SPHERELIKE DEFORMATIONS OF A BALLOON*

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1. Introduction. Let (r, θ, z) be a fixed cylindrical coordinate system and let S measure the arc length along a curve C defined by

$$C : \begin{matrix} r = R(S) \\ z = Z(S) \end{matrix} , \quad \begin{matrix} R(0) = R(1) = 0 \\ Z(0) = 0 \end{matrix} , \quad 0 \leq S \leq 1, \tag{1.1}$$

characterizing the meridian curve of a closed membrane of revolution at its undeformed state. By a closed membrane we mean that the curve C, together with the axis of symmetry, $r = 0$, encloses a simply connected domain.

Suppose the membrane, composed of a Mooney material [2], is inflated by a properly nondimensionalized pressure p. Isaacson [1] has shown that as $p \to \infty$ the shape of the inflated membrane becomes spherical. We develop formal asymptotic series to represent the inflated membrane and find the series are power series in the parameter p^{-2} and that the lowest-order term verifies the formula in [1]. The inflated membrane approaches a spherical surface of radius R_∞ asymptotically as $p \to \infty$, with

$$R_\infty/p = C_1 R_c/8C_2 , \tag{1.2}$$

where C_1 and C_2 are the two constants defining a Mooney material, R_c is the distance between the centroid of C and the axis of symmetry and the pressure p in (1.2) is nondimensionalized by the constant C_1.

We shall say that a closed membrane of revolution defined by the meridian curve C is tubelike if $R_c \ll 1$, otherwise spherelike. Our result shows that the inflated shape of an initially spherelike or tubelike membrane of revolution tends to a spherical surface asymptotically as $p \to \infty$. On the other hand, since (1.2) is only the first term of an asymptotic series of powers of p^{-2}, in order for the assumed asymptotic expansion to be valid the condition $R_c \gg 1/p^2$ must be satisfied. It follows that for an initially spherelike membrane the asymptotic solution is valid for "moderately large" pressures, while for an initially tubelike membrane the asymptotic solution is valid for "very large" pressures.

Thus, from a practical point of view, an initially tubelike membrane may have burst long before the asymptotic solution begins to be valid. Only in this sense do we say that the asymptotic solution presented here is not applicable to initially tubelike membranes of revolution. It appears, however, that a different asymptotic expansion may be obtained for an initially tubelike membrane. This result will be reported in a future paper.

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2. Basic equations and a spherical membrane as a lead. Let the meridian curve of a membrane of revolution be defined by (1.1) where R and Z are continuous but may have discontinuous derivatives. We further assume that

$$R(S) \neq 0, \qquad 0 < S < 1, \tag{2.1}$$

and

$$\begin{aligned} R(S) &= S + O(S^3), \qquad S \to 0, \\ R(S) &= (1 - S) + O(\delta^3), \qquad \delta \equiv (1 - S) \to 0. \end{aligned} \tag{2.2}$$

Conditions (2.2) are needed to ensure the convergence of the solution near the poles. These conditions exclude the possibility that C has infinite curvature* or a cusp at $S = 0$ and 1.

Suppose the membrane is inflated by a dimensionless pressure p, nondimensionalized by the constant C_1 defining a Mooney material [2]. The deformed shape of the membrane can be characterized by a meridian curve

$$c : \begin{matrix} r = X(S) \\ z = Y(S) \end{matrix}, \qquad 0 \leq S \leq 1.$$

If we denote by Λ_1 and Λ_2 the principal extension ratio in the meridional and azimuthal directions, respectively, then

$$\Lambda_1 = dL/dS, \tag{2.4}$$

$$\Lambda_2 = X/R, \tag{2.5}$$

where $L = L(S)$ measures the arc length along the curve c. A Mooney material is characterized by a strain energy function W defined by

$$W(\Lambda_1, \Lambda_2) = \left(\Lambda_1^2 + \Lambda_2^2 + \frac{1}{\Lambda_1^2\Lambda_2^2}\right) + k\left(\Lambda_1^2\Lambda_2^2 + \frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2}\right) \tag{2.6}$$

where $k = C_2/C_1$, the ratio of the two Mooney constants, and W is nondimensionalized by the quantity C_1H, H being the constant thickness of the undeformed membrane.

The fundamental equations can be derived from (2.6) by using the principle of virtual work. We prefer to use the set of equations given in [3]. These are

$$T_1 = (1/\Lambda_2)W_{\Lambda_1}, \tag{2.7}$$

$$T_2 = (1/\Lambda_1)W_{\Lambda_2}, \tag{2.8}$$

$$X(dT_1/dS) = (T_2 - T_1)(dX/dS), \tag{2.9}$$

$$(T_1/\Lambda_1)(d\phi/dS) + (T_2/X)\sin\phi = p, \tag{2.10}$$

$$(1/\Lambda_1)(dX/dS) = \cos\phi, \tag{2.11}$$

$$(1/\Lambda_1)(dY/dS) = \sin\phi, \tag{2.12}$$

where the subscripts on W denote partial differentiation with respect to the indicated argument, T_1 and T_2 are, respectively, meridional and azimuthal stress resultants, and

* The curvature is $R''/(1 - R'^2)^{1/2}$.

ϕ is the angle between the tangent to c and the r-axis. Eqs. (2.4)–(2.12), together with the boundary conditions

$$L(0) = X(0) = X(1) = Y(0) = dT_1(0)/dS = dT_1(1)/dS = 0, \qquad (2.13)^*$$

constitute the complete formulation of the problem.

For the present problem, it is convenient to integrate (2.10) once. Multiplying (2.10) by XdX/dL and using (2.9) and (2.11), we obtain

$$(d/dL)(XT_1 \sin \phi) = pX(dX/dL)$$

and hence

$$T_1 \sin \phi = \tfrac{1}{2}pX \qquad (2.14)$$

by (2.13). We shall replace (2.10) by (2.14). Our objective is to solve the above equations asymptotically for large values of p.

The asymptotic behavior can be readily visualized from the explicit solution of a spherical membrane. To this end we introduce a spherical membrane defined by

$$C : \begin{array}{l} r = R(S) = \dfrac{1}{\pi} \sin \pi S \\ z = Z(S) = \dfrac{1}{\pi}(1 - \cos \pi S) \end{array}, \qquad 0 \le S \le 1. \qquad (2.15)$$

The inflated shape can be characterized by the meridian curve

$$c : \begin{array}{l} r = X(S) = \dfrac{\rho}{\pi} \sin \pi S \\ z = Y(S) = \dfrac{\rho}{\pi}(1 - \cos \pi S) \end{array}, \qquad 0 \le S \le 1, \qquad (2.16)$$

where ρ/π is the deformed radius. The explicit solution is [4]

$$\begin{aligned} \Lambda_1 &= \Lambda_2 = \rho \\ T_1 &= T_2 = 2\left(1 - \frac{1}{\rho^6}\right)(1 + k\rho^2) \\ p &= \frac{4}{\rho}\left(1 - \frac{1}{\rho^6}\right)(1 + k\rho^2). \end{aligned} \qquad (2.17)$$

It follows that, as $p \to \infty$,

$$\begin{aligned} \rho/p = \Lambda_1/p = \Lambda_2/\tfrac{1}{2} = L/p &= (1/4k) - (4/p^2) + O(p^{-4}) \\ T_1/p^2 = T_2/p^2 &= (1/8k) - (2/p^2) + O(p^{-4}). \end{aligned} \qquad (2.18)$$

We note that $k \simeq 0.2$ is satisfied by some materials [4].

3. Asymptotic solution of a closed membrane. Consider now an arbitrary closed membrane of revolution inflated by a large pressure p. Based on (2.18) we write

$$\begin{aligned} \Lambda_1 = p\lambda_1 , \quad \Lambda_2 = p\lambda_2 , \quad T_1 = p^2 t_1 , \quad T_2 = p^2 t_2 , \\ L = pl, \quad X = px, \quad Y = py. \end{aligned} \qquad (3.1)$$

* The last two conditions are derived from the symmetry conditions at the poles (see, e.g., [4]).

(2.6) can now be written as

$$W(\Lambda_1, \Lambda_2) = p^4 w(\lambda_1, \lambda_2, \epsilon) \tag{3.2}$$

where

$$w(\lambda_1, \lambda_2, \epsilon) = k\lambda_1^2\lambda_2^2 + \epsilon(\lambda_1^2 + \lambda_2^2) + \epsilon^3 k\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right) + \epsilon^4 \frac{1}{\lambda_1^2\lambda_2^2} \tag{3.3}$$

and $\epsilon = p^{-2}$ is assumed to be a small parameter. Because of (3.3), we shall consider all the newly introduced quantities in (3.1) and ϕ as functions of S and ϵ and write $f = f(S, \epsilon)$ where f is a generic symbol.

The system of equations (2.4)–(2.13), with (2.9) replaced by (2.14), now becomes

$$\lambda_1 = dl/dS, \tag{3.4}$$

$$\lambda_2 = x/R, \tag{3.5}$$

$$t_1 = (1/\lambda_2)\, w_{\lambda_1}, \tag{3.6}$$

$$t_2 = (1/\lambda_1)\, w_{\lambda_2}, \tag{3.7}$$

$$x(dt_1/ds) = (t_2 - t_1)(dx/dS), \tag{3.8}$$

$$t_1 \sin\phi = \tfrac{1}{2}x, \tag{3.9}$$

$$(1/\lambda_1)\,(dx/dS) = \cos\phi, \tag{3.10}$$

$$(1/\lambda_1)\,(dy/dS) = \sin\phi, \tag{3.11}$$

$$l(0, \epsilon) = x(0, \epsilon) = x(1, \epsilon) = y(0, \epsilon) = \frac{dt_1(0, \epsilon)}{dS} = \frac{dt_1(1, \epsilon)}{dS} = 0. \tag{3.12}$$

We shall assume that the solution can be expanded as an asymptotic series of powers of ϵ and adopt the convenient notation $f_0(S) \equiv f(S, \epsilon)|_{\epsilon=0}$ and $f_n(S) \equiv (\partial^n/\partial\epsilon^n)f(S, \epsilon)|_{\epsilon=0}$.

Letting $\epsilon = 0$, the system of equations (3.4)–(3.12) becomes

$$\lambda_{10} = dl_0/dS, \tag{3.13}$$

$$\lambda_{20} = x_0/R, \tag{3.14}$$

$$t_{10} = 2k\lambda_{10}\lambda_{20}, \tag{3.15}$$

$$t_{20} = 2k\lambda_{10}\lambda_{20}, \tag{3.16}$$

$$x_0(dt_{10}/dS) = (t_{20} - t_{10})(dx_0/dS), \tag{3.17}$$

$$t_{10} \sin\phi_0 = \tfrac{1}{2}x_0, \tag{3.18}$$

$$(1/\lambda_{10})\,(dx_0/dS) = \cos\phi_0, \tag{3.19}$$

$$(1/\lambda_{10})\,(dy_0/dS) = \sin\phi_0, \tag{3.20}$$

$$l_0(0) = x_0(0) = x_0(1) = y_0(0) = \frac{dt_{10}(0)}{dS} = \frac{dt_{10}(1)}{dS} = 0. \tag{3.21}$$

It follows from (3.15), (3.16) and (3.17) that

$$t_{10} = t_{20} = \text{constant}. \tag{3.22}$$

Using (3.14), (3.15) and (3.20), (3.18) becomes

$$dy_0/dS = (1/4k)R(S)$$

and hence

$$y_0 = \frac{1}{4k}\int_0^S R(S')\,dS'. \tag{3.23}$$

Differentiating (3.18), keeping in mind that t_{10} is a constant and using (3.19), we obtain

$$dl_0 = 2t_{10}\,d\phi_0 \tag{3.24}$$

or

$$l_0 = 2t_{10}\phi_0 \tag{3.25}$$

by (3.21). (3.19) and (3.20), together with (3.13), (3.24) and (3.21), yield

$$x_0 = 2t_{10}\sin\phi_0\ , \tag{3.26}$$

$$y_0 = 2t_{10}(1 - \cos\phi_0), \tag{3.27}$$

which, in turn, yield

$$x_0^2 + (y_0 - 2t_{10})^2 = (2t_{10})^2. \tag{3.28}$$

Thus, as $p \to \infty$, the inflated membrane is a spherical surface of radius $2t_{10}$. Eq. (3.28) is essentially the result obtained by Isaacson. We must now determine t_{10}. (3.23) and (3.27) together with (3.21) imply

$$y_0(1) = \frac{1}{4k}\int_0^1 R(S)\,dS = 4t_{10}$$

and hence

$$t_{10} = (1/16k)R_c \tag{3.29}$$

where $R_c = \int_0^1 R(S)\,dS$ is the distance between the centroid of C and the z-axis. (3.29) reduces to (2.18) for an initially spherical membrane.

The function x_0 can now be solved from (3.28) and (3.23):

$$x_0 = \frac{1}{4k}\left[\int_0^S R(S')\,dS' \int_S^1 R(S')\,dS'\right]^{1/2}. \tag{3.31}$$

For future purposes we need to know the behavior of x_0 near $S = 0$ and 1. This can be determined by using (2.2), (3.29) and (3.31). We have

$$\begin{aligned} x_0 &= (t_{10}/2k)^{1/2}S, && S \to 0, \\ x_0 &= (t_{10}/2k)^{1/2}(1 - S), && S \to 1. \end{aligned} \tag{3.32}$$

We complete the zeroth-order solution by obtaining l_0 from (3.13), (3.14), (3.15) and (3.21):

$$l_0 = 2t_{10}\int_0^S R(S')\left[\int_0^{S'} R(S'')\,dS'' \int_{S'}^1 R(S'')\,dS''\right]^{-1/2} dS'. \tag{3.33}$$

It can easily be shown that (3.32) is integrable for $0 \le S \le 1$.

Differentiating (3.4)–(3.12) with respect to ϵ and then setting $\epsilon = 0$, we obtain

$$\lambda_{11} = dl_1/dS, \tag{3.34}$$

$$\lambda_{21} = x_1/R, \tag{3.35}$$

$$t_{11} = 2k(\lambda_{10}\lambda_{21} + \lambda_{11}\lambda_{20}) + 2(\lambda_{10}/\lambda_{20}), \tag{3.36}$$

$$t_{21} = 2k(\lambda_{10}\lambda_{21} + \lambda_{11}\lambda_{20}) + 2(\lambda_{20}/\lambda_{10}), \tag{3.37}$$

$$x_0(dt_{11}/dS) = (t_{21} - t_{11})(dx_0/dS), \tag{3.38}$$

$$t_{11} \sin \phi_0 + t_{10}\phi_1 \cos \phi_0 = \tfrac{1}{2}x_1\,, \tag{3.39}$$

$$-\frac{\lambda_{11}}{\lambda_{10}^2}\frac{dx_0}{dS} + \frac{1}{\lambda_{10}}\frac{dx_1}{dS} = -\phi_1 \sin \phi_0\,, \tag{3.40}$$

$$-\frac{\lambda_{11}}{\lambda_{10}^2}\frac{dy_0}{dS} + \frac{1}{\lambda_{10}}\frac{dy_1}{dS} = \phi_1 \cos \phi_0\,, \tag{3.41}$$

$$l_1(0) = x_1(0) = x_1(1) = y_1(0) = \frac{dt_{11}(0)}{dS} = \frac{dt_{11}(1)}{dS} = 0. \tag{3.42}$$

Substituting (3.36), (3.37) into (3.38) and making other appropriate substitutions we obtain

$$dt_{11}/dS = F(S) \tag{3.43}$$

where

$$F(S) = \frac{2}{R}\left(1 - \frac{y_0}{2t_{10}}\right)\left(1 - \frac{t_{10}^2R^4}{4k^2x_0^4}\right)$$

$$= \frac{2}{R}\left[1 - \frac{2\int_0^S R\,dS'}{\int_0^1 R\,dS}\right]\left[1 - \frac{R^4\left(\int_0^1 R\,dS\right)^2}{4\left(\int_0^S R\,dS'\right)^2\left(\int_S^1 R\,dS'\right)^2}\right]. \tag{3.44}$$

It can easily be shown by using (2.2) that

$$\lim_{S\to 0,1} F(S) = 0.$$

It follows from (2.1) and the above that $F(S)$ is continuous in [0, 1] and $F(0) = F(1) = 0$. Integrating (3.43) yields

$$t_{11} = t_{11}(0) + \int_0^S F(S')\,dS' \tag{3.45}$$

where the constant $t_{11}(0)$ remains to be determined.

We need another form of (3.43) for further deductions. First, (3.13), (3.14), (3.15), (3.19) and (3.24) imply

$$dS = 8kt_{10} \sin \phi_0 \frac{1}{R}\,d\phi_0 \tag{3.46}$$

Substituting (3.36), (3.37) into (3.38) and making use of the known zeroth-order solution and (3.46), we get

$$\sin \phi_0 \frac{dt_{11}}{d\phi_0} = 16kt_{10}\frac{\sin^2 \phi_0 \cos \phi_0}{R^2} - \frac{1}{4kt_{10}}\frac{R^2 \cos \phi_0}{\sin^2 \phi_0}. \tag{3.47}$$

This is the relation needed and is equivalent to (3.43).

(3.35) and (3.36) imply

$$\lambda_{11} = \frac{1}{2k}\frac{t_{11}}{\lambda_{20}} - \frac{1}{k}\frac{\lambda_{10}}{\lambda_{20}^2} - \frac{\lambda_{10}}{\lambda_{20}}\frac{x_1}{R}. \tag{3.48}$$

Substituting (3.48) into (3.40) and solving for ϕ_1, we obtain

$$\phi_1 = -\frac{4k}{R}\frac{dx_1}{dS} - \frac{4k}{Rx_0}\frac{dx_0}{dS}x_1 + \frac{4k}{Rt_{10}}\frac{dx_0}{dS}t_{11} - \frac{4R}{x_0^2}\frac{dx_0}{dS}. \tag{3.49}$$

Substituting (3.49) into (3.39) and making use of the zeroth-order solution repeatedly, we obtain

$$\frac{2k}{R}(y_0 - 2t_{10})\frac{dx_1}{dS} - \frac{2t_{10}^2}{x_0^2}x_1 = -\frac{2t_{10}}{x_0}t_{11} + \frac{8k}{x_0}\left(\frac{dx_0}{dS}\right)^2 \tag{3.50}$$

or, after applying (3.46), (3.26) and (3.27),

$$\frac{\cos\phi_0}{\sin\phi_0}\frac{dx_1}{d\phi_0} + \frac{1}{\sin^2\phi_0}x_1 = \frac{2t_{11}}{\sin\phi_0} - \frac{1}{2kt_{10}}R^2\frac{\cos^2\phi_0}{\sin^3\phi_0}. \tag{3.51}$$

Multiplying (3.51) by $\tan^2\phi_0$, we have

$$\frac{d}{d\phi_0}\left(x_1\tan\phi_0 - \frac{2}{\cos\phi_0}t_{11}\right) = -\frac{2}{\cos\phi_0}\frac{dt_{11}}{d\phi_0} - \frac{1}{2kt_{10}}\frac{R^2}{\sin\phi_0} \tag{3.52}$$

or, after applying (3.47), (3.46), (3.26) and (3.27),

$$d\left(x_1\tan\phi_0 - \frac{2}{\cos\phi_0}t_{11}\right) = G(S)\,dS, \tag{3.53}$$

$$G(S) = \frac{t_{10}^2}{k^2}\frac{R^3}{x_0^4} - \frac{4}{R} - \frac{1}{4k^2}\frac{R^3}{x_0^2}. \tag{3.54}$$

It can easily be shown by using (2.1), (2.2) and (3.32) that $G(S)$ is continuous on [0, 1] and

$$\lim_{S\to 0,1} G(S) = 0. \tag{3.55}$$

Integrating (3.53) and using (3.26) and (3.27) yields

$$x_1 = 4t_{10}\frac{t_{11}}{x_0} - \frac{y_0 - 2t_{10}}{x_0}\int_0^S G(S')\,dS' + c\,\frac{y_0 - 2t_{10}}{x_0} \tag{3.56}$$

where c is a constant to be determined and $t_{11}(S)$ is given by (3.45).

It can easily be shown by using (2.2) that

$$\lim_{S\to 0}\frac{1}{x_0}\int_0^S F(S')\,dS', \qquad \frac{1}{x_0}\int_0^S G(S')\,dS' = 0. \tag{3.57}$$

Thus, to satisfy the condition $x_1(0) = 0$, we must have

$$4t_{10}\frac{t_{11}(0)}{x_0(0)} - c\,\frac{2t_{10}}{x_0(0)} = 0 \quad\text{or}\quad c = 2t_{11}(0). \tag{3.58}$$

The constant $t_{11}(0)$ can now be determined by using the condition $x_1(1) = 0$, viz.

$$t_{11}(0) = \frac{1}{4}\int_0^1 (G - 2F)\,dS. \tag{3.59}$$

Using (3.58) and (3.59), (3.45) and (3.56) become

$$t_{11} = \frac{1}{4}\int_0^1 (G - 2F)\, dS + \int_0^S F(S')\, dS', \tag{3.60}$$

$$x_1 = \frac{4t_{10}t_{11}}{x_0} + \frac{y_0 - 2t_{10}}{x_0}\left[\frac{1}{2}\int_0^1 (G - 2F)\, dS - \int_0^S G(S')\, dS'\right]. \tag{3.61}$$

The function t_{21} can now be solved from (3.38). We have

$$t_{21} = t_{11} - \frac{4kx_0^2F(S)}{R(S)(y_0 - 2t_{10})}. \tag{3.62}$$

It can easily be seen by using (3.44) that $t_{21} = t_{11}$ at the poles and hence the conditions of symmetry are satisfied. (3.44) also shows that the second term in (3.62) is finite at the equator of the asymptotic sphere, i.e. at $y_0 = 2t_{10}$.

Eliminating dx_1/dS from (3.49) and (3.50) and using (3.60), (3.61) and the zeroth-order solution repeatedly, we obtain

$$\phi_1 = -\frac{y_0t_{11}}{x_0t_{10}} + \frac{1}{x_0}\int_0^S [G(S') + 2F(S')]\, dS'. \tag{3.63}$$

It follows from (3.57), (3.60) and (3.63) that ϕ_1 vanishes at $S = 0$ and 1 and hence the conditions of symmetry at the poles are again satisfied.

Eliminating λ_{11} from (3.40) and (3.41), applying (3.49) and (3.50) to remove ϕ_1 and dx_1/dS and using the zeroth-order solution repeatedly, we get

$$dy_1/dS = -R^3/4k^2x_0^2, \tag{3.64}$$

and hence

$$y_1 = -\frac{1}{4k^2}\int_0^S \frac{R^3(S')}{x_0^2(S')}\, dS' \tag{3.65}$$

by (3.42). Finally, the quantity l_1 can be obtained from (3.34) and (3.48). Thus, we have formally completed a two-term asymptotic expansion.

4. Examples: a circular tube with two flat end membranes and the sphere. Consider a closed membrane of revolution characterized by a meridian curve

$$\begin{aligned} C : r = R(S) &= S, & 0 \le S \le \alpha \\ &= \alpha, & \alpha \le S \le 1 - \alpha, \\ &= 1 - S, & 1 - \alpha \le S \le 1 \end{aligned} \tag{4.1}$$

where $0 < \alpha \le \frac{1}{2}$ characterizes the slenderness of the "tubular balloon." The special case $\alpha = \frac{1}{2}$ consists of two flat circular membranes sealed along the circumferences.

We inflate the balloon by a large pressure p and the solution is given by

$$\begin{aligned} T_1(S) &= p^2[t_{10} + p^{-2}t_{11} + 0(p^{-4})], \\ T_2(S) &= p^2[t_{20} + p^{-2}t_{21} + 0(p^{-4})], \\ X(S) &= p[x_0 + p^{-2}x_1 + 0(p^{-4})], \\ Y(S) &= p[y_0 + p^{-2}y_1 + 0(p^{-4})]. \end{aligned} \tag{4.2}$$

Because of the symmetry, $R(S) = R(1 - S)$, we shall give the solution for the interval $0 \le S \le \frac{1}{2}$. The functions involved in the two-term expansion are, for the interval $0 \le S \le \alpha$,

$$t_{10} = t_{20} = \alpha(1 - \alpha)/16k\,, \tag{4.3}$$

$$x_0 = \frac{S}{8k}\,[2\alpha(1-\alpha) - S^2]^{1/2}, \tag{4.4}$$

$$y_0 = (1/8k)\,S^2, \tag{4.5}$$

$$\begin{aligned} x_1 = {} & \left(3 - \frac{1}{\alpha} + 3\ln\frac{1 - \frac{3}{2}\alpha}{1-\alpha} - \frac{\alpha}{1-\alpha}\ln\frac{2-3\alpha}{\alpha}\right)\frac{2S}{[2\alpha(1-\alpha) - S^2]^{1/2}} \\ & + \frac{8\alpha(1-\alpha) - 6S^2}{S[2\alpha(1-\alpha) - S^2]^{1/2}}\ln\frac{2\alpha(1-\alpha) - S^2}{2\alpha(1-\alpha)}\,, \end{aligned} \tag{4.6}$$

$$y_1 = 8\ln\frac{2\alpha(1-\alpha) - S^2}{2\alpha(1-\alpha)}\,, \tag{4.7}$$

$$\begin{aligned} t_{11} = {} & 2 - \frac{1}{\alpha} + 3\ln\frac{1-\frac{3}{2}\alpha}{1-\alpha} - \frac{\alpha}{1-\alpha}\ln\frac{2-3\alpha}{\alpha} - \frac{S^2}{\alpha(1-\alpha)} \\ & + \ln\frac{2\alpha(1-\alpha) - S^2}{2\alpha(1-\alpha)} + \frac{2\alpha(1-\alpha)}{2\alpha(1-\alpha) - S^2}\,, \end{aligned} \tag{4.8}$$

$$t_{21} = t_{11} - \frac{S[2\alpha(1-\alpha) - S^2]}{2[S^2 - \alpha(1-\alpha)]}\left\{-\frac{2S}{\alpha(1-\alpha)} - \frac{2S}{2\alpha(1-\alpha) - S^2} + \frac{4\alpha(1-\alpha)S}{[2\alpha(1-\alpha) - S^2]^2}\right\}, \tag{4.9}$$

and, for the interval $\alpha \le S \le \frac{1}{2}$,

$$t_{10} = t_{20} = \alpha(1-\alpha)/16k \tag{4.10}$$

$$x_0 = \frac{\alpha}{4k}\left\{\left(S - \frac{\alpha}{2}\right)\left[\left(1 - \frac{\alpha}{2}\right) - S\right]\right\}^{1/2} \tag{4.11}$$

$$y_0 = (\alpha/4k)\left(S - \frac{\alpha}{2}\right) \tag{4.12}$$

$$\begin{aligned} x_1 = {} & \frac{1-\alpha}{\left\{\left(S - \frac{\alpha}{2}\right)\left[\left(1-\frac{\alpha}{2}\right) - S\right]\right\}^{1/2}}\left\{2 - \frac{1}{\alpha} - \frac{\alpha}{1-\alpha} + \frac{(1-2\alpha)^2}{2\alpha(1-\alpha)}\right. \\ & \left. + 4\ln\frac{1-\frac{3}{2}\alpha}{1-\alpha} - \frac{\alpha}{1-\alpha}\ln\frac{2-3\alpha}{\alpha} - \frac{2(\frac{1}{2} - S)^2}{\alpha(1-\alpha)} + \frac{\frac{\alpha}{2}}{S - \frac{\alpha}{2}} + \frac{\frac{\alpha}{2}}{\left(1-\frac{\alpha}{2}\right) - S}\right\} \\ & + \frac{S - \frac{1}{2}}{\left\{\left(S-\frac{\alpha}{2}\right)\left[\left(1-\frac{\alpha}{2}\right) - S\right]\right\}^{1/2}}\left\{-\frac{2}{\alpha} + \frac{4S}{\alpha} - \frac{2\alpha}{1-\alpha}\ln\frac{\left(1-\frac{\alpha}{2}\right) - S}{S - \frac{\alpha}{2}}\right. \\ & \left. + \frac{\alpha}{S - \frac{\alpha}{2}} - \frac{\alpha}{\left(1-\frac{\alpha}{2}\right) - S}\right\}, \end{aligned} \tag{4.13}$$

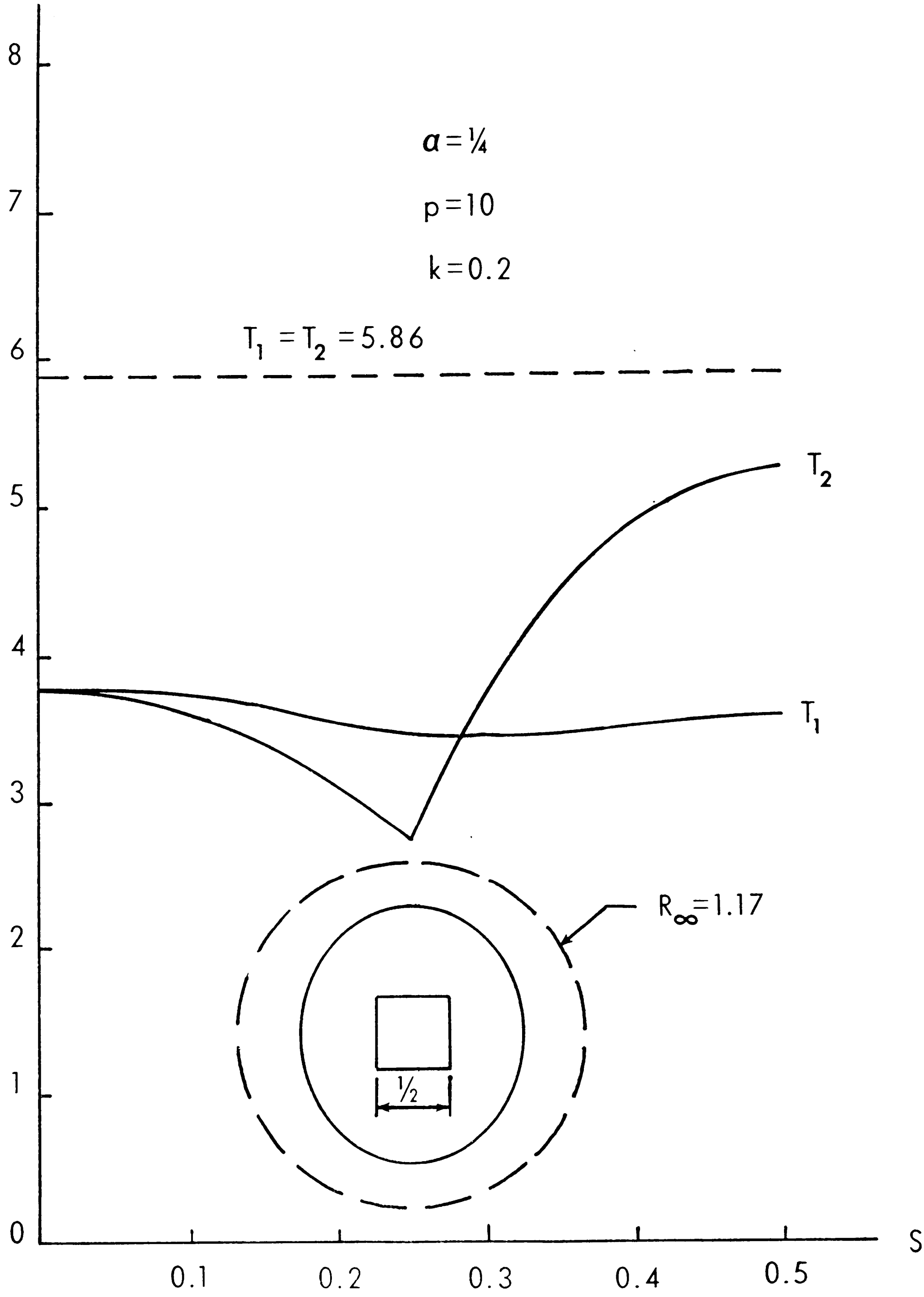

FIG. 1. A tubular balloon with the height equal to the diameter.

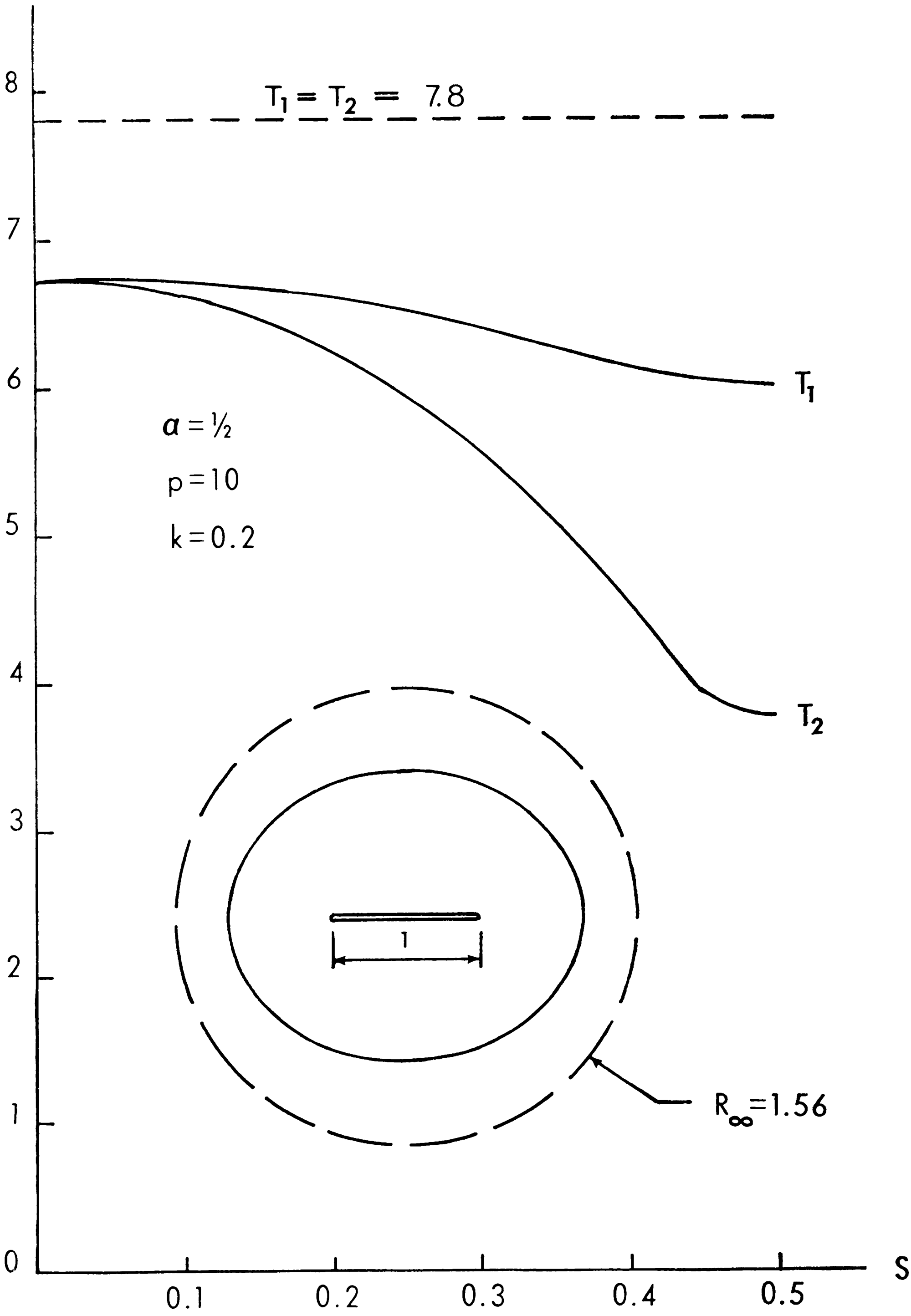

FIG. 2. A balloon made of two flat membranes sealed along the edges.

$$y_1 = 8 \ln \frac{1 - \frac{3}{2}\alpha}{1 - \alpha} - \frac{4\alpha}{1 - \alpha} \ln \frac{S - \frac{\alpha}{2}}{\frac{\alpha}{2}} - \ln \frac{\left(1 - \frac{\alpha}{2}\right) - S}{(1 - \frac{3}{2}\alpha)}, \tag{4.14}$$

$$t_{11} = 2 - \frac{1}{\alpha} - \frac{\alpha}{1 - \alpha} + \frac{(1 - 2\alpha)^2}{2\alpha(1 - \alpha)} + 4 \ln \frac{1 - \frac{3}{2}\alpha}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \ln \frac{2 - 3\alpha}{\alpha}$$

$$- \frac{2(\frac{1}{2} - S)^2}{\alpha(1 - \alpha)} + \frac{\frac{\alpha}{2}}{S - \frac{\alpha}{2}} + \frac{\frac{\alpha}{2}}{\left(1 - \frac{\alpha}{2}\right) - S}, \tag{4.15}$$

$$t_{21} = t_{11} + \frac{4}{\alpha(1 - \alpha)} \left(S - \frac{\alpha}{2}\right)\left[\left(1 - \frac{\alpha}{2}\right) - S\right] - \frac{\alpha(1 - \alpha)}{\left(S - \frac{\alpha}{2}\right)\left[\left(1 - \frac{\alpha}{2}\right) - S\right]}. \tag{4.16}$$

We note that the last term in (4.6) tends to zero as S tends to zero. Also, the condition $R(S) = R(1 - S)$ and (3.44) imply $F(S) = -F(1 - S)$. This condition simplifies the above calculation considerably.

Two sets of data for the cases $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{2}$ are given in Figs. 1 and 2. The broken lines indicate the results of a one-term expansion which yields the asymptotic spherical surface. The solid lines indicate the results of a two-term expansion.

We have not tried to compare the above asymptotic solution for a tubular balloon with a numerical solution which is not readily available. The two-term asymptotic solution for an initially spherical membrane, however, does appear to be in very good agreement with the exact solution for large pressures. The asymptotic solution for the spherical membrane calculated by using the equations given in Sec. 3 is exactly the same as (2.18). For $k = 0.2$ and $p = 10$, the results are

$$\rho = 10[1.25 - \tfrac{4}{100} + 0(10^{-4})],$$

$$T_1 = T_2 = 100[0.625 - \tfrac{2}{100} + 0(10^{-4})].$$

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