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ON VARIATIONAL METHODS IN FINITE AND INCREMENTAL ELASTIC DEFORMATION PROBLEMS WITH DISCONTINUOUS FIELDS*

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Abstract. With a view toward a numerical solution by means of the finite-element method, we give here a variational statement for large elastic deformations at finite strains which involves independent variation of the displacement, the (nonsymmetric first Piola–Kirchhoff) stress, and the deformation-gradient fields, and which includes both the boundary and the jump conditions. Then we present, for small deformations superimposed on the large, three variational statements, each involving three independent fields and each including both the boundary and the jump conditions. These statements are such that the first variation of the corresponding functional yields the field equations which characterize the equilibrium of the finitely-deformed state considered and also the field equations that pertain to the incremental deformations. Several specializations of these results are discussed. By way of illustration, finally, we present a finite-element formulation of the large deformation problem, using three independent fields, where each field is approximated by a piecewise-linear function within each element.

I. Introduction. It appears that a variational statement in which both the displacement and the stress fields are given independent variation was first developed by Hellinger [1, Sec. 7e, Eqs. (21) and (22a, b)], for finite elastic deformation problems. Hellinger uses a nonsymmetric stress tensor which is now commonly referred to as the first Piola–Kirchhoff (or Lagrangian) stress tensor (denoted in the present work by T^R_{Aa}), together with the deformation gradient (denoted here by $x_{a,A}$). In the first two chapters of his encyclopedic article he discusses the virtual work theorem for the statics and dynamics of one-, two-, and three-dimensional continua, including the case of polar (oriented) media, and presenting results in terms of both what is now commonly referred to as the Eulerian and the Lagrangian formulation. In the third chapter of his paper, Hellinger introduces the assumption of potential loads and the strain-energy function, developing the above-mentioned variational theorem in Sec. 7e. However, he does not include explicitly in his variational functional the boundary conditions. Hellinger's results, with further generalization and clarification, are presented in Secs. 231–238 of [2] in a more modern notation. A more general statement of a variational theorem for finite elasticity was given independently by Reissner [3] who formulates his results in terms of what is now commonly referred to as the symmetric Piola–Kirchhoff stress tensor (denoted here by S_{AB}) and the Lagrangian strain tensor (denoted here by E_{AB}). In this regard, therefore, Reissner's formulation is essentially different from that of Hellinger. Moreover, he includes explicit boundary data in his formulation by permitting

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the independent variation of the surface tractions. Koppe[1] [4], on the other hand, recasts Reissner's formulation in terms of the field quantities used by Hellinger, and hence arrives at a more general result than Hellinger's [1]. A yet more general statement occurs in Washizu [5] where the same variables as those considered by Reissner are used, but where, in addition to the displacement and stress fields, the strain field is also given independent variation.

A short historical account of the subject matter is given by Reissner in [6, Sec. 4]. It appears that Prange [7] was the first to give a variational theorem for the linear elasticity problem in which the displacement and the stress fields are given independent variations, and which also explicitly included the boundary data. Prange's work, however, remained unknown, since it was not published. Thus it appears that the first general variational statement of the kind mentioned above was published by Reissner [8]. In 1955, Hu [9] published a paper in which he developed variational methods for small strain problems, which involved arbitrary variations of the displacement, the stress, *and* the strain fields. Similar results were independently reported by Washizu [10], and later on more general statements for linear problems were given by Naghdi[2] [11] and Prager [12]. For linear elasticity without couple stresses, the most general discussion is given by Prager, who not only permits independent variations of displacement, strain, and stress fields, but also gives all the boundary and the jump conditions as the Euler equations and the boundary data of the corresponding variational statement.

For the problem of small elastic deformations superimposed on the large, it appears that variational methods of the generality mentioned above have not been considered, although the work of Biot [14], [15] and Washizu [5, Chapter 5] should be mentioned. Biot considers an independent displacement field only, while Washizu considers variational statements in the presence of initial strain and initial stress.

The use of variational theorems for the numerical solution of elasticity problems has recently been stressed by a number of authors (see for example, [12], [16–22]). For finite elasticity, however, it is the incremental formulation that appears to be more effective. Moreover, from a purely computational point of view, different formulations of the corresponding variational theorems would lead to different numerical results. In this connection it therefore appears useful to present various possible variational theorems concerning the incremental deformations of nonlinear elasticity problems. This is one of the aims of the present work.

In Sec. 2 we define our basic notation, and in Sec. 3 we give a variational statement for large elastic deformations, which has the same generality as that given by Prager [12] for the linear case. Here we use the (nonsymmetric) first Piola–Kirchhoff stress tensor and the deformation gradient. We permit independent variation for displacement, stress, and deformation-gradient fields, and include both the boundary and the jump conditions. Moreover, we compare this functional with the one which is expressed in terms of the second Piola–Kirchhoff stress tensor and the Green deformation tensor. In Sec. 4 we consider three formulations of the problem of small deformation superimposed on the large, namely the Lagrangian, the Eulerian, and a mixed, giving the corresponding variational statements with three independent fields and including both the boundary and the jump conditions. These functionals are such that their first

[1] I am indebted to Professor Reissner for calling my attention to Koppe's paper.

[2] Here a reference should also be made to [13].

variation yields the field equations characterizing the equilibrium of the finitely-deformed state considered and also the field equations pertaining to the incremental deformations. Here several specializations of the results are also considered. By way of illustration, we present, in Sec. 5, a finite-element formulation of the large deformation problem, using three independent fields, where each field is approximated by a piecewise-linear function within each element.

