NUMERICAL DETERMINATION OF DISPLACEMENT FIELDS IN ELASTIC DISKS WITH STRESS BOUNDARY CONDITIONS*

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Abstract. An earlier method [5] of establishing consistent finite difference equations is extended to obtain equations of higher degrees of precision. Again, the main advantage over the customary finite element method is that this new technique uses a uniform grid so that the boundary of the disk need not consist of grid segments. Numerous displays of pertinent quantities are given to facilitate the application of the method to specific stress boundary value problems.

1. Introduction. The generalized plane stress problem of two-dimensional elasticity with stress boundary conditions is usually handled via a finite difference treatment using Airy's stress function. This leads to a standard boundary value problem: the stress function is biharmonic in the region of the disk, and its value as well as those of its normal derivative along the edge of the disk are readily obtained from the given edge tractions. However, the values of the stress function at the nodes of a grid have no direct physical meaning. The numerical determination of the displacements from the stress function is awkward, while that of the stresses involves numerical evaluation of second derivatives and therefore requires use of a rather fine grid for reasonable accuracy.

These difficulties may be circumvented and/or minimized within the finite difference technique developed in this paper.

2. Formulation of the problem. To keep the following discussion specific, it is restricted to the problem of generalized plane stress under stress boundary conditions. As is well known, however, the fundamental equations for plane strain under stress boundary conditions have the same form, the only difference being the numerical value of an elastic constant.

Using the terminology of [1], boundary conditions will be classified as *statical* or *kinematical* according to whether they concern components of the surface traction or the surface displacement.

The problem considered in this paper concerns the determination of the displacement field in an elastic disk of constant thickness that is subject to a given equilibrated system of edge tractions, i.e., statical boundary conditions. For simplicity, the disk will be assumed to be simply connected.

The fundamental equations for this problem will be referred to the rectangular

* Received May 22, 1968. This article is based on part of the dissertation *Finite difference methods in elasticity with high degrees of precision* submitted to the University of California, San Diego in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Applied Mechanics.

Cartesian coordinates x_1, x_2 in the median plane of the disk. Lower-case Roman subscripts will have the range 1, 2, and the summation convention of tensor calculus will be used.

If the displacement field in the median plane is denoted by u_i , the strain field is given by

$$e_{ij} = (\partial_i u_j + \partial_j u_i)/2, \quad (1)$$

where $\partial_i = \partial/\partial x_i$ [2]. The stress field σ_{ij} is related to the strain field by Hooke's law:

$$\sigma_{ij} = \frac{E}{1+\nu} \left(e_{ij} + \frac{\nu}{1-\nu} e_{kk} \delta_{ij} \right). \quad (2)$$

Here E is Young's modulus, ν is Poisson's ratio, and δ_{ij} is the unit tensor.

To avoid carrying two independent elastic constants throughout the analysis, we will take Poisson's ratio as $\frac{1}{3}$, which is a representative value for many engineering materials. With the shear modulus $G = E/2(1+\nu)$, Eq. (2) then takes the form

$$\sigma_{ij} = G(2e_{ij} + e_{kk} \delta_{ij}). \quad (2')$$

In the absence of body forces, the equation of equilibrium is

$$\partial_i \sigma_{ij} = 0. \quad (3)$$

Substitution of (2') into (3) and use of (1) furnishes the Navier equation

$$\partial_{ii} u_j + 2\partial_{ij} u_i = 0 \quad (4)$$

or, explicitly,

$$3\partial_{11} u_1 + \partial_{22} u_1 + 2\partial_{12} u_2 = 0, \quad 3\partial_{22} u_2 + \partial_{11} u_2 + 2\partial_{12} u_1 = 0. \quad (5)$$

The partial differential equations (5) for the displacement components u_1 and u_2 can be shown to form an *elliptic* system.

Stress boundary conditions specify the *surface traction* $T_i = n_i \sigma_{ij}$ along the boundary, n_i being the unit vector along the exterior normal to the boundary. In view of (2') and (1), stress boundary conditions involve the first derivatives of the unknown displacement components and are therefore of the Neumann type. By these boundary conditions and the differential equations (5), the displacement field is only determined to within a rigid-body displacement. To eliminate this ambiguity, it is customary to single out the solution for which, say, both displacement components vanish at a point P_1 and the displacement component normal to $P_1 P_2$ vanishes at a second point P_2 .

If the boundary conditions are such that the displacement field can be found in closed form, different choices of P_1 and P_2 yield displacement fields that differ *exactly* by a rigid-body displacement. One would naturally like to meet the same situation when finite difference methods must be used because a closed-form solution is not available. This will, however, not be the case unless special precautions are taken in the derivation of the finite difference equations. A system of finite difference equations that meets this condition will be called *consistent*. It is an advantage of the method described in this paper that it automatically yields consistent finite difference equations.

