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THE ERROR IN NUMERICAL INTEGRATION OF ANALYTIC FUNCTIONS*

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1. Introduction. In the numerical calculation of the integral

$$I(f) = \int_a^b f(x)\,dx, \tag{1}$$

by means of a linear quadrature formula

$$Q(f) = \sum_{r=1}^{n} a_i f(x_i); \qquad a_i \text{ real}, \qquad x_i \in [a, b], \qquad i = 1, \cdots, n, \tag{2}$$

when f belongs to a certain Banach space B, it may happen that the error functional E^Q—defined by $E^Q(f) = I(f) - Q(f)$—is bounded; and then an upper bound for the quadrature error may be given in the form

$$|I(f) - Q(f)| \leq ||E^Q|| \cdot ||f|| \tag{3}$$

(where "$||\ \ ||$" stands for the norm in B or in B^*, as appropriate).

Davis [1] introduced the idea of considering the bound (3) in the case that B is the Hardy space H^2 of functions analytic in the unit disk $|z| < 1$ and having finite norm defined by

$$||f||^2 = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \tag{4}$$

(where $f(z) = \sum_{n=0}^{\infty} a_n z^n$). For such f, the radial limit, $\lim_{r\to 1-} f(re^{i\theta})$, exists for almost all θ, and may be taken to define f on the unit circle C_1; f is then in L^2 on the unit circle, and in fact

$$||f||^2 = \int_{C_1} |f(z)|^2\,ds. \tag{4'}$$

In this case B is a Hilbert space, and Davis considered integration over various intervals $[a, b]$ with $-1 < a < b < 1$. It seems more convenient, however, to fix the integration interval as $[-1, 1]$, and to consider functions analytic inside the circle $C_R: |z| = R$, for values of R greater than one. I shall denote this space "$H^2(C_R)$"; if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm is now defined by

$$||f||^2 = 2\pi R \sum_{n=0}^{\infty} |a_n|^2 R^{2n} = \int_{C_R} |f(z)|^2\,ds \tag{4''}$$

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or, equivalently, the inner product is defined by

$$(f, g) = \int_{C_R} f(z)\overline{g(z)}\, ds. \tag{5}$$

Davis pointed out that an advantage of the error bound (3) over the classical error bounds, in this case, is that it does not involve derivatives of the integrand f and so may be much easier to use.

A little while later, Davis and Rabinowitz [2] made use of the spaces $L^2(\mathcal{E}_\rho)$ for the same purpose. $\mathcal{E}_\rho$, for $\rho > 1$, is the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

where

$$a = \tfrac{1}{2}(\rho^{1/2} + \rho^{-1/2}), \qquad b = \tfrac{1}{2}(\rho^{1/2} - \rho^{-1/2}).$$

All these ellipses have their foci at ± 1 on the x axis; if $\rho' < \rho$ then $\mathcal{E}_{\rho'}$ is inside $\mathcal{E}_\rho$, and as ρ decreases to 1, $\mathcal{E}_\rho$ shrinks down to the interval $[-1, 1]$ on the x axis. The advantage of this family of regions is that any function that is analytic on the segment $[-1, 1]$ is necessarily analytic inside $\mathcal{E}_\rho$ for some $\rho > 1$, so that the resulting analysis applies to the numerical integration of any function analytic on the (closed) interval of integration. $L^2(\mathcal{E}_\rho)$ consists of all functions f analytic in the interior D_ρ of $\mathcal{E}_\rho$ and such that

$$\iint_{D_\rho} |f(z)|^2\, dx\, dy < \infty;$$

it is a Hilbert space with inner product

$$(f, g) = \iint_{D_\rho} f(z)\overline{g(z)}\, dx\, dy.$$

In this paper $L^2(\mathcal{E}_\rho)$, like $H^2(C_R)$, will always be used in connection with integration over $[-1, 1]$.

Davis and Rabinowitz, Hämmerlin, and others have given a number of evaluations and bounds for the error norms $\|E^Q\|$, for the space $H^2(C_R)$ and $L^2(\mathcal{E}_\rho)$, and various quadrature formulas Q [1]–[8].

