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A NOTE ON A PAPER BY SPILLERS AND HICKERSON*

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This note reports an example for which the heuristic algorithm proposed by Spillers and Hickerson [1] for finding an efficient ordering in Gaussian elimination fails to produce an optimal one.

Consider the graph in Fig. 1 and the two orderings 1, 2, 3, 4, 5, 6, 7 and 2, 3, 4, 1, 5, 6, 7. The graphs after the original one thus obtained are given in Figs. 2 and 3 respectively.

FIG. 1. A graph.

(a)

(b)

(c)

(d)

(e)

(f)

FIG. 2. The graphs obtained from the graph in Fig. 1 by the successive elimination of the vertices 1 (a), 2 (b), 3 (c), 4 (d), 5 (e) and 6 (f).

* Received March 3, 1970; revised version received April 12, 1970.

FIG. 3. The graphs obtained from the graph in Fig. 1 by the successive elimination of the vertices 2 (a), 3 (b), 4 (c), 1 (d), 5 (e) and 6 (f).

TABLE I.

Eliminated vertices	d_i degree of the eliminated vertices plus one	$d_i(d_i + 1) - 1$ number of multiplications
1	4	19
2	6	41
3	5	29
4	4	19
5	3	11
6	2	5
7	1	1

TABLE II.

Eliminated vertices	d_i degree of the eliminated vertices plus one	$d_i(d_i + 1) - 1$ number of multiplications
2	5	29
3	5	29
4	5	29
1	4	19
5	3	11
6	2	5
7	1	1

FIG. 4. A family of graphs.

Following [1], let d_i be one plus the degree of the eliminated vertex at the time of its elimination. Then the number of multiplications required at each step for the first ordering is given in Table I for a total of 125. For the second ordering we have the array shown in Table II, for a total of 123. This shows that the first ordering which initiates with the only vertex of degree 3 of the original graph is nonoptimal.

It is also interesting to point out that it is possible to construct a family of graphs $G(X, E)$ for which the difference between the number of multiplications performed with the heuristic algorithm and the optimal one grows with the number of vertices. This family (Fig. 4) is defined by

$$X = \{a\} \cup \{y_1, y_2, \dots, y_N\} \cup \{z_1, z_2, \dots, z_N\},$$

$$E = \left(\bigcup_{i=1}^N (a, y_i) \right) \cup \left(\bigcup_{i,j=1; i \neq j}^N (z_i, z_j) \right) \cup \left(\bigcup_{i,j=1}^N (y_i, z_j) \right).$$

The example in Fig. 1 is a special case for $N = 3$.

These examples have been derived from considerations about nonserial dynamic programming [2]–[7]. The secondary optimization problem of nonserial dynamic programming is very closely related to the problem of finding elimination orders for sparse symmetric systems.

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