2. Statement of problem and notational preliminaries. We consider a body $\mathcal{B}$ with the regular boundary $\partial\mathcal{B}$ which in its initial (virgin) state occupies the volume $\mathcal{V}$ having a regular surface $\mathcal{S}$. We assume that the body is deformed from this initial configuration $\mathcal{C}_0$ to a deformed configuration $\mathcal{C}$ by means of a set of applied dead body forces $\mathbf{F}$ prescribed per unit mass, and the following set of boundary conditions: on the surface $\mathcal{S}$ which bounds the solid, some components of the dead surface tractions, T_α, and the *complementary* components, $W_{\mathbf{a}}$, of the surface displacements, are defined; the prescribed components of the tractions will be denoted by the subscript α, and the prescribed components of the surface displacements by the subscript $\mathbf{a}$. For example, if on a part of the boundary the components T_1, W_2, and W_3 are given, then T_α denotes T_1 and $W_{\mathbf{a}}$ represents W_2 and W_3. We note that we do not exclude cases in which surface tractions $\mathbf{T}$ are prescribed on a portion $\mathcal{S}_T$, and surface displacements $\mathbf{W}$ on the remaining part $\mathcal{S}_U$ of $\mathcal{S}$. Although our results will hold for a certain class of conservative loads, we confine our attention to dead-loads only, since for other surface loads which may depend on the local deformations of the material neighborhood on which they act one is in general not able to define a unique work function (or a potential), as has been discussed by Sewell [23] and this writer [24].

We use a fixed rectangular Cartesian coordinate system, and denote the particle positions in the initial configuration $\mathcal{C}_0$ by X_A and those in the deformed configuration $\mathcal{C}$ by x_a. We assume that x_a can be defined in terms of X_A, and hence we write $x_a = x_a(X_A)$, $A, a = 1, 2, 3$. Furthermore, we suppose that the configuration $\mathcal{C}_0$ can be divided into a finite number of regular subregions in each of which the mapping defined above is one-to-one and invertible so that $X_A = X_A(x_a)$. Hence, in each domain of regularity, the Jacobian $J = \det |x_{a,A}|$ is neither zero nor infinity, where a comma followed by a subscript index letter denotes partial differentiation with respect to the corresponding coordinate. We let these regions of regularity be separated from each other by discontinuity surfaces across which some components of the traction and some components of the displacement vector (at a point on a discontinuity surface no more than six jumps can be prescribed) may suffer finite discontinuities or jumps. If Σ is the collection of all these discontinuity surfaces in the reference configuration $\mathcal{C}_0$, we denote by $\mathbf{N}$ its unit normal which points outward from one domain of regularity, say domain 1, toward the adjacent subregion, say domain 2, and define the jump $\langle q \rangle$ of a field quantity q at a point P on Σ by $\langle q \rangle = q^{(1)} - q^{(2)}$, where $q^{(1)}$ and $q^{(2)}$ are the limiting values of q at P as this point is approached along $\mathbf{N}$ from the interior of domains 1 and 2, respectively.

We confine our attention to cases in which the body $\mathcal{B}$ consists of an elastic material which admits a strain-energy-density function defined by $\Phi = \Phi(C_{AB})$, where $C_{AB} = x_{a,A}x_{a,B}$ is the Green deformation tensor and where the summation convention on repeated indices is used and will be employed all through the rest of this paper. For future use, we express this strain-energy density function as $\Phi = \Phi(x_{a,A})$, but it should be carefully noted that Φ depends on the deformation-gradient $x_{a,A}$ in a special manner, as defined above.

To define the state of stress in $\mathcal{B}$ we may employ any one of the three stress tensors[3] S_{AB} , T^R_{Aa} , and T_{ab} , which are respectively called the second Piola–Kirchhoff (or Kirchhoff), the first Piola–Kirchhoff (or Lagrangian), and the Cauchy (or the true) stress tensor. We have $\frac{1}{2}S_{AB} = \partial\bar{\Phi}/\partial C_{AB}$, $T^R_{Aa} = \partial\Phi/\partial x_{a,A}$, and $T_{ab} = (1/J)x_{a,A}T^R_{Ab}$.

For our incremental formulation we consider the increments f_a of body forces, t_α of surface tractions, and w_{a} of surface displacements, all expressed in terms of the particle positions in $\mathcal{C}_0$, and denote the increment of the displacement, the displacement gradient, the first Piola–Kirchhoff stress tensor, and the second Piola–Kirchhoff stress tensor, respectively, by v_a , $v_{a,A}$, t^R_{Aa} , and s_{AB} , which are also regarded as functions of X_A .

In what follows we shall have occasion to consider the displacement, the deformation gradient, and the stress tensors as independent fields with arbitrary variation. When the deformation gradient $x_{a,A}$ is obtained from the displacement field U_a by differentiation, we shall say that it *corresponds* to this displacement field. In this case the variation in the deformation gradient is given, once the variation in the displacement field is defined. On the other hand, when the deformation gradient does not correspond to the displacement field, we shall denote it by x_{aA} (without a comma between a and A) and observe that it may have an independent variation. Similar remarks hold in connection with the stress tensor and the corresponding deformation gradient, or the strain tensors. For example, when T^R_{Aa} corresponds to the deformation gradient x_{aA} , we have $T^R_{Aa} = \partial\Phi/\partial x_{aA}$; otherwise T^R_{Aa} will be regarded as an independent field with arbitrary variation.

In the next section we shall present a variational theorem for finite deformations with discontinuous fields which has all the simplicity associated with the linear elasticity theory. Then in Sec. 4 we shall give for the incremental loading three new variational statements which not only yield all the field equations of small deformations superimposed on the initial finitely-deformed state of the solid, but also give the equilibrium equations for that state.

