic whether initial conditions (7.7) or (7.8) were used. Subsequent calculations were made in which the initial values (7.8) were individually increased by a factor of 100 and it was found that the preferred harmonic changed by at most one. The particular case for $\alpha = -100/\text{sec}$, $2h = 0.7 \text{ in.}$, $P = 42,000 \text{ psi}$, $H = 500,000 \text{ psi}$ and initial conditions (7.7) is illustrated in Fig. 3. The curves show clearly the development of the responsive modes as time increases and are typical of all the cases that were considered.

Results of preferred mode number for three values of α (axial strain rate) are summarized in Table 1. Note that if V is the relative inward velocity of the ends of the plate, then $-\alpha = V/L$. Hence, with $L = 10$ inches it follows that $V = -\alpha/1.2 \text{ ft/sec}$.

The critical loads for the plates, as calculated from Euler's formula for elastic buckling ($n = 1$), are 25,000 psi and 50,000 psi respectively. Consequently the thinner plate will

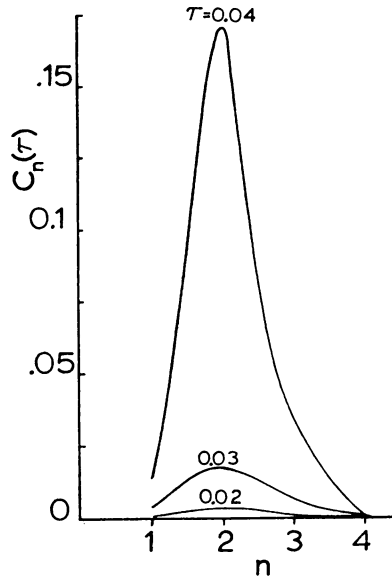


FIG. 3. Growth of preferred harmonic.

buckle elastically as an Euler strut for low values of α and in Table 1 it is not meaningful to include results for $\alpha = -1$. However for the higher values of $|\alpha|$ the effects of lateral inertia are sufficiently great to allow the compressive stress to exceed the elastic limit so that plastic flow occurs. The thicker plate will always allow some plastic deformation to develop even for quasi-static loading since the Euler buckling load exceeds the yield stress of the material.

As the ratio of plate-length/plate-thickness increases the rate of loading must also increase in order that plastic flow may occur. The results of Table 1 indicate that the mode number will also increase. Plates with length/thickness ratios of 80 have been examined by Goodier [2] for relatively high rates of loading. His experimental results are compared in Table 2 with the corresponding results obtained using the present theory.

TABLE 1
Preferred mode number for 10-inch plate.

| $E = 10^7$ psi | $\alpha = -400/\text{sec}$ $2h = 0.5'' \quad 2h = 0.7''$ | $\alpha = -100/\text{sec}$ $2h = 0.5'' \quad 2h = 0.7''$ | $\alpha = -1/\text{sec}$ $2h = 0.5'' \quad 2h = 0.7''$ |
|-------------------------------------|---|---|---|
| $P = 42,000$ psi $H = 500,000$ " | 4 3 | 3 2 | — 1 |
| $P = 45,000$ " $H = 100,000$ " | 4 3 | 3 2 | — 1 |
| $P = 50,000$ " $H = 50,000$ " | 4 3 | 3 2 | — 1 |

TABLE 2

Comparison of Theory and Experiment.

$$L/(2h) = 80, P = 30,000 \text{ psi}, H = 50,000 \text{ psi}, E = 10^7 \text{ psi}$$

Experimental Results obtained from Reference [2], Table 2. Theoretical Results of Present Theory

| Tube Number | Impact Velocity (ft/sec) | Mode Number* | Impact Velocity (ft/sec) | Mode Number |
|-------------|--------------------------|--------------|--------------------------|-------------|
| SAC-2 | 300 | 15 | 300 | 11 |
| SAC-1 | 400 | 10 | 400 | 12 |
| LAC-1 | 184 | 12 | 200 | 11 |
| LAC-2 | 310 | 10 | 300 | 11 |
| LAC-3 | 344 | 10 | 300 | 11 |
| 4CSC-3 | 59 | 8 | 50 | 8 |
| 4CSC-3 | 100 | 10 | 100 | 9 |
| 4CSC-3 | 115 | 10 | 100 | 9 |

* obtained by dividing buckled length by half wavelength.

8. Discussion. Table 1 shows that the predicted mode numbers into which the plates buckle are not affected by the chosen values of P and H . However, this does not imply that the time t_0 at which the perturbation is introduced is not important. It merely means that accurate data for a stress-strain curve are not required and the only material constant required for mode number prediction is the yield stress. This has already been observed by Goodier [2] for rigid-plastic plates. He showed that bending is resisted primarily by a moment independent of the hardening modulus and called it the directional moment. The relative magnitudes of the directional and hardening moments (the latter due to strain hardening) have been examined by Vaughan and Florence [9] for rigid-plastic cylindrical shells. For short shells (corresponding to the plane stress state of the plate considered here) the directional moment was found to be dominant. It is interesting to find the same effect for an elastic-plastic material for which the values of H (point A) were many times higher than those considered in [9].

In the solution of the differential equations, the strain rate α was taken as a constant. This is reasonable since the motion is followed for just a short interval of time. However, in reality α changes with time. In particular, in the experiments reproduced in Table 2 the plates were compressed by a large uniform mass producing almost uniform deceleration in the strain rates. The choice of t_0 then corresponds to different α . However, Table 1 shows that the mode number is highly dependent upon α . The question therefore arises as to which value of α to choose, or equivalently at what stage during the motion to introduce the perturbation. The following argument may be used to obtain an answer which agrees very well with existing experimental evidence.

Since plastic strains are residual in nature, we would expect that the initial buckling mode numbers are the ones that develop throughout the ensuing motion. Thus one would take the value of α which corresponds to the instant t_0 when the load exceeds the yield stress. In most cases this value of α may be taken as the initial value since the change during the elastic deformation will generally be small. This conception certainly agrees with the observations in [1] for cylindrical shells in which the buckled wave form was

selected very early in the motion. More immediate verification is given in Table 2. By choosing α to correspond to the initial impact velocity of the plates, very good agreement has been obtained between the present theory and Goodier's experiments. Any value of α less than this initial value could have been chosen and then the mode numbers would have diminished.

It appears that mode number selection is insensitive to the slope of the stress-strain curve but is sensitive to the rate of loading. Mode number prediction is accurate when the theoretical rate of loading is taken equal to the rate occurring at the inception of plastic flow.

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