A few years ago Wilf [11] considered the space $H^2(C_1)$ in connection with integration over the interval [0, 1]. This case differs greatly from those mentioned above, in that the closed interval of integration is not contained in the region of analyticity of the functions—integrands are admitted which are not analytic at one endpoint. I shall refer to this case by the symbol "$H^2(C_1; 0, 1)$". The study of integration over $[-1, 1]$ of functions in $H^2(C_1)$ is essentially the same [12].

In all cases a problem of obvious interest is the determination, for each specific number n of quadrature abscissas, of that quadrature formula Q for which $\|E^Q\|$ is as small as possible. There is a unique Q with this property, whenever the function space B is a Hilbert space, or, more generally, when B^* is strictly convex. Such a formula is called a "minimum-norm" formula (relative to the specified number n); we shall denote it by "Q_n^{MN}" and its error functional by "E_n^{MN}". It is defined by

$$|E_n^{\mathrm{MN}}| = \inf_{\substack{a_1, a_2, \cdots, a_n \\ a \le x_1, x_2, \cdots, x_n \le b}} \|E^Q\| \tag{6}$$

where Q is given by (2). The explicit determination of the a_i and x_i of such formulas is quite difficult and has been done only in very few cases [8], [10]. Valentin [13] proved that for the spaces $H^2(C_R)$ the abscissas and coefficients of Q_n^{MN}, for any fixed n, approach those of the n-point Gauss–Legendre quadrature formulas as $R \to \infty$; proof of the same fact for $L^2(\mathcal{E}_\rho)$, as $\rho \to \infty$, can be found in [14]. (For information about the Gaussian quadrature formulas see, e.g., [16, chapter 8].)

2. Bounds on $||E_n^{MN}||$. In some recent papers [11], [14], [15] bounds on $||E_n^{MN}||$, and on the asymptotic size of $||E_n^{MN}||$ as $n \to \infty$, have been derived by the technique of calculation of the norm of the error functional of some convenient n-point quadrature formula Q_n, using the inequality

$$||E_n^{MN}|| \le ||E^{Q_n}|| \tag{7}$$

(which follows from (6)). Stetter [15] used the repeated midpoint (or "Euler's") rule,

$$M_n(f) = \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{n - 2r + 1}{n}\right), \tag{8}$$

for the spaces $H^2(C_R)$ and $L^2(\mathcal{E}_\rho)$, and showed that in both cases

$$||E_n^{MN}|| = O(n^{-2}) \tag{9}$$

for any $R, \rho > 1$. Barnhill [14] found a bound on $||E^{G_n}||$ for $L^2(\mathcal{E}_\rho)$, where G_n is the n-point Gauss–Legendre quadrature formula, and applied it to $||E_n^{MN}||$. For large values of ρ his bound approaches zero exponentially as $n \to \infty$ (i.e., it is $O(c^{-n})$, for some $c > 1$) and thus improves on (9); but for $\rho < e^2$ it approaches infinity as n increases.

For the case $H^2(C_1\,; 0, 1)$, Wilf showed that

$$||E_n^{MN}|| = O(\log n/n)^{1/2}. \tag{10}$$

This stands in considerable contrast to what can be shown in the other cases. I give a slightly better result below, but I do not think any great improvement is possible.

THEOREM 1. *Let D be a simply-connected open set in the complex plane that contains the closed interval $[a, b] = [-1, 1]$ on the x-axis. Let S be a Hilbert space of functions analytic on D with the following property: If D_c is any compact subset of D, there is a positive number $M = M(D_c)$ such that*

$$\max_{z \in D_c} |f(z)| \le M\ ||f|| \quad \text{for all} \quad f \in S. \tag{11}$$

Then if Q is of the form (2), the functional E^Q is bounded; and if

$$\rho = \rho(D) = \sup\,\{\rho' : \mathcal{E}_{\rho'} \subset D\}, \tag{12}$$

and ϵ is any number between 0 and $\rho - 1$, there is a positive number $A = A(\epsilon)$ such that

$$||E_n^{MN}|| \le ||E^{G_n}|| \le A(\rho - \epsilon)^{-n}. \tag{13}$$

(This theorem can be applied to integration over any finite interval $[a, b]$, by a linear change of variables. The interval $[-1, 1]$ was used so as not to complicate the forms of (12) and (13).)

Proof. Since the interval $[-1, 1]$ is a compact subset of D, the assumption (11) immediately implies the boundedness of the functionals I and Q, and so of E^Q. The rest of the theorem follows from a classical result on polynomial approximation: Any $f \in S$ is analytic on and inside $\mathcal{E}_{\rho-\epsilon}$; it then follows from the proof of Theorem 7, p. 76, of [17][1] that there is a constant $A_1 = A_1(\rho, \epsilon)$ such that for any positive integer m there exists a polynomial $P_m(z)$, of degree m, such that

$$\max_{z \in [-1,1]} |f(z) - P_m(z)| \leq A_1(\rho - \epsilon)^{-m/2}\mu, \tag{14}$$

where μ is the maximum of the absolute value of f on $\mathcal{E}_{\rho-\epsilon}$. By (11), there is an M such that $\mu \leq M \,||f||$, so that we can write

$$\max_{z \in [-1,1]} |f(z) - P_m(z)| \leq A_2(\rho - \epsilon)^{-m/2} \,||f||. \tag{15}$$

Now G_n integrates polynomials of degree $2n - 1$ exactly, and the sum of the absolute values of the coefficients in G_n is 2; so that

$$\begin{aligned} |E^{G_n}(f)| &= |E^{G_n}(f - P_{2n-1})| \\ &\leq |I(f - P_{2n-1})| + |G_n(f - P_{2n-1})| \\ &\leq 4A_2(\rho - \epsilon)^{-(2n-1)/2}||f||. \end{aligned}$$

(13) now follows, with $A = 4A_2\rho^{1/2}$.

It is easy to see that the spaces $L^2(\mathcal{E}_\rho)$ and $H^2(C_R)$ satisfy the hypothesis (11). If $f \in L^2(\mathcal{E}_\rho)$, let z_0 be a point such that

$$|f(z_0)| \geq \max_{z \in D_\epsilon} |f(z)|.$$

There is a neighborhood of z_0—of area α, say—in which $|f(z)| \geq \frac{1}{2} |f(z_0)|$. Then

$$||f||^2 \geq \frac{\alpha}{4} |f(z_0)|^2.$$

A similar argument, using the maximum modulus theorem to locate z_0 on C_R, works for $H^2(C_R)$.

$H^2(C_R)$ has the orthonormal basis

$$\varphi_n(z) = (2\pi R)^{-1/2}(z/R)^n, \qquad n = 0, 1, 2, \cdots; \tag{16}$$

$L^2(\mathcal{E}_\rho)$ has the orthonormal basis

$$\varphi_n(z) = \left(\frac{2(n+1)}{\pi}\right)^{1/2}(\rho^{n+1} + \rho^{-n-1})^{-1/2}U_n(z), \qquad n = 0, 1, 2, \cdots, \tag{17}$$

where U_n is the nth degree Chebyshev polynomial of the second kind (see, e.g., [2]). In each case, the functions in the orthonormal basis are real when z is real. For such spaces, Theorem 1 can be given a more specific form. Define the "kernel function" K (using a standard notation) by

$$K(z, \bar{w}) = \sum_{r=0}^{\infty} \varphi_r(z)\overline{\varphi_r(w)}. \tag{18}$$

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(1) Note that in describing ellipses the "ρ" of [17] is the square root of the ρ of the present paper. There is a slight misstatement in the proof in [17]: the function g used there is not necessarily analytic on the ring R, as stated, but is only continuous on R and analytic in the subrings $\rho^{-1} \leq |z| < 1$ and $1 < |z| \leq \rho$. The rest of the proof is not affected.