A finite difference method will be said to have the *degree of precision* [3, p. 101] n if, regardless of the mesh size, it furnishes exact values of the unknown functions at the

grid points provided that the boundary conditions are such that these functions are polynomials in x_1 and x_2 of a joint degree not exceeding n .

3. Finite difference methods based on piecewise displacement fields.

3.1. *Derivation of the difference equation.* Using rectangular Cartesian coordinates x_1 , x_2 in the median plane of the disk, we denote the region covered by the disk by R and its boundary by B , the area element of R by dA , and the arc element of B by ds . To discretize the problem, we cover the x_1 , x_2 -plane by a triangular grid formed by equidistant parallels to the coordinate axes and the diagonals of positive slope in the resulting squares (see Fig. 1). The displacement field of the disk is approximated by a linear combination of special displacement fields, in the spirit of the Ritz method.

However, similar to Pólya's approach for the Laplace equation [4], we use highly localized coordinate (or Rayleigh) functions $u_i^{\alpha P}$ that vanish throughout the entire region R except for the *domain of influence* of node P , i.e. the hexagon formed by the nodes that are connected to P by a grid segment (see Fig. 1). Even within this domain of influence, $u_i^{\alpha P}$ is nonzero only when the displacement component subscript j ($j = 1, 2$) equals the superscript α ($\alpha = 1, 2$).

The case of the coordinate function $u_i^{\alpha P}$ being piecewise linear, leading to piecewise constant strain $e_{ij}^{\alpha P}$ and stress $\sigma_{ij}^{\alpha P}$ fields, has been covered by Tanrikulu and Prager [5] and the resulting equations have the degree of precision 1. We now consider the consequences of assuming $u_i^{\alpha P}$ to be piecewise quadratic in the coordinates x_1 , x_2 .

For the numerical determination of the displacement field of the disk, only those nodes are relevant whose domain of influence has a common area with the region R . Let S be the set of these nodes, which have been circled in Fig. 1, and let N be their number. Denoting the displacement components of the typical node P in S by U^{1P} , U^{2P} , we consider piecewise quadratic displacement fields of the form

$$u_i = \sum_{P \in S} \sum_{\alpha=1}^2 U^{\alpha P} u_i^{\alpha P}. \quad (6)$$

FIG. 1. Region R covered by disk with boundary B . Domain of influence of node P (shaded). Nodes appearing in finite difference equations (circled).

The corresponding piecewise linear fields of strain and stress are

$$e_{ij} = \sum_{P \in S} \sum_{\alpha=1}^2 U^{\alpha P} e_{ij}^{\alpha P}, \quad \sigma_{ij} = \sum_{Q \in S} \sum_{\beta=1}^2 U^{\beta Q} \sigma^{\beta Q}. \quad (7)$$

The unknown displacements at the nodes will now be determined to minimize the potential energy which, in the absence of body forces, is

$$\Pi = \frac{1}{2} \int_R \sigma_{ij} e_{ij} t \, dA - \int_B T_i u_i t \, ds. \quad (8)$$

In view of the principle of minimum potential energy, this procedure will yield the exact displacement field if this is contained in the class (6). Since the only displacement fields of this class that can occur in an elastic disk that is free from body forces are fully quadratic, rather than piecewise quadratic, the procedure will furnish the exact displacement field if the displacement vector is a polynomial in x_1 and x_2 of a joint degree not exceeding 2. The system of finite difference equations that is obtained in this manner therefore has the degree of precision 2.

Substituting (6) and (7) into (8), using Betti's reciprocal identity $e_{ij}^{\alpha P} \sigma_{ij}^{\beta Q} = \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q}$, and differentiating with respect to a given $U^{\alpha P}$, we obtain as a typical minimum condition $\partial \Pi / \partial U^{\alpha P} = 0$ a linear equation for the displacement $U^{\beta Q}$:

$$\sum_{Q \in S} \sum_{\beta=1}^2 U^{\beta Q} \int_R \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} \, dA = \int_B T_i u_i^{\alpha P} \, ds. \quad (\alpha = 1, 2; P \in S). \quad (9)$$

Since there will be an equation of this kind for each of the values $\alpha = 1, 2$ and for each node $P \in S$, we have $2N$ equations for the $2N$ unknown node displacements $U^{\beta Q}$.

The variation of a polynomial of joint degree 2 in x_1 and x_2 along a grid segment may be specified by its values at three points along this segment. This remark suggests that, in addition to the vertices of the triangular mesh (hereafter called *grid points*), we also use the centers of the sides (hereafter called *midway points*). Since the most general quadratic displacement field is of the form

$$\begin{aligned} u_1 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1^2 + a_5 x_2^2, \\ u_2 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2 + b_4 x_1^2 + b_5 x_2^2, \end{aligned} \quad (10)$$

the values of u_1 and u_2 at the six *nodes*, three *grid points* and the three *midway points*, enable us to evaluate the 12 coefficients in (10). Since, moreover, the variations of u_1 and u_2 in (10) along a grid segment are completely determined by the values of these displacement components at the two grid points and the midway point on this segment, continuity of the displacement components from one mesh to the next is assured.

The piecewise quadratic u_1^{1P} component has unit intensity at the node P and vanishes at all other nodes, while u_2^{1P} vanishes identically. Depending on whether the node P is a grid point or a midway point on a slanting grid segment or a midway point on a grid segment in a coordinate direction, we have three kinds of domain of influence as shown in Figs. 2, 3, and 4.

With respect to the coordinate axes shown in these figures and for a grid spacing of 2, the u_1 -fields within the domain of influence are specified by the following expressions:

FIG. 2. a. Domain of influence (shaded) when P is a grid point (circled); midway points are indicated with an "x".
b. Variation of quadratic displacement u along a grid line through P .

Fig. 2.

$$\begin{aligned}
 u_1 &= 1 \mp \frac{3}{2}x_1 + \frac{1}{2}x_1^2 && \text{in } \left\{ \begin{array}{l} \text{1st} \\ \text{5th} \end{array} \right\} \text{ octant,} \\
 u_1 &= 1 \mp \frac{3}{2}x_2 + \frac{1}{2}x_2^2 && \text{in } \left\{ \begin{array}{l} \text{2nd} \\ \text{6th} \end{array} \right\} \text{ octant,} \\
 u_1 &= 1 \pm \frac{3}{2}(x_1 - x_2) + \frac{1}{2}(x_1 - x_2)^2 && \text{in } \left\{ \begin{array}{l} \text{2nd} \\ \text{4th} \end{array} \right\} \text{ quadrant.}
 \end{aligned}$$

Fig. 3.

$$\begin{aligned}
 u_1 &= 2x_2 - x_1x_2 && \text{in 1st octant,} \\
 u_1 &= 2x_1 - x_1x_2 && \text{in 2nd octant.}
 \end{aligned}$$

Fig. 4.

$$\begin{aligned}
 u_1 &= 2(x_1 - x_2) - x_1^2 + x_1x_2 && \text{in 1st octant,} \\
 u_1 &= 2x_1 - x_1^2 + x_1x_2 && \text{in 4th quadrant.}
 \end{aligned}$$

It should be noted that Eqs. (9) are consistent, in the sense defined earlier. This may be shown by considering an arbitrary rigid-body displacement $\bar{u}_i$ with displacement components $\bar{U}^{\alpha P}$ at the node P , so

$$\bar{u}_i = \sum_{P \in S} \sum_{\alpha=1}^2 \bar{U}^{\alpha P} u_i^{\alpha P}. \quad (11)$$

Each equation of the system (9) corresponds to a certain value of α and to a certain node P . Let us now multiply each equation by the corresponding $\bar{U}^{\alpha P}$ and form the

FIG. 3. a. Domain of influence (shaded) when P is a midway point of diagonal grid segment.
b. Variation of quadratic displacement u along the grid line through P .

FIG. 4. a. Domain of influence (shaded) when P is a midway point on grid segment in coordinate direction.
b. Variation of quadratic displacement u along the grid line through P .

sum of the equations obtained in this manner. Using (11) to write the sum of the right-hand side of the multiplied equations as $\int_B T_i \bar{u}_i ds$, we see that this vanishes by the principle of virtual displacements for any rigid-body displacement $\bar{u}_i$. Hence, consistency has been demonstrated.

3.2. *Evaluation of $\int_R \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dA$.* Application of (9) to obtain the set of linear simultaneous equations for the displacements requires evaluation of the integral $\int_R \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dA$, which represents the interaction of the domains of influence of nodes P and Q . P may be a grid point, a midway point on a slanting grid segment, or a midway point on a grid segment in a coordinate direction. The same possibilities exist for Q , and all possible combinations must be discussed in the evaluation of the integral. Moreover, the stresses $\sigma_{ij}^{\alpha P}$ and the strains $e_{ij}^{\beta Q}$ are linear functions of position. This means that the stress field throughout a mesh is determined by the values of the stress component at the vertices of this mesh, and a similar remark applies to the strains. The desired integral over a mesh may therefore be written in terms of the stress and strain components at the vertices. Denoting the vertices of the right triangle with sides of length 2 by V_1, V_2, V_3 , and the area element of the triangular mesh by dS , one finds

$$\int_S \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dS = \frac{1}{6} \left[\sum_{N=1}^3 \sigma_{ij}^{\alpha P}(V_N) e_{ij}^{\beta Q}(V_N) + \sum_{N=1}^3 \sigma_{ij}^{\alpha P}(V_N) \sum_{N=1}^3 e_{ij}^{\beta Q}(V_N) \right]. \quad (12)$$

Using this formula, we construct basic "building blocks" of which there are a set of three (for $\alpha = \beta = 1$; $\alpha \neq \beta$; and $\alpha = \beta = 2$) for P being each of the four types of nodes: a grid point (see Figs. 5, 6, and 7), a midway point on a horizontal grid segment (see

FIG. 5. Values of $(6h^2/G) \int_S \sigma_{ij}^{1P} e_{ij}^{1Q} dS$ when P is a grid point.

FIG. 6. Values of $(6h^2/G) \int_S \sigma_{ij}^{1P} e_{ij}^{2Q} dS = (6h^2/G) \int_S \sigma_{ij}^{2P} e_{ij}^{1Q} dS$ when P is a grid point.

Figs. 8, 9, and 10), a midway point on a slanting grid segment (see Figs. 11, 12, and 13), and a midway point on a vertical grid segment (see Figs. 14, 15, and 16). For each relevant position of Q , the value proportional to $\int_S \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dS$ for a mesh that has Q as a vertex is displayed inside this mesh and close to Q . By keeping the contributions of the individual meshes separate and not presenting just the total result at any node Q , we may use these same basic "building blocks" to handle boundary nodes as well.

From these figures, note that the results for $\alpha = \beta = 2$ may be obtained from the $\alpha = \beta = 1$ values by reflection of the latter values on the line with slope of unity passing through the point P (i.e., Figs. 7, 16, 13, and 10 are obtainable from Figs. 5, 8, 11, and 14, respectively). With this feature in mind, we have minimized the number of figures subsequently presented in this article.

3.3. *Sample applications.* The application of the four separate sets of building blocks to the difference equation (9) may be expected to yield four distinct pairs ($\alpha = 1$ and $\alpha = 2$) of equations. However, when P is an interior node (i.e., its domain of influence lies in R), all three possible midway point positions yield the same result. Therefore, only two distinct pairs of difference equations emerge when P is an interior node: one for P a grid point (Fig. 17) and one for P a midway point (Fig. 18). Note that the equa-

FIG. 7. Values of $(6h^2/G) \int_S \sigma_{ij}^{2P} e_{ij}^{2Q} dS$ when P is a grid point.

FIG. 11. Values of $(3h^2/2G) \int_S \sigma_{ij}^1 e_{ij}^1 dS$ when P is a midway point on a slanting grid segment.

FIG. 12. Values of $(3h^2/2G) \int_S \sigma_{ij}^1 e_{ij}^2 dS = (3h^2/2G) \int_S \sigma_{ij}^2 e_{ij}^1 dS$ when P is a midway point on a slanting grid segment.

FIG. 13. Values of $(3h^2/2G) \int_S \sigma_{ij}^2 e_{ij}^2 dS$ when P is a midway point on a slanting grid segment.

FIG. 14. Values of $(3h^2/2G) \int_S \sigma_{ij}^1 e_{ij}^1 dS$ when P is a midway point on a vertical grid segment.

FIG. 15. Values of $(3h^2/2G) \int_S \sigma_{ij}^1 e_{ij}^2 dS = (3h^2/2G) \int_S \sigma_{ij}^2 e_{ij}^1 dS$ when P is a midway point on a vertical grid segment.

FIG. 16. Values of $(3h^2/2G) \int_S \sigma_{ij}^2 \epsilon_{ij}^2 dS$ when P is a midway point on a vertical grid segment.

tions are homogeneous since P is an interior node and there are no body forces. In Fig. 18, we have not indicated by circles or crosses whether specific nodes are grid or midway points since this information varies depending on which of the three types of midway points P happens to be.

Both Figs. 17 and 18 clearly show that the equations for $\alpha = 2$ are easily obtainable from those for $\alpha = 1$. Because of the symmetry of the stress tensor and Betti's reciprocity theorem, the second block in Figs. 17a and 18a (corresponding to $\alpha = 1, \beta = 2$) is identical to the first block in Figs. 17b and 18b (corresponding to $\alpha = 2, \beta = 1$). As noted previously, the values for $\alpha = \beta = 2$ are obtainable from those for $\alpha = \beta = 1$ so the second block of Figs. 17b and 18b may be obtained from the first block of 17a and 18a by a reflection in the $x_1 = -x_2$ line through point P . Hence all subsequent figures are for $\alpha = 1$ only.

Comparison of Fig. 18a with Fig. 5 of the Tanrikulu and Prager paper [5] reveals that the finite difference equation for P an interior midway point with this piecewise quadratic displacement field analysis is identical with that previously obtained for the piecewise linear displacement field (since the equations are homogeneous, the differing multiplicative factor has already been divided out in Fig. 18). This clearly indicates that, at least for interior nodes, the finite difference equations previously derived are valid

FIG. 17. Finite difference equations when P is an interior grid point.