3. A variational theorem with discontinuous fields. The simplest and perhaps the most general variational statement in finite elasticity results if we use the first Piola–Kirchhoff stress tensor T^R_{Aa} , the deformation gradient x_{aA} , and the displacement U_a , as our independent fields with arbitrary variations, and consider the following functional:

$$\begin{aligned} \mathfrak{F} = \int_{\mathcal{V}} [\Phi(x_{aA}) - \rho_0 F_a U_a]\, d\mathcal{V} - \int_{\mathcal{V}} T^R_{Aa}[x_{aA} - x_{a,A}]\, d\mathcal{V} \\ - \int_{\mathcal{S}} T_\alpha U_\alpha\, d\mathcal{S} - \int_{\Sigma} \langle T_\alpha\rangle U_\alpha\, d\Sigma \qquad (1) \\ - \int_{\mathcal{S}} T_{\mathrm{a}}(U_{\mathrm{a}} - W_{\mathrm{a}})\, d\mathcal{S} - \int_{\Sigma} T_{\mathrm{a}}[\langle U_{\mathrm{a}}\rangle - \langle W_{\mathrm{a}}\rangle]\, d\Sigma, \end{aligned}$$

where $\langle T_\alpha\rangle$ denotes the prescribed jump on the components of the tractions across the discontinuity surfaces Σ, $\langle W_{\mathrm{a}}\rangle$ represents the prescribed jump of the components of the displacements across these surfaces, and ρ_0 is the initial mass density. In (1), the α repeated subscript is to be summed over only the prescribed components of the surface tractions, while the a's are to be summed over the prescribed displacement components. Observe that in (1) T^R_{Aa} and T_{a} may be regarded as the Lagrangian multipliers. Taking

[3] For simplicity, we choose to refer to a tensor by its Cartesian components.

the first variation of $\mathfrak{F}$, using the Gauss theorem, and noting that $\delta x_a = \delta U_a$, we obtain

$$\begin{aligned}
\delta\mathfrak{F} = &\int_{\mathcal{V}} \left[\frac{\partial\Phi}{\partial x_{aA}} - T^R_{Aa}\right] \delta x_{aA}\, d\mathcal{V} - \int_{\mathcal{V}} [T^R_{Aa,A} + \rho_0 F_a]\, \delta U_a\, d\mathcal{V} \\
&- \int_{\mathcal{V}} [x_{aA} - x_{a,A}]\, \delta T^R_{Aa}\, d\mathcal{V} \\
&- \int_{\mathcal{S}} [T_a - T^R_{Aa} N_A]\, \delta U_a\, d\mathcal{S} - \int_{\Sigma} [\langle T_a\rangle - \langle T^R_{Aa} N_A\rangle]\, \delta U_a\, d\Sigma \\
&- \int_{\mathcal{S}} [U_{\mathbf{a}} - W_{\mathbf{a}}]\, \delta T_{\mathbf{a}}\, d\mathcal{S} - \int_{\Sigma} [\langle U_{\mathbf{a}}\rangle - \langle W_{\mathbf{a}}\rangle]\, \delta T_{\mathbf{a}}\, d\Sigma.
\end{aligned} \tag{2}$$

We observe that the consequence of the arbitrariness of the variation of the deformation gradient x_{aA}, displacement U_a, stress field T^R_{Aa}, the surface displacements U_a corresponding to the prescribed components of the tractions, and the tractions $T_{\mathbf{a}}$ corresponding to the prescribed components of surface displacements, respectively, is the definition of the first Piola–Kirchhoff stress tensor, equations of equilibrium, the definition of the deformation gradient, the prescribed traction boundary data, and the prescribed displacement boundary data. In addition, in the fifth integral, the arbitrariness of the variation of the displacement components U_a of points on the discontinuity surfaces Σ yields the jump condition on the corresponding traction components. Similarly, the jump condition on the components of the displacement vector on Σ is given by the last integral in (2). If the jumps $\langle T_a\rangle$ and $\langle W_{\mathbf{a}}\rangle$ are prescribed to be zero, then the corresponding integrals in Eq. (2) yield, respectively, the continuity conditions on the tractions and the displacements across Σ, i.e. $\langle T^R_{Aa} N_A\rangle = 0$ and $\langle U_{\mathbf{a}}\rangle = 0$.

A variational theorem with the generality of the one just given but with prescribed zero jump conditions on interior surfaces Σ has been given by Prager [12] for linear elasticity. The most general variational theorem for nonlinear elasticity, which includes boundary conditions but neither the definition of strain (as given by the third integral in the right-hand side of (2)) nor the jump conditions, was first given by Reissner [3] who used the complementary energy functional together with the second Piola–Kirchhoff stress tensor (here denoted by S_{AB}) and the Lagrangian strain (here given by $E_{AB} = \frac{1}{2}(C_{AB} - \delta_{AB})$, where δ_{AB} is the Kronecker delta). Reissner's variational theorem generalizes that of Hellinger [1] who, however, did not give the boundary conditions explicitly; Reissner's results have been rewritten by Koppe [4] in terms of the field quantities used by Hellinger. A more general statement is found in Washizu's book [5, Sec. 3.9] which is expressed in terms of the same quantities as that of Reissner, and which also considers the variation in the strain field, but does not include the jump conditions.

For the sake of completeness and in order to permit comparison in the finite-element applications, we now rewrite functional (1) in terms of the second Piola–Kirchhoff stress tensor S_{AB} and the Green deformation tensor C_{AB} as follows:

$$\begin{aligned}
\mathcal{G} = &\int_{\mathcal{V}} [\Phi(C_{AB}) - \rho_0 F_a U_a]\, d\mathcal{V} - \int_{\mathcal{V}} \frac{1}{2} S_{AB}[C_{AB} - x_{a,A} x_{a,B}]\, d\mathcal{V} \\
&- \int_{\mathcal{S}} T_a U_a\, d\mathcal{S} - \int_{\Sigma} \langle T_a\rangle U_a\, d\Sigma \\
&- \int_{\mathcal{S}} T_{\mathbf{a}}(U_{\mathbf{a}} - W_{\mathbf{a}})\, d\mathcal{S} - \int_{\Sigma} T_{\mathbf{a}}[\langle U_{\mathbf{a}}\rangle - \langle W_{\mathbf{a}}\rangle]\, d\Sigma.
\end{aligned} \tag{3}$$

The vanishing of the first variation of $\mathcal{G}$ for arbitrary fields S_{AB}, C_{AB}, and U_a in $\mathcal{V}$ and for U_a and T_a on $\mathcal{S}$ and Σ yields all the field equations, boundary conditions, and jump conditions for the equilibrium of the body $\mathcal{B}$. (Observe here that S_{AB} and T_a in (3) may again be regarded as the Lagrangian multipliers.) Since these field equations will occur subsequently in connection with the incremental formulation of the problem, we shall not report them here (see Eq. (7)).

In certain situations one may wish to deal with a variational statement in which only the displacement and the stress fields, but not the strain field, are considered as independent with arbitrary variation. For finite elasticity, one then considers the variational theorem given by Hellinger [1] and its extension given by Koppe [4, Eq. (15)], or that given by Reissner [3], depending on whether the first or the second Piola–Kirchhoff stress tensor is employed. To obtain these formulations, however, one must invoke the Legendre transformation, and thus the corresponding formulation is valid only where such transformation is applicable. For the sake of completeness, we shall now briefly consider this.

To this end we regard the strain-energy-density function Φ as a function of the Lagrangian strain E_{AB} and note that $S_{AB} = \partial\Phi/\partial E_{AB}$. This equation may now be regarded as defining the Lagrangian strain E_{AB} as a function of the stress tensor S_{AB}. We write

$$\mathrm{H} = S_{AB}E_{AB} - \Phi,$$

where H is now viewed as a function of S_{AB}. We then have

$$E_{AB} = \partial\mathrm{H}/\partial S_{AB}.$$

With U_a and S_{AB} as independent fields, we write

$$\begin{aligned}
\mathcal{G}_1 = &\int_{\mathcal{V}} [S_{AB}E_{AB} - \mathrm{H} - \rho_0 F_a U_a]\, d\mathcal{V} \\
&- \int_{\mathcal{S}} T_a U_a \, d\mathcal{S} - \int_{\Sigma} \langle T_a \rangle U_a \, d\Sigma \\
&- \int_{\mathcal{S}} T_a (U_a - W_a)\, d\mathcal{S} - \int_{\Sigma} T_a[\langle U_a\rangle - \langle W_a\rangle]\, d\Sigma,
\end{aligned}$$

whose first variation is given by

$$\begin{aligned}
\delta\mathcal{G}_1 = &\int_{\mathcal{V}} \left[E_{AB} - \frac{\partial\mathrm{H}}{\partial S_{AB}}\right] \delta S_{AB}\, d\mathcal{V} \\
&- \int_{\mathcal{V}} [(S_{AB}x_{a,A})_{,B} + \rho_0 F_a]\, \delta U_a \, d\mathcal{V} - \text{B.C.} - \text{J.C.},
\end{aligned}$$

where B.C. and J.C. denote, respectively, the boundary and the jump conditions which can readily be obtained by analogy with Eq. (2) (see also Eq. (7)). In the above equation we have used the fact that $E_{AB} = \frac{1}{2}(x_{a,A}x_{a,B} - \delta_{AB})$. A similar expression can be obtained if we use the first Piola–Kirchhoff stress tensor. For this, however, we refer the reader to Eq. (15) of [4].

4. Incremental variational statements. We shall now consider incremental small deformations superimposed on a finitely deformed configuration $\mathcal{C}$, and, using the notation defined in Sec. 2, we write the following incremental functional:

$$
\begin{aligned}
\mathfrak{f} = {} & \int_{\mathcal{V}} \left[\frac{\partial \Phi}{\partial x_{aA}} v_{aA} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial x_{aA}\, \partial x_{bB}} v_{aA} v_{bB} \right] d\mathcal{V} - \int_{\mathcal{V}} \rho_0 (F_a + f_a) v_a \, d\mathcal{V} \\
& - \int_{\mathcal{V}} [T^R_{Aa} + t^R_{Aa}](v_{aA} - v_{a,A}) \, d\mathcal{V} - \int_{\mathcal{V}} [T'^R_{Aa} + t^R_{Aa}](x_{aA} - x_{a,A}) \, d\mathcal{V} \\
& - \int_{\mathcal{S}} (T_a + t_a) v_a \, d\mathcal{S} - \int_{\Sigma} \langle T_a + t_a \rangle v_a \, d\Sigma \\
& - \int_{\mathcal{S}} (T_{\mathbf{a}} + t_{\mathbf{a}})[(U_{\mathbf{a}} + v_{\mathbf{a}}) - (W_{\mathbf{a}} + w_{\mathbf{a}})] \, d\mathcal{S} \\
& - \int_{\Sigma} (T_{\mathbf{a}} + t_{\mathbf{a}})[\langle U_{\mathbf{a}} + v_{\mathbf{a}} \rangle - \langle W_{\mathbf{a}} + w_{\mathbf{a}} \rangle] \, d\Sigma.
\end{aligned}
\tag{4}
$$

In this equation the quantity inside the first integral may be interpreted as the change in the strain-energy for the increment in the displacement gradient v_{aA}, where one assumes that Φ is sufficiently smooth so that the expansion indicated is valid. The quantities $T^R_{Aa} + t^R_{Aa}$ and $T_{\mathbf{a}} + t_{\mathbf{a}}$ may be regarded as the Lagrangian multipliers. We note that the variations of these multipliers are given by δt^R_{Aa} and $\delta t_{\mathbf{a}}$, respectively, since T^R_{Aa} and $T_{\mathbf{a}}$ have zero variation. Now, the first variation of $\mathfrak{f}$, for the independently variable fields v_a, v_{aA}, t^R_{Aa}, $v_{\mathbf{a}}$, and $t_{\mathbf{a}}$, gives

$$
\begin{aligned}
\delta \mathfrak{f} = {} & \left\{ \int_{\mathcal{V}} \left[\frac{\partial \Phi}{\partial x_{aA}} - T^R_{Aa} \right] \delta v_{aA} \, d\mathcal{V} - \int_{\mathcal{V}} [T^R_{Aa,A} + \rho_0 F_a] \, \delta v_a \, d\mathcal{V} \right. \\
& - \int_{\mathcal{V}} [x_{aA} - x_{a,A}] \, \delta t^R_{Aa} \, d\mathcal{V} \\
& - \int_{\mathcal{S}} [T_a - T^R_{Aa} N_A] \, \delta v_a \, d\mathcal{S} - \int_{\Sigma} [\langle T_a \rangle - \langle T^R_{Aa} N_A \rangle] \, \delta v_a \, d\Sigma \\
& \left. - \int_{\mathcal{S}} [U_{\mathbf{a}} - W_{\mathbf{a}}] \, \delta t_{\mathbf{a}} \, d\mathcal{S} - \int_{\Sigma} [\langle U_{\mathbf{a}} \rangle - \langle W_{\mathbf{a}} \rangle] \, \delta t_{\mathbf{a}} \, d\Sigma \right\} \\
& + \left\{ \int_{\mathcal{V}} \left[\frac{\partial^2 \Phi}{\partial x_{aA}\, \partial x_{bB}} v_{bB} - t^R_{Aa} \right] \delta v_{aA} \, d\mathcal{V} \right. \\
& - \int_{\mathcal{V}} [t^R_{Aa,A} + \rho_0 f_a] \, \delta v_a \, d\mathcal{V} - \int_{\mathcal{V}} [v_{aA} - v_{a,A}] \, \delta t^R_{Aa} \, d\mathcal{V} \\
& - \int_{\mathcal{S}} [t_a - t^R_{Aa} N_A] \, \delta v_a \, d\mathcal{S} - \int_{\Sigma} [\langle t_a \rangle - \langle t^R_{Aa} N_A \rangle] \, \delta v_a \, d\Sigma \\
& \left. - \int_{\mathcal{S}} [v_{\mathbf{a}} - w_{\mathbf{a}}] \, \delta t_{\mathbf{a}} \, d\mathcal{S} - \int_{\Sigma} [\langle v_{\mathbf{a}} \rangle - \langle w_{\mathbf{a}} \rangle] \, \delta t_{\mathbf{a}} \, d\Sigma \right\}.
\end{aligned}
\tag{5}
$$

We observe that the quantity inside of the first braces in the right-hand side of (5) corresponds to the field equations that characterize the equilibrium condition of the finitely deformed configuration $\mathcal{C}$, as can be seen by comparison to Eq. (2). The vanishing of the integrals inside of the second braces for arbitrary variations in the corresponding field quantities, on the other hand, yields all the field equations, boundary data, and jump conditions across discontinuity surfaces Σ for the small incremental deformations superimposed on configuration $\mathcal{C}$. For this reason the incremental functional f given by (4) appears to be useful for the numerical solution (by the aid of the finite-element method, see for example [21]) of finite deformation problems in which a sequence of incremental loading is considered.

If one assumes that the field equations corresponding to configuration $\mathcal{C}$ are identically satisfied, then the functional (4) can be simplified to yield the incremental field equations only. To this end one needs to consider the first variation of the following expression:

$$
\begin{aligned}
\mathrm{f}_1 = & \int_{\mathcal{V}} \left[\frac{1}{2} \frac{\partial^2 \Phi}{\partial x_{aA}\, \partial x_{bB}} v_{aA} v_{bB} - \rho_0 f_a v_a \right] d\mathcal{V} \\
& - \int_{\mathcal{V}} t^R_{Aa} (v_{aA} - v_{a,A})\, d\mathcal{V} \\
& - \int_{\mathcal{S}} t_a v_a \, d\mathcal{S} - \int_{\Sigma} \langle t_a \rangle v_a \, d\Sigma \\
& - \int_{\mathcal{S}} \mathrm{t_a} (v_{\mathrm{a}} - w_{\mathrm{a}})\, d\mathcal{S} - \int_{\Sigma} \mathrm{t_a} [\langle v_{\mathrm{a}} \rangle - \langle w_{\mathrm{a}} \rangle]\, d\Sigma.
\end{aligned} \tag{5a}
$$

This variation is, of course, given by the quantity inside the second braces in the right-hand side of Eq. (5).

We now give an alternative representation for the variational statement defined by Eqs. (4) and (5), using the second Piola–Kirchhoff stress tensor, the Lagrangian strain tensor, and their increments. Referring again to Sec. 2 for the definition of the notation, we consider the functional

$$
\begin{aligned}
\mathrm{g} = & \int_{\mathcal{V}} \left[\frac{\partial \Phi}{\partial C_{AB}} c_{AB} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial C_{AB}\, \partial C_{DE}} c_{AB} c_{DE} \right] d\mathcal{V} - \int_{\mathcal{V}} \rho_0 (F_a + f_a) v_a \, d\mathcal{V} \\
& - \frac{1}{2} \int_{\mathcal{V}} (S_{AB} + s_{AB}) [c_{AB} - (x_{a,A} v_{a,B} + x_{a,B} v_{a,A})]\, d\mathcal{V} \\
& - \frac{1}{2} \int_{\mathcal{V}} (S_{AB} + s_{AB}) [C_{AB} - x_{a,A} x_{a,B} + v_{a,A} v_{a,B}]\, d\mathcal{V} - \text{B.C.} - \text{J.C.}
\end{aligned} \tag{6}
$$

where, for simplicity in presentation, we have denoted the boundary and the jump conditions by B.C. and J.C., respectively. These conditions are the same as those occurring in the right-hand side of Eq. (4). In Eq. (6) c_{AB} denotes the components of the increment of the Green deformation tensor C_{AB}. The first variation of g, for the independently variable fields v_a, c_{AB}, s_{AB} in $\mathcal{V}$, and v_a and $\mathrm{t_a}$ on $\mathcal{S}$, is given by

$$
\begin{aligned}
\delta \mathrm{g} = & \left\{ \int_{\mathcal{V}} \left[\frac{\partial \Phi}{\partial C_{AB}} - \frac{1}{2} S_{AB} \right] \delta c_{AB}\, d\mathcal{V} - \int_{\mathcal{V}} [(S_{AB} x_{a,A})_{,B} + \rho_0 F_a]\, \delta v_a \, d\mathcal{V} \right. \\
& - \frac{1}{2} \int_{\mathcal{V}} [C_{AB} - x_{a,A} x_{a,B}]\, \delta s_{AB}\, d\mathcal{V}
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\mathcal{S}} [T_a - S_{AB} x_{a,A} N_B] \, \delta v_a \, d\mathcal{S} \\
&- \int_{\mathcal{S}} [U_a - W_a] \, \delta t_a \, d\mathcal{S} - \text{J.C.} \Big\} \\
&+ \left\{ \int_{\mathcal{V}} \left[\frac{\partial^2 \Phi}{\partial C_{AB} \, \partial C_{DE}} c_{DE} - \tfrac{1}{2} s_{AB} \right] \delta c_{AB} \, d\mathcal{V} \right. \qquad (7) \\
&- \int_{\mathcal{V}} \{ [(S_{AB} + s_{AB}) v_{a,B} + s_{AB} x_{a,A}]_{,B} + \rho_0 f_a \} \, \delta v_a \, d\mathcal{V} \\
&- \frac{1}{2} \int_{\mathcal{V}} [c_{AB} - (x_{a,A} v_{a,A} + x_{a,B} v_{a,A})] \, \delta s_{AB} \, d\mathcal{V} \\
&- \int_{\mathcal{S}} \{ t_a - [(S_{AB} + s_{AB}) v_{a,A} + s_{AB} x_{a,A}] N_B \} \, \delta v_a \, d\mathcal{S} \\
&- \int_{\mathcal{S}} [v_a - w_a] \, \delta t_a \, d\mathcal{S} - \text{J.C.} \Big\} ,
\end{aligned}
$$

where the jump conditions are not written down explicitly but may readily be obtained by analogy with Eq. (5). As is clear, the vanishing of the quantity inside of the first braces in the right-hand side of (7) yields all the field equations corresponding to the equilibrium configuration $\mathcal{C}$, and the vanishing of the second braces provides us with the corresponding incremental field equations. We observe that the field equations given by the quantity in the first braces correspond to those that would result from $\delta\mathcal{G} = 0$; see Eq. (3).

So far we have used the so-called Lagrangian variables, and have hence developed all our results with reference to the initial configuration $\mathcal{C}_0$. Although for the numerical solution of the elasticity problems (by means of an incremental method) this is most suitable, we shall now present an Eulerian variational counterpart in which the current configuration $\mathcal{C}$ is used for the purpose of referencing. To this end we denote by u_a and t_{ab} displacement and the Cauchy stress tensor, and by F_a, T_a, and W_a, respectively, the body forces, the prescribed surface tractions, and the prescribed surface displacements, all expressed as functions of the particle positions x_a in the current configuration $\mathcal{C}$. Furthermore, we denote by the lower-case Italic letters the incremental quantities that are expressed as functions of x_a, so that v_a, t_{ab}, f_a, w_a, and t_a, are, respectively, the increments of displacement, stress tensor referred to configuration $\mathcal{C}$, body forces, prescribed components of surface displacements, and finally the prescribed components of the surface tractions. With the additional notation

$$
\Omega_{ab} = \frac{\rho}{\rho_0} \frac{\partial \Phi}{\partial x_{a,A}} x_{b,A} , \qquad \Gamma_{acbd} = \frac{\rho}{\rho_0} \frac{\partial^2 \Phi}{\partial x_{a,A} \, \partial x_{b,B}} x_{c,A} x_{d,B} , \qquad (8)
$$

where ρ is the mass density in configuration $\mathcal{C}$ and where Ω_{ab} and Γ_{acbd} are viewed as functions of the particle positions in this configuration, we now write

$$
\begin{aligned}
\mathrm{h} = & \int_{\mathrm{v}} [\Omega_{ab}\mathrm{v}_{ab} + \tfrac{1}{2}\Gamma_{abcd}\mathrm{v}_{ab}\mathrm{v}_{cd}]\, d\mathrm{v} - \int_{\mathrm{v}} \rho(\mathrm{F}_a + \mathrm{f}_a)\mathrm{v}_a\, d\mathrm{v} \\
& - \int_{\mathrm{v}} (t_{ab} + \mathrm{t}_{ab})(\mathrm{v}_{ab} - \mathrm{v}_{a,b})\, d\mathrm{v} \\
& - \int_{\mathrm{s}} (\mathrm{T}_a + \mathrm{t}_a)\mathrm{v}_a\, d\mathrm{s} - \int_{\sigma} \langle \mathrm{T}_a + \mathrm{t}_a \rangle \mathrm{v}_a\, d\sigma \\
& - \int_{\mathrm{s}} (\mathrm{T}_a + \mathrm{t}_a)[(\mathrm{u}_a + \mathrm{v}_a) - (\mathrm{W}_a + \mathrm{w}_a)]\, d\mathrm{s} \\
& - \int_{\sigma} (\mathrm{T}_a + \mathrm{t}_a)[\langle \mathrm{u}_a + \mathrm{v}_a \rangle - \langle \mathrm{W}_a + \mathrm{w}_a \rangle]\, d\sigma,
\end{aligned}
\tag{9}
$$

where v and s are, respectively, the volume and the surface of $\mathcal{B}$ in configuration $\mathcal{C}$, and σ denotes the discontinuity surfaces in this configuration. The first variation of h for the independently variable fields v_a , t_{ab} , v_{ab} , t_a , and v_a now yields

$$
\begin{aligned}
\delta\mathrm{h} = & \left\{ \int_{\mathrm{v}} [\Omega_{ab} - t_{ab}]\, \delta\mathrm{v}_{ab}\, d\mathrm{v} - \int_{\mathrm{v}} [t_{ab,b} + \rho\mathrm{F}_a]\, \delta\mathrm{v}_a\, d\mathrm{v} \right. \\
& - \int_{\mathrm{s}} [\mathrm{T}_a - t_{ab}n_b]\, \delta\mathrm{v}_a\, d\mathrm{s} - \int_{\sigma} [\langle \mathrm{T}_a \rangle - \langle t_{ab}n_b \rangle]\, \delta\mathrm{v}_a\, d\sigma \\
& \left. - \int_{\mathrm{s}} [\mathrm{u}_a - \mathrm{W}_a]\, \delta\mathrm{t}_a\, d\mathrm{s} - \int_{\sigma} [\langle \mathrm{u}_a \rangle - \langle \mathrm{W}_a \rangle]\, \delta\mathrm{t}_a\, d\sigma \right\} \\
& + \left\{ \int_{\mathrm{v}} [\Gamma_{abcd}\mathrm{v}_{cd} - \mathrm{t}_{ab}]\, \delta\mathrm{v}_{ab}\, d\mathrm{v} \right. \\
& - \int_{\mathrm{v}} [\mathrm{t}_{ab,b} + \rho\mathrm{f}_a]\, \delta\mathrm{v}_a\, d\mathrm{v} \\
& \left. - \int_{\mathrm{v}} [\mathrm{v}_{ab} - \mathrm{v}_{a,b}]\, \delta\mathrm{t}_{ab}\, d\mathrm{v} - \mathrm{B.C.} - \mathrm{J.C.} \right\},
\end{aligned}
\tag{10}
$$

where n_b denotes the components of the exterior unit normal to s and where, as before, B.C. and J.C. stand for the boundary and jump conditions. As is evident, the vanishing of the first and second sets of braces in the right-hand side of (10) yields, respectively, the field equations corresponding to the equilibrium of configuration $\mathcal{C}$ and the field equations for the incremental deformation superimposed on this configuration. These equations are all expressed in terms of the particle positions in $\mathcal{C}$. Note that the incremental Lagrangian (or nominal [25]) stress tensor t_{ab} is given by

$$
\mathrm{t}_{ab} = \Gamma_{abcd}\mathrm{v}_{cd} = [\mathrm{t}_{bd}\, \delta_{ac} + \gamma_{abcd}]\mathrm{v}_{cd}\ , \tag{11a}
$$

where

$$
\gamma_{abde} = 4\, \frac{\rho}{\rho_0}\, \frac{\partial^2 \Phi}{\partial C_{AB}\, \partial C_{DE}}\, x_{a,A} x_{b,B} x_{d,D} x_{e,E}\ , \tag{11b}
$$

and where γ_{abde} is the so-called tangent elastic modulus tensor. Hence the configuration $\mathcal{C}$ is stable as long as the functional

$$
\int_{\mathrm{v}} [t_{ab}\mathrm{v}_{ca}\mathrm{v}_{cb} + \gamma_{abcd}\mathrm{v}_{ab}\mathrm{v}_{cd}]\, d\mathrm{v}
$$

is positive-definite. For further discussion of this question, see [21], [24], [25].

We note that, if only the incremental field equations are needed, the functional h should be modified to read

$$\begin{aligned} \mathrm{h}_1 = \int_V [\tfrac{1}{2}\Gamma_{abcd}\mathrm{v}_{ab}\mathrm{v}_{cd} - \rho \mathrm{f}_a \mathrm{v}_a]\, dv - \int_V \mathrm{t}_{ab}(\mathrm{v}_{ab} - \mathrm{v}_{a,b})\, dv \\ - \int_S \mathrm{t}_a \mathrm{v}_a\, ds - \int_S \mathrm{t}_a(\mathrm{v}_a - \mathrm{w}_a)\, ds - \text{J.C.} \end{aligned} \tag{12}$$

The first variation of this functional is, of course, given by the quantity inside the second braces in the right-hand side of (9).

5. Application. As has been pointed out before, the variational theorems discussed in the previous sections can be used most effectively in the numerical treatment of finite elasticity problems by means of the so-called finite-element technique. While it is not our purpose to develop here systems of equations which would result from the application of the various variational statements considered above, for the purpose of illustration we shall briefly discuss one case. It should, however, be noted that although the various variational theorems developed here are formally equivalent, from the computational point of view they lead to totally different systems of equations which may involve different computational errors.

In the finite-element technique one divides the body into a finite number of subregions and uses a given function containing some unknown parameters to describe a certain field quantity in each subregion. The unknown parameters are then calculated in such a manner that the corresponding field equations are satisfied. When a variational method is used, one obtains these parameters by minimizing the corresponding functional. In finite deformation problems, one may subdivide either the initial configuration $\mathcal{C}_0$ or the final one $\mathcal{C}$. In the latter case, which has been discussed by Felippa[4] [26], one is dealing with a continually changing sequence of configurations, and hence one must update these configurations during the process of calculation. On the other hand, in the former case, which has been considered by a number of writers (see, for example, [21], [22], [27] for further references), the fixed initial configuration is used throughout the analysis, leading to a much more convenient and effective numerical scheme. In the following, we shall deal with this latter case.

Let us divide the configuration $\mathcal{C}_0$ into a finite number of tetrahedra, and consider a typical tetrahedron, say S_1 , with four vertices P_1 , P_2 , P_3 and P_4 , For the node (vertex), say, P_1 of S_1 , let us define a *unit field* as

$$\Psi^{P_1S_1} = A_0^{P_1S_1} + A_B^{P_1S_1}X_B , \qquad B = 1, 2, 3, \tag{13}$$

and calculate the four constants in this linear equation by requiring that $\Psi^{P_1S_1}$ have the value 1 at the node P_1 and vanish at the other three vertices of tetrahedron S_1 . If there are n tetrahedra meeting at node P_1 , we define the corresponding unit field by

$$\Psi^{P_1} = \sum_{\alpha=1}^{n} \Psi^{P_1S_\alpha}. \tag{14}$$

The displacement field U_a , for example, may now be approximated by a piecewise-

[4] Felippa's results are not, however, quite correct, as has been pointed out in [21].

linear function as

$$U_a = \mathrm{U}_a^\alpha \Psi^\alpha, \qquad a = 1, 2, 3, \quad \alpha = 1, 2, \cdots, M, \tag{15}$$

where it is assumed that there are a total of M nodes for the entire body. Here the U_a^α denotes the displacement vector at node α, and summation on α is implied.

In a similar manner, piecewise-linear approximations of stress and strain tensors can be obtained. Since each one of these fields can be regarded as independent in the corresponding (suitable) variational statement, the assumption that both the displacement and the strain fields have linear variations in each element is not contradictory. The contradiction arises only if we demand that they should *correspond* to each other. As has been demonstrated by Dunham and Pister [20] for linear plane strain elasticity problems, the use of independent displacement and stress fields may lead to more accurate results. Here we propose that all three fields, namely the displacement, the stress, and the corresponding strain fields, be regarded as independent with arbitrary variations. We therefore write, for example, in connection with functional $\mathcal{G}$,

$$S_{AB} = \mathrm{S}_{AB}^\alpha \Psi^\alpha, \qquad C_{AB} = \mathrm{C}_{AB}^\alpha \Psi^\alpha, \tag{16}$$

where S_{AB}^α and C_{AB}^α at each node α represent, respectively, the six (since they are symmetric) components of the nodal values of the second Piola–Kirchhoff stress and the Green deformation tensors. To obtain an approximate expression for $x_{a,A}$ we note that $x_a = U_a + X_B\,\delta_{aB}$, where δ_{aB} is the Kronecker delta. Hence, from (15) we calculate

$$x_{a,A} = \mathrm{U}_a^\alpha \Psi_{,A}^\alpha + \delta_{aA}\,. \tag{17}$$

The approximate fields (15) and (16) *are all continuous* over the entire body, and therefore, if there are no prescribed jumps within the considered continuum, the integrals corresponding to these jumps would not occur in functional (3). Upon substitution from (15), (16), and (17), into (3), we obtain

$$\begin{aligned} \mathcal{G} = & \int_{\mathcal{V}} [\bar{\Phi}(\mathrm{S}_{AB}^\alpha \Psi^\alpha) - \rho_0 F_a \mathrm{U}_a^\alpha \Psi^\alpha]\, d\mathcal{V} \\ & - \frac{1}{2} \int_{\mathcal{V}} \mathrm{S}_{AB}^\alpha \Psi^\alpha [\mathrm{C}_{AB}^\beta \Psi^\beta - (\mathrm{U}_a^\beta \Psi_{,A}^\beta + \delta_{aA})(\mathrm{U}_a^\gamma \Psi_{,B}^\gamma + \delta_{aB})]\, d\mathcal{V} - \int_{\mathcal{S}} T_a \mathrm{U}_a^\alpha \Psi^\alpha\, d\mathcal{S}, \\ & a, A, B = 1, 2, 3, \quad \alpha, \beta, \gamma = 1, 2, \cdots, M, \end{aligned} \tag{18}$$

where we have assumed that the approximate displacement field (15) is a priori adjusted so as to satisfy the prescribed displacement boundary data. It should be observed that, since the unit field Ψ^α is completely defined as soon as the coordinates of all the nodes are known,[5] all the integrals in (18) can be calculated explicitly, and hence $\mathcal{G}$ becomes an explicit function of the nodal values of the displacement field U_a^α, stress field S_{AB}^α, and the deformation tensor C_{AB}^α. To arrive at the system of discretized equations which can then be used to calculate the above nodal values, we minimize $\mathcal{G}$ given by (18) by considering

$$\frac{\partial \mathcal{G}}{\partial \mathrm{U}_a^\alpha} = 0, \quad \frac{\partial \mathcal{G}}{\partial \mathrm{S}_{AB}^\alpha} = 0, \quad \frac{\partial \mathcal{G}}{\partial \mathrm{C}_{AB}^\alpha} = 0, \quad \alpha = 1, 2, \cdots, M, \quad a, A, B = 1, 2, 3, \tag{19}$$

[5] Explicit expression of this unit field in terms of the nodal coordinates can be found in [21].

which are the required equations for the corresponding unknowns. These equations, however, are nonlinear and therