A STABILITY CONDITION FOR LINEAR COMPARISON SYSTEMS*

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Introduction. The following class of functions plays a particularly important role in the comparison theorem (vector Liapunov function) alternative to the classical theorems of the direct method of Liapunov.

Definition. A function $f: R^n \rightarrow R^n$ is said to be of type W on a set $S \subset R^n$ if $f_i(a) \leq f_i(b)$ ($i = 1, 2, \dots, n$) for all a, b in S such that $a_i = b_i$, $a_k \leq b_k$ ($i \neq k$).

In the comparison theorems, one attempts to investigate the behavior of solutions to

$$\dot{x} = f(t, x) \quad (x, f \in R^n) \quad (1)$$

by constructing a system of first-order differential inequalities

$$\dot{v}_i(t, x) \leq g_i(t, v(t, x)), \quad i = 1, 2, \dots, m, \quad (2)$$

satisfied by the vector Liapunov function $v = (v_1, v_2, \dots, v_m)$, where g has the important restriction to being of type W . Providing f, g and v possess certain additional properties, it is then possible to conclude that the existence of a solution property (e.g. asymptotic stability) for the comparison system

$$\dot{u} = g(t, u) \quad (3)$$

implies the existence of the corresponding property for (1) (cf. [1]).

Linear Systems of Type W . Since very little is accomplished if the comparison system (3) cannot itself be analyzed, it follows that a most important class of comparison systems are those whose right sides are linear with constant coefficients. The following theorem provides a very useful and convenient condition for the stability of such systems:

THEOREM 1. *Let the right side of the n th order constant coefficient linear system*

$$\dot{u} = Au, \quad A = \|a_{ik}\|_1^n \quad (4)$$

be of type W on R^n . Then (4) will be asymptotically stable if and only if

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, (-1)^n \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} > 0. \quad (5)$$

Proof. The proof follows immediately from a result of Kotelyanskii [2, pp. 71-73]

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which gives the conditions (5) as a necessary and sufficient condition for all the characteristic values of a real matrix A with nonnegative nondiagonal elements to have negative real parts. But it is easy to see that the right side of (4) will be of type W if and only if A satisfies the assumptions of Kotelyanskii's theorem.

As an illustration of the usefulness of Theorem 1, it is possible to provide the following short proof to a theorem given by Bailey in [3] which was established there using a comparatively lengthy root locus type argument. The result itself plays a central role in one of the few results concerned with providing a constructive procedure for obtaining the comparison equation.

THEOREM 2. *The null solution of the system of linear differential equations*

$$\begin{aligned}\dot{x}_1 &= -a_1x_1 + b_1x_n \\ \dot{x}_2 &= -a_2x_2 + b_2x_1 \\ &\vdots \\ \dot{x}_n &= -a_nx_n + b_nx_{n-1}\end{aligned}$$

with a_i and b_i real, $a_i > 0$ and $b_i \geq 0$ ($i = 1, 2, \dots, n$), is asymptotically stable if and only if

$$\prod_{i=1}^n \frac{b_i}{a_i} < 1.$$

Proof. It is clear that the sign of the first $n - 1$ principal diagonal minors of the matrix

$$\begin{bmatrix} -a_1 & 0 & \cdots & 0 & b_1 \\ b_2 & -a_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & b_n & -a_n \end{bmatrix}$$

satisfy the conditions of Theorem 1. The conditions of the theorem are satisfied then if and only if

$$(-1)^n \left[(-a_n) \prod_{i=1}^{n-1} (-a_i) + (-1)^{n+1} b_1 \prod_{i=2}^n (b_i) \right] = \prod_{i=1}^n a_i - \prod_{i=1}^n b_i > 0,$$

that is, if and only if

$$\prod_{i=1}^n \frac{b_i}{a_i} < 1.$$

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