FIG. 18. Finite difference equations when P is an interior midway point.

for a higher-order displacement field than that for which they were originally constructed. We will return to this point later in Sec. 4.

a. *Boundary along grid lines.* To illustrate the way in which boundary nodes are handled, we consider a node P that lies on a boundary grid line $x_2 = \text{constant}$ of the region R . This means that only the numbers in the lower halves of Figs. 5 and 6 are used to obtain the left-hand side of the difference equation (9) for the value $\alpha = 1$ and the node P . The result is shown in Fig. 19. Of course, the right-hand side of (9) is no longer zero but must be evaluated from the specified tractions T_i .

Similarly, when P is a midway point, we readily obtain the appropriate finite difference equation (for example, see Fig. 20).

b. *Boundary not following grid lines.* A big advantage of this approach is that boundaries do *not* have to follow grid lines. While a triangular element in a plane state of stress with a parabolically varying displacement distribution has already been considered by Fraeijs de Veubeke [6, p. 162] and Argyris [7, p. 27] for the conventional finite element method, that method approximates the boundary with sides of irregular elements, resulting in a mosaic of nonuniform elements. Here, we simply adjust the intensities of the coefficients by a weighting scheme and maintain our uniform net over the irregular region R .

Fig. 21 shows a mesh that has the boundary B cutting through it, forming an

FIG. 19. Finite difference equation when P is a grid point on a boundary B parallel to x_1 -axis ($\alpha = 1$).

FIG. 20. Finite difference equation when P is a midway point on a boundary B parallel to x_2 -axis ($\alpha = 1$).

area $A < 2$. All nodes are numbered and a local coordinate system η_1, η_2 is used. A system of lumped "areas" at the nodes is to be established that is completely equivalent for our purpose to the actual area A .

Since both stresses and strains are linear throughout the mesh, we may readily obtain an expression for $\int_A \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dA$ as a function of η_1 and η_2 in terms of the stresses and strains at the midway point nodes (4, 5, and 6). This expression involves six integrals that represent the area $A = \int dS$, and its first and second order moments such as $\bar{\eta}_1 = \int \eta_1 dS/A$, $I_{\eta_1} = \int \eta_1^2 dS$, $I_{\eta_1 \eta_2} = \int \eta_1 \eta_2 dS$. We now determine the magnitudes of the six lumped "areas" a_k at the six nodes in such a manner that they furnish the same total area and first and second moments. The six equations that uniquely determine the a_k for an arbitrary area A are:

$$\begin{aligned} A &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = \sum_{k=1}^6 a_k, \\ \bar{\eta}_1 A &= a_1 + a_3 + a_4 - a_2, \\ \bar{\eta}_2 A &= a_1 + a_2 + a_5 - a_3, \\ I_{\eta_1} &= a_1 + a_2 + a_3 + a_4, \\ I_{\eta_1} &= a_1 + a_2 + a_3 + a_5, \\ I_{\eta_1 \eta_2} &= a_1 - a_2 - a_3. \end{aligned} \quad (13)$$

Substitution of these expressions into the equation for $\int_A \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dA$ and introduction of the stresses and strains at grid points in some of the terms involving the stresses and strains at the midway points (e.g., $\sigma_{ij}^{\alpha P}|_4 = \frac{1}{2}(\sigma_{ij}^{\alpha P}|_1 + \sigma_{ij}^{\alpha P}|_3)$, etc.) finally lead to the expression

FIG. 21. Mesh with boundary B with reduced area A (shaded) and numbering of nodes.

 a. $\alpha = \beta = 1$.

 b. $\alpha \neq \beta$.

 FIG. 22. $8\sigma_{ij}^{\alpha P} e_{ij}^{\beta Q}/G$ when Q is a grid point coincident with P .

$$\int_A \sigma_{ij}^{\alpha P} e_{ij}^{\beta Q} dA = \sum_{k=1}^6 a_k (\sigma_{ij}^{\alpha P} e_{ij}^{\beta Q})_k. \quad (14)$$

For the special case of A being the full mesh, (14) of course reduces to (12).

To make (14) easily applicable, the values of $\sigma_{ij}^{\alpha P} e_{ij}^{\beta Q}$ at each node for all possible relative positions of P and Q and all values of α and β are displayed in Figs. 22 through 26. (Recall that $\alpha = \beta = 2$ results are obtainable from those for $\alpha = \beta = 1$.)

Using the a_k for a full mesh of width 2 ($a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = a_6 = \frac{2}{3}$) with these displays in (14), we again obtain the basic building blocks shown in Figs. 5 through 7. The material presented earlier in this section is simply a special case of the more general approach being taken now.

 a. $\alpha = \beta = 1$.

 b. $\alpha \neq \beta$.

 FIG. 23. $8\sigma_{ij}^{\alpha P} e_{ij}^{\beta Q}/G$ when Q is a grid point.

FIG. 27. Quarter-circle boundary.

To illustrate the setting up of finite difference equations when the boundary does not follow grid lines, we consider the quarter-circle boundary of Fig. 27 with a radius of 2. The values of the area in the first octant and its first and second moments, referred to its local coordinates (see Fig. 21), are given by

$$A = \pi/2, \quad \bar{\eta}_1 A = 0.7897, \quad \bar{\eta}_2 A = 0.3148, \\ I_{\eta_1} = 0.5795, \quad I_{\eta_2} = 0.3704, \quad I_{\eta_1 \eta_2} = 0.0959.$$

Those for the area sector in the second octant are the same except η_1 and η_2 must be interchanged (due to the role reversal in the local coordinates). Substitution of these values into (13) yields the discrete lumped "areas" for each mesh:

$$\begin{aligned} \text{for sector I: } a_1 &= 0.0186, & a_2 &= 0.0278, & a_3 &= -0.1051, \\ & a_4 &= 0.4291, & a_5 &= 0.6382, & a_6 &= 0.5622, \\ \text{for sector II: } a_1 &= 0.0186, & a_2 &= -0.1051, & a_3 &= 0.0278, \\ & a_4 &= 0.6382, & a_5 &= 0.4291, & a_6 &= 0.5622. \end{aligned}$$

These are shown in Fig. 28.

Using these values as weighting factors for the displays in Figs. 22 through 26, we obtain the partial finite difference expression shown in Fig. 29. Using this with the difference equation shown in Fig. 17a (only now nonhomogeneous and with the factor of $\frac{1}{3}$ restored) leads to the finite difference equation shown in Fig. 30 for $\alpha = 1$.

FIG. 28. Lumped "areas" for quarter-circle boundary.

FIG. 29. Partial finite difference expression for quarter-circle boundary ($\alpha = 1$).

c. *Handling of right-hand side of difference equation.* Up to this point, no mention has been made of how to handle the right-hand side of the difference equation (9), except that it vanishes when P is an interior node (i.e., its domain of influence does not extend beyond the boundary B). We will now indicate how $\int_B T_i u_i^{\alpha P} ds$ may be handled for boundary nodes.

Of course, for a particular boundary B and a particular pair of edge tractions T_i , the integration may be performed directly. For example, for a uniform tensile load of intensity σ in the x_1 -direction for the quarter-circle boundary of Fig. 27 with radius 2, $\int_B T_i u_i^{\alpha P} ds = -0.1370\sigma$. However, it is desirable to have such information in a form in which it may be specified only at discrete nodes so that the set of finite difference equations for the entire grid may be submitted to a computer for simultaneous solution.

By analogy with the preceding development, we set out to determine a system of discrete "masses" that are lumped at the six nodes of a mesh that are completely equivalent to the $\int_B T_i u_i^{\alpha P} ds$ for this mesh (see Fig. 21). Indeed, for $\alpha = 1$ and P at node 4, we have the general quadratic $u_1^{14} = a_2 + a_1 x_1 + a_2 x_2 + \dots$ so that

$$\begin{aligned} \int_B T_i u_i^{14} ds &= \int_B T_1 u_1^{14} ds = \int_B T_1 (a_0 + a_1 x_1 + a_2 x_2 + \dots) ds \\ &= a_0 \int_B T_1 ds + a_1 \int_B T_1 x_1 ds + a_2 \int_B T_1 x_2 ds + \dots \quad (\text{since } u_2^{14} = 0). \end{aligned} \quad (15)$$

Treating T_1 as a line density, we now define a system of masses

FIG. 30. Finite difference equation for quarter-circle boundary ($\alpha = 1$).

$$M = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 = \sum_{k=1}^6 m_k \equiv \int_B T_1 ds,$$

$$M\bar{x}_1 = m_1x_1^1 + m_2x_1^2 + \cdots + m_6x_1^6 = \sum_{k=1}^6 m_kx_1^k \equiv \int_B T_1x_1 ds,$$

(note: superscripts are node numbers, not powers)

$$M\bar{x}_2 = \sum_{k=1}^6 m_kx_2^k \equiv \int_B T_1x_2 ds,$$

$$I_{x_1} = \sum_{k=1}^6 m_k(x_2^k)^2 \equiv \int_B T_1x_2^2 ds,$$

$$I_{x_2} = \sum_{k=1}^6 m_k(x_1^k)^2 \equiv \int_B T_1x_1^2 ds,$$

$$I_{x_1x_2} = \sum_{k=1}^6 m_kx_1^kx_2^k \equiv \int_B T_1x_1x_2 ds.$$

(16)

Substitution of (16) into (15) provides

$$\begin{aligned} \int_B T_1u_1^{14} ds &= a_0m_1 + a_0m_2 + \cdots + a_0m_6 \\ &+ a_1m_1x_1^1 + a_1m_2x_1^2 + \cdots + a_1m_6x_1^6 \\ &+ a_2m_1x_2^1 + a_2m_2x_2^2 + \cdots + a_2m_6x_2^6 + \cdots \\ &= m_1(a_0 + a_1x_1^1 + a_2x_2^1 + \cdots) + m_2(a_0 + a_1x_1^2 + a_2x_2^2 + \cdots) \\ &+ \cdots + m_4(a_0 + a_1x_1^4 + a_2x_2^4 + \cdots) + \cdots \\ &= m_1u_1^{11} + m_2u_1^{12} + \cdots + m_4u_1^{14} + \cdots. \end{aligned}$$

But, by definition (since we are here considering only P at 4),

$$u_1^{11} = u_1^{12} = u_1^{13} = u_1^{15} = u_1^{16} = 0 \quad \text{and} \quad u_1^{14} = 1$$

so $\int_B T_1u_1^{14} ds = m_4$; i.e., the system of lumped masses defined in (16) is such that the value of the mass at each node represents that node's contribution to $\int_B T_1u_i^{\alpha P} ds$ for any T_i and α .

To illustrate the use of this technique, we return to the example of uniform tensile load of σ in the x_1 -direction for the quarter-circle. Here, $T_1 = \sigma \sin \theta$ and $T_2 = 0$ so the masses of mesh I (for $\alpha = 1$) are

$$m_1 = 0.3191\sigma, \quad m_2 = -0.0850\sigma, \quad m_3 = -0.0286\sigma,$$

$$m_4 = 0.7811\sigma, \quad m_5 = 0.1515\sigma, \quad m_6 = 0.2761\sigma.$$

Specification of these intensities at the six nodes of mesh I is the completely equivalent finite difference form for the $\int_B T_1u_i^{1P} ds$ for this mesh.

When the boundary B follows a grid line, it is just a special case of the more general development just presented.

It may be noted in closing this section that substitution of the known exact solution for this problem ($u_1 = (3\sigma/8G)x_1$ and $u_2 = -(\sigma/8G)x_2$) into the left-hand side of the

finite difference equation shown in Fig. 30 yields -0.1370σ , shown above to be the value of the right-hand side ($\int_B T_i u_i^{1P} ds$), thus providing a check on the handling of both sides of the equation.

4. Degrees of precision. It has already been mentioned (Sec. 3.3) that, contrary to what might be expected, the finite difference equations for interior points derived in reference [5] are exactly fulfilled not only by linear but also by quadratic or cubic displacement fields. To indicate this fact, these equations will be said to have the *local* degree of precision 3. In the same sense, the equation displayed in Fig. 6 of reference [5] for a grid point on a boundary that follows grid lines is found to have the local degree of precision 2. More important, however, than these local degrees of precision is the *global* degree of precision of the entire system of finite difference equations. The system discussed in [5] has, in general, the global degree of precision 1, as is to be expected from its derivation. Only in the exceptional case that the entire boundary follows grid lines does this system have the global degree of precision 2. These results and the corresponding ones for the difference equations derived from piecewise quadratic displacement fields are displayed in Table 1.

Note that all finite difference equations derived for interior nodes have the local degree of precision 3. Note also that while the local degree of precision for boundary nodes in the piecewise linear case depends on whether, as regards this node, the boundary follows grid lines or not, this is not the case for the piecewise quadratic analysis where the local degree of precision 2 is obtained for both cases.

We now investigate how good an approximation the piecewise displacement field (6) is to the true field. For the piecewise linear analysis of [5], let u_i^0 , σ_{ii}^0 , and e_{ii}^0 represent the true displacement, stress and strain fields, respectively; u_i^* any piecewise linear displacement field and σ_{ii}^* , e_{ii}^* the piecewise constant stress and strain fields derived

TABLE 1
Degrees of precision of finite difference equations.

Piecewise Degree of $u_i^{\alpha P}$		Type of Node P		Sample Equation	Degree of Precision	
					Local	Global
linear		Interior		Fig. 5. of [5]	3	1
		Boundary	B along grid lines	Fig. 6. of [5]	2	
			B not along grid lines	Fig. 8 of [5]	1	
quadratic	P a grid point	Interior		Fig. 17	3	2
		Boundary		Fig. 19	2	
	P a mid-way point	Interior		Fig. 18	3	
		Boundary		Fig. 20	2	

from u_i^* ; and, finally, u_i that piecewise linear displacement field resulting from the application of the principle of minimum potential energy, with its σ_{ij} and e_{ij} . Because u_i comes from the class u_i^* by application of the principle of minimum potential energy, we have (for the case of no body forces and constant thickness)

$$\frac{1}{2} \int \sigma_{ij}^* e_{ij}^* dA - \int T_i u_i^* ds \geq \frac{1}{2} \int \sigma_{ij} e_{ij} dA - \int T_i u_i ds.$$

Adding $\frac{1}{2} \int \sigma_{ij}^0 e_{ij}^0 dA$ to both sides and using

$$\int T_i u_i^* ds = \int n_i \sigma_{ij}^0 u_i^* ds = \int \partial_i (\sigma_{ij}^0 u_i^*) dA = \int \sigma_{ij}^0 e_{ij}^* dA$$

with Betti's reciprocity theorem, we obtain

$$\frac{1}{2} \int (\sigma_{ij}^* e_{ij}^* - \sigma_{ij}^0 e_{ij}^* + \sigma_{ij}^0 e_{ij}^0 - \sigma_{ij}^* e_{ij}^0) dA \geq \frac{1}{2} \int (\sigma_{ij} e_{ij} - \sigma_{ij}^0 e_{ij} + \sigma_{ij}^0 e_{ij}^0 - \sigma_{ij} e_{ij}^0) dA$$

or

$$\mathcal{E}_u(e_{ij}^* - e_{ij}^0) \geq \mathcal{E}_u(e_{ij} - e_{ij}^0),$$

where $\mathcal{E}_u(\dots)$ is the elastic strain energy corresponding to the strain field displayed in the parenthesis. In other words, the strain energy associated with the error $e_{ij} - e_{ij}^0$ of the strain field e_{ij} is smaller than the strain energy associated with the error $e_{ij}^* - e_{ij}^0$ of the strain field e_{ij}^* . Because of (2'), $\mathcal{E}_u$ is positive definite

$$\begin{aligned} \mathcal{E}_u &= \frac{1}{2} \int_V \sigma_{ij} e_{ij} dV = \frac{t}{2} \int_R \sigma_{ij} e_{ij} dA = \frac{tG}{2} \int_R (2e_{ij} + e_{kk} \delta_{ij}) e_{ij} dA \\ &= \frac{tG}{2} \int (2e_{ij} e_{ij} + e_{ii} e_{jj}) dA \end{aligned}$$

(since the integrand is the sum of squares). Therefore, in the sense that the strain energy associated with the error of the strain field e_{ij} is smaller than that associated with the error of the strain field of any other piecewise linear displacement field, u_i may be said to be the best approximation to u_i^* within the class considered.

The identical argument may be used when u_i is the particular piecewise quadratic displacement field obtained from all piecewise quadratic displacement fields (u_i^*) by applying the principle of minimum potential energy. Hence, the displacement fields considered in [5] and in Sec. 3 of this article are the best possible piecewise linear or quadratic approximations to the true displacement field.

In conclusion, we state that the technique presented in this article could be extended to provide equations of higher degrees of precision. Note, however, that nodes not along grid segments must be included for such higher order analyses. For example, the general cubic displacement field has ten coefficients and therefore requires ten nodes (three at the vertices, two at the one-third points of each of the three sides, and one at the centroid) of the triangular mesh.

It should be noted that this scheme for constructing higher order meshes to connect the values of a polynomial at the nodes with the coefficients of this polynomial may be extended and is never singular. This may be proved by considering the cubic case.

If the system is not singular, then requiring the function to vanish at *all* nodes should

lead to the identically vanishing function, i.e., zero coefficients. Let us write the most general cubic function as

$$u(x_1, x_2) = p_3(x_1) + [p_2(x_1)]x_2 + [p_1(x_1)]x_2(x_2 - 1) + p_0x_2(x_2 - 1)(x_2 - 2) \quad (17)$$

where $p_3(x_1)$ is a cubic polynomial in x_1 , $p_2(x_1)$ is a quadratic polynomial in x_1 , $p_1(x_1)$ is a linear function of x_1 , and p_0 is a constant. Along the line $x_2 = 0$, $u(x_1, 0) = p_3(x_1)$. If this is to vanish at the four nodes on $x_2 = 0$, then $p_3(x_1) \equiv 0$. Hence $p_3(x_1)$ must be deleted from (17). Then, for $x_2 = 1$, $u(x_1, 1) = p_2(x_1)$. If this is to vanish at the three nodes along $x_2 = 1$, then $p_2(x_1) \equiv 0$. Hence, $p_2(x_1)$ must be deleted from (17). Similarly, this type of argument shows that $p_1(x_1)$ and p_0 likewise must vanish so that (17) becomes $u(x_1, x_2) = 0$. Therefore, the system is not singular.

This proof may be extended for polynomials of higher degree, thus showing that the technique is never singular.

Acknowledgments. The author is greatly indebted to Dr. William Prager (University Professor of Engineering and Applied Mathematics at Brown University) for guidance and inspiration and to the National Aeronautics and Space Administration for the Trainee Fellowship under which the author pursued his graduate studies.

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