K is real for real z and w; and we have

$$\|E^Q\|^2 = \sum_{r=0}^{\infty} |E^Q(\varphi_r)|^2 = \sum_{r=0}^{\infty} E^Q_{(z)}(\varphi_r(z))E^Q_{(w)}\overline{(\varphi_r(w))} = E^Q_{(z)}E^Q_{(w)}(K(z, \bar{w})). \tag{19}$$

(Here the subscript "(z)" or "(w)" indicates with respect to which variable the functional is operating.) For $0 < \epsilon < \rho - 1$, let M_ϵ be the maximum of $|K(z, \bar{w})|$ for $z, w \in \mathcal{E}_{\rho-\epsilon}$. By Theorem 4 of [19], it follows that

$$|E^{G_n}_{(w)}(K(z, w))| < \frac{32}{\pi} M_\epsilon(\rho - \epsilon)^{-n}$$

for all z inside $\mathcal{E}_{\rho-\epsilon}$. $E^{G_n}_{(w)}(K(z, \bar{w}))$ is itself analytic in D, and real for real z; applying the same theorem again, we have

$$\|E^{G_n}\|^2 < \left[\frac{32}{\pi}(\rho - \epsilon)^{-n}\right]^2 M_\epsilon$$

and

$$\|E^{\mathrm{MN}}_n\| \leq \|E^{G_n}\| \leq \frac{32}{\pi} M_\epsilon^{1/2}(\rho - \epsilon)^{-n}. \tag{20}$$

For $H^2(C_R)$, the ρ of (12) is equal to $2R^2 - 1 + 2R(R^2 - 1)^{1/2}$, and

$$K(z, \bar{w}) = \frac{R}{2\pi(R^2 - z\bar{w})}, \tag{21}$$

K obviously attains its maximum on $\mathcal{E}_{\rho-\epsilon}$ at

$$z = \bar{w} = \max\{\mathrm{Re}(z) : z \in \mathcal{E}_{\rho-\epsilon}\} = \frac{(\rho - \epsilon)^{1/2} + (\rho - \epsilon)^{-1/2}}{2}.$$

Since $R = (\rho^{1/2} + \rho^{-1/2})/2$, we have that

$$M_\epsilon = \frac{\rho^{1/2} + \rho^{-1/2}}{\pi}\left(\epsilon - \frac{\epsilon}{\rho(\rho - \epsilon)}\right)^{-1} \leq \frac{\rho^{1/2} + \rho^{-1/2}}{\pi}\left(\epsilon - \frac{\epsilon}{\rho}\right)^{-1} = \frac{\epsilon^{-1}\rho^{-1/2}}{\pi}\frac{\rho^{1/2} + \rho^{-1/2}}{\rho^{1/2} - \rho^{-1/2}}. \tag{22}$$

Thus

$$\|E^{G_n}\| \leq 32\pi^{-3/2}\rho^{1/4}\epsilon^{-1/2}\left(\frac{\rho^{1/2} + \rho^{-1/2}}{\rho^{1/2} - \rho^{-1/2}}\right)^{1/2}(\rho - \epsilon)^{-n}$$

This last expression is least when $\epsilon = \rho/(2n + 1)$; if $\rho > 1 + 1/(2n)$ this is an admissible value of ϵ. Since

$$\left(\rho - \frac{\rho}{2n + 1}\right)^{-n} = \rho^{-n}\left(1 + \frac{1}{2n}\right)^n < e^{1/2}\rho^{-n},$$

that value of ϵ gives us the bound

$$||E^{G_n}|| \leq B_1(2n+1)^{1/2}\rho^{-n}$$

where

$$B_1 = 32e^{1/2}\pi^{-3/2}\rho^{-1/4}\left(\frac{\rho^{1/2}+\rho^{-1/2}}{\rho^{1/2}-\rho^{-1/2}}\right)^{1/2}.$$

For $L^2(\mathcal{E}_\rho)$, the ρ of (12) is just ρ, and

$$K(z, \bar{w}) = \frac{4}{\pi}\sum_{r=0}^{\infty}(r+1)\frac{U_r(z)\overline{U_r(w)}}{\rho^{r+1}-\rho^{-r-1}}. \qquad (23)$$

For $z \in [-1, 1]$, $|U_r(z)| \leq r+1$, since then

$$U_r(z) = \frac{\sin[(r+1)\arccos z]}{\sin[\arccos z]};$$

and it follows, by a classical theorem of Berstein [17, page 42] that

$$|U_r(z)| \leq (r+1)(\rho-\epsilon)^{r/2}$$

throughout $\mathcal{E}_{\rho-\epsilon}$. Thus

$$M_\epsilon \leq \frac{4}{\pi}\sum_{r=0}^{\infty}(r+1)^3\frac{(\rho-\epsilon)^r}{\rho^{r+1}-\rho^{-r-1}} < \frac{4}{\pi(\rho-\rho^{-1})}\sum_{r=0}^{\infty}(r+1)^3(1-\epsilon/\rho)^r.$$

Since

$$\sum_{r=0}^{\infty}(r+1)^3x^r = \frac{1+4x+x^2}{(1-x)^4} < \frac{6}{(1-x)^4},$$

for $|x| < 1$, we have

$$M_\epsilon < \frac{24}{\pi(\rho-\rho^{-1})}\rho^4\epsilon^{-4}.$$

We can then minimize with respect to ϵ, as was done above; and in sum we obtain:

THEOREM 2: *For $H^2(C_R)$,*

$$||E_n^{\mathrm{MN}}|| \leq ||E^{G_n}|| \leq B_1(2n+1)^{1/2}\rho^{-n} \qquad (24)$$

if $\rho > 1 + 1/(2n)$, where $\rho = 2R^2 - 1 + 2R(R^2-1)^{1Z2}$, and

$$B_1 = 32e^{1/2}\pi^{-3/2}\rho^{-1/4}\left(\frac{\rho^{1/2}+\rho^{-1/2}}{\rho^{1/2}-\rho^{-1/2}}\right)^{1/2}.$$

For $L^2(\mathcal{E}_\rho)$,

$$||E_n^{\mathrm{MN}}|| \leq ||E^{G_n}|| \leq B_2(n+2)^2\rho^{-n} \qquad (25)$$

if $\rho > 1 + 2/n$, where

$$B_2 = \frac{2^8 6^{1/2}e^2}{\pi^{3/2}(\rho-\rho^{-1})^{1/2}}.$$

3. The case $H^2(C_1; 0, 1)$. If $0 \leq X < 1$, let P_x denote the "point functional" defined by

$$P_x(f) = f(x), \qquad f \in H^2(C_1).$$

Then

$$P_x(f) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-x}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}f(e^{i\theta})}{e^{i\theta}-x}\,d\theta.^{(2)}$$

So, applying the Schwartz inequality,

$$|P_x(f)| \le \frac{1}{2\pi}\left(\int_0^{2\pi} \frac{d\theta}{|e^{i\theta}-x|^2}\right)^{1/2} \cdot \|f\|$$

and it follows that P_x is bounded. Thus quadrature functionals Q of the form (2) (with $[a, b] = [0, 1]$) are bounded, if we exclude the use of the point $x = 1$ as an abscissa in Q. (This must be done in any case, if Q is to be used to integrate arbitrary functions in $H^2(C_1)$, as these are not all well defined at 1.) To show that E^Q is bounded, we show that I is:

$$I(f) = \int_0^1 f(t)\,dt = \lim_{x\to 1-}\int_0^x f(t)\,dt$$

and

$$\int_0^x f(t)\,dt = \frac{1}{2\pi i}\int_0^x \int_{C_1} \frac{f(z)}{z-t}\,dz\,dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}f(e^{i\theta})\int_0^x \frac{dt}{e^{i\theta}-t}\,d\theta.$$

Therefore

$$\left|\int_0^x f(t)\,dt\right| \le \frac{1}{2\pi}\cdot\|f\|\cdot\left(\int_0^{2\pi}\left|\log\frac{e^{i\theta}}{e^{i\theta}-x}\right|^2 d\theta\right)^{1/2}.$$

The last integrand is bounded except in the neighborhood of 0 and of 2π, so we can write (letting "C" denote a generic constant, different in different occurrences, but in each case independent of x)

$$\int_0^{2\pi}\left|\log\frac{e^{i\theta}}{e^{i\theta}-x}\right|^2 d\theta \le C + C\int_0^{\pi/2} |\log(e^{i\theta}-x)|^2\,d\theta$$
$$\le C + C\int_0^{\pi/2} |\log\sin\theta|^2\,d\theta$$
$$\le C + C\int_0^{\pi/2} \log^2\theta\,d\theta \le C^*.$$

Thus

$$\left|\int_0^x f(t)\,dt\right| \le \frac{C^*}{2\pi}\,\|f\|,$$

for all x, and it follows that the functional I is also bounded.

A complete orthonormal sequence for $H^2(C_1)$ is given by (16), with $R = 1$, and $K(z, \bar{w}) = 1/2\pi(1 - z\bar{w})$. If we set

$$F(z) = I_{(w)}(K(z,\bar{w})) = \frac{1}{2\pi z}\log\frac{1}{1-z} \tag{26}$$

(2) For the validity of the Cauchy integral formula for functions in H^2 see, e.g., [21, p. 332].

then $F(z) = \sum_{r=0}^{\infty} I(\varphi_r)\varphi_r(z)$. Noting that

$$I(\varphi_r) = I(\bar{\varphi}_r) = \overline{I(\varphi_r)},$$

and similarly for Q, we may write

$$\begin{aligned} \|E^Q\|^2 &= \sum_{r=0}^{\infty} |E^Q(\varphi_r)|^2 = \sum_{r=0}^{\infty} (I(\varphi_r) - Q(\varphi_r))^2 \\ &= \sum_{r=0}^{\infty} [I(I(\varphi_r)\varphi_r) - Q(I(\varphi_r)\varphi_r)] - \sum_{r=0}^{\infty} Q(\varphi_r)[I(\varphi_r) - Q(\varphi_r)] \qquad (27) \\ &= E^Q(F) - Q_{(z)}E^Q_{(w)}(K). \end{aligned}$$

Here I shall take, for the comparison formula Q, Euler's rule

$$M_n(f) = \frac{1}{n}\sum_{r=0}^{n} f\left(\frac{2r-1}{2n}\right). \qquad (28)$$

To estimate E^{M_n} I shall use the Peano remainder form (see, e.g., [20, p. 108])

$$E^{M_n}(f) = \sum_{r=1}^{n} \int_{r-1/n}^{r/n} f''(t)\left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum{}^{*}\left(\frac{2k-1}{2n} - t\right)\right] dt \equiv \sum_{r=1}^{n} e_r(f) \qquad (29)$$

where the sum Σ^* is over those values of k between 1 and n for which $(2k-1)/2n \geq t$. (The statement of the Peano theorem in [20] assumes that $f^{(n+1)}$—in the present case f''—is continuous on the closed integration interval $[a, b]$. However, the usual proof is valid whenever $f^{(n+1)}$ is continuous on (a, b) and $\int_a^b (b-t)^{n+1} f^{(n+1)}(t)\,dt$ is finite. F and K satisfy these conditions.)

Looking first at e_n, we have

$$\begin{aligned} e_n(F) &= \int_{1-1/n}^{1-1/2n} F''(t)\left[\frac{(1-t)^2}{2} - \frac{1}{n}\left(1 - t - \frac{1}{2n}\right)\right] dt + \int_{1-1/2n}^{1} F''(t)\left[\frac{(1-t)^2}{2}\right] dt \\ &= \int_{1-1/n}^{1} \frac{(1-t)^2}{2} F''(t)\,dt - \frac{1}{n}\int_{1-1/n}^{1-1/2n} (1-t)F''(t)\,dt + \frac{1}{2n^2}\int_{1-1/n}^{1-1/2n} F''(t)\,dt. \end{aligned} \qquad (30)$$

Substituting

$$F''(t) = \frac{1}{2\pi}\left[\frac{1}{t(1-t)^2} - \frac{2}{t^2(1-t)} + \frac{2}{t^3}\log\frac{1}{1-t}\right] \qquad (31)$$

in the last 3 integrals and seeing that each of the last 2 terms in (31) contributes only $O(n^{-2})$ to each of the 3 terms in (30), we obtain

$$\begin{aligned} 2\pi e_n(F) &= \frac{1}{2}\int_{1-1/n}^{1} \frac{dt}{t} - \frac{1}{n}\int_{1-1/n}^{1-1/2n} \frac{dt}{t(1-t)} + \frac{1}{2n^2}\int_{1-1/n}^{1-1/2n} \frac{dt}{t(1-t)^2} + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{2}\int_{1-1/n}^{1} dt - \frac{1}{n}\int_{1-1/n}^{1-1/2n} \frac{dt}{1-t} + \frac{1}{2n^2}\int_{1-1/n}^{1-1/2n} \frac{dt}{(1-t)^2} + O\left(\frac{1}{n^2}\right) \qquad (32) \\ &= \frac{1-\log 2}{n} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

For $r < n$,

$$e_r(F) = \int_{(r-1)/n}^{(r-1/2)/n} F''(t)\left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum_{k=r}^{n}\left(\frac{2k-1}{2n} - t\right)\right] dt$$
$$+ \int_{(r-1/2)/n}^{r/n} F''(t)\left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum_{k=r+1}^{n}\left(\frac{2k-1}{2n} - t\right)\right] dt. \qquad (33)$$

The quantities in brackets in (29), and so in (33), are nonnegative for all t; in the case $N = 1$ this can be determined by inspection, and it follows for all n by the corollary on p. 111 of [20]. Therefore

$$e_r(F) = F''(\xi) \int_{(r-1)/n}^{(r-1/2)/n} \left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum_{k=r}^{n}\left(\frac{2k-1}{2n} - t\right)\right] dt$$
$$+ F''(\xi_2) \int_{(r-1/2)/n}^{r/n} \left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum_{k=r+1}^{n}\left(\frac{2k-1}{2n} - t\right)\right] dt$$
$$= F''(\xi_3) \int_{r-1/n}^{r/n} \left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum{}^{*}\left(\frac{2k-1}{2n} - t\right)\right] dt \qquad (34)$$
$$= \frac{F''(\xi_3)}{24n^3},$$

where ξ_1, ξ_2, and ξ_3 are numbers in $((r-1)/n, r/n)$. Now

$$F(t) = \frac{1}{2\pi}\left(1 + \frac{t}{2} + \frac{t^2}{3} + \cdots\right),$$

so that $F''(t)$ is continuous at $t = 0$, and increasing in t. It follows from (31) that

$$0 \le F''(t) \le C/(1-t)^2$$

for some constant C, and this and (33) imply that

$$|e_n(F)| \le C/24n(n-r)^2, \qquad r = 1, 2, \cdots, n-1.$$

Thus

$$\left|\sum_{r=1}^{n-1} e_n(F)\right| \le \frac{C}{24n}\sum_{r=1}^{n-1}\frac{1}{(n-r)^2} \le \frac{C'}{n};$$

this, together with (32), implies that $|E^{M_n}(F)| = O(1/n)$.

To bound the other term on the right side of (27), we note that for $z \in [0, 1]$

$$E^{M_n}_{(w)}(K(z, \bar{w})) = \frac{1}{2\pi} E^{M_n}_{(t)}\left(\frac{1}{1-zt}\right)$$
$$= \frac{1}{2\pi}\int_0^1 \frac{2z^2}{(1-zt)^3}\left[\frac{(1-t)^2}{2} - \frac{1}{n}\sum{}^{*}\left(\frac{2k-1}{2n} - t\right)\right] dt.$$

Call this last quantity "$g(z)$", for convenience. We have seen that the Peano kernel of M_n—the bracketed quantity in the last integral—is nonnegative and its integral over each interval $(r/n, (r+1)/n)$ is $1/(24n^3)$. Furthermore it is continuous, and its derivative, at each point of differentiability, is < 2 in absolute value. It is therefore $< C/n^2$, for some constant C, for all t in $[0, 1]$. It follows that

$$|g(z)| < \frac{C}{n^2} \int_0^1 \frac{2z^2}{(1 - zt)^3} \, dt = \frac{C}{n^2} \left(\frac{z}{(1 - z)^2} - z \right) < \frac{C}{n^2} \frac{1}{(1 - z)^2}.$$

Then

$$|Q_{(z)} E^{M_n}_{(w)}(K(z, \bar{w}))| = \left| \frac{1}{n} \sum_{r=1}^{n} g\left(\frac{2r - 1}{2n} \right) \right| < \frac{C}{n^3} \sum_{r=1}^{n} \left[1 - \left(\frac{2r - 1}{2n} \right) \right]^{-2}$$

$$< \frac{4C}{n} \sum_{r=1}^{n} (2n - 2r + 1)^{-2} < \frac{C'}{n}.$$

This completes the proof of

THEOREM 3. *In the case of* $H^2(C_1; 0,1,)$,

$$||E_n^{MN}||^2 \leq ||E^{M_n}||^2 = O\left(\frac{1}{n} \right).$$

This looks very weak by comparison with the previous theorems, but I think that it cannot be improved as to order.

REFERENCES

[1] Philip J. Davis, *Errors of numerical approximation for analytic functions,* J. Rational Mech. Anal. 2, 303–313 (1953)

[2] Philip J. Davis and Philip Rabinowitz, *On the estimation of quadrature errors for analytic functions,* MTAC 8, 193–203 (1954)

[3] Günther Hämmerlin, *Über ableitungsfreie Schranken für Quadraturfehler,* Numer. Math. 5, 226–233 (1963)

[4] Günther Hämmerlin, *Über ableitungsfreie Schranken für Quadraturfehler.* II, Numer. Math. 7, 232–237 (1965)

[5] Günther Hämmerlin, *Zur Abschätzung von Quadraturfehlern für analytische Funktionen,* Numer. Math. 8, 334–344 (1966)

[6] Y. T. Lo, S. W. Lee and B. Sun, *On Davis' method of estimating qudrature errors,* Math. Comp .19, 133–138 (1965)

[7] R. E. Barnhill and J. A. Wixom, *Quadratures with remainders of minimum norm.* I, Math. Comp. 21, 66–75 (1967)

[8] R. E. Barnhill and J. A. Wixom, *Quadratures with remainders of minimum norm.* II, Math. Comp. 21, 382–387 (1967)

[9] R. E. Barnhill, *Optimal quadratures in* $L^2(\mathcal{E}_\rho)$. I, SIAM J. Numer. Anal. 4, 390–397 (1967)

[10] R. E. Barnhill, *Optimal quadratures in* $L^2(\mathcal{E}_\rho)$. II, SIAM J. Numer. Anal. 4, 534–541 (1967)

[11] H. S. Wilf, *Exactness conditions in numerical quadrature,* Numer. Math. 6, 315–319 (1964)

[12] H. S. Wilf, *Advances in numerical quadrature,* Mathematical Methods for Digital Computers, vol. II, Wiley, New York, 1967

[13] R. A. Valentin, *Applications of functional analysis to optimal numerical approximation for analytic functions,* Doctoral Thesis, Brown University, Providence, R. I., 1965

[14] R. E. Barnhill, *Asymptotic properties of minimum norm and optimal quadratures,* Numer. Math. 12, 384–393 (1968)

[15] F. Stetter, *On best quadrature of analytic functions,* Quart. Appl. Math. 27, 270–272 (1969)

[16] F. B. Hildebrand, *Introduction to numerical analysis,* McGraw-Hill, New York, 1956

[17] G. G. Lorentz, *Approximation of functions,* Holt, Rinehart and Winston, New York, 1966

[18] N. I. Ahiezer, *Lectures in the theory of approximation,* OGIZ, Moscow, 1947; English transl., Ungar, New York, 1956

[19] F. Stenger, *Bounds on the error of Gauss-type quadratures,* Numer. Math. 8, 150–160 (1966)

[20] P. J. Davis and P. Rabinowitz, *Numerical integration,* Blaisdell, Waltham, Mass., 1967

[21] W. Rudin, *Real and complex analysis,* McGraw-Hill, New York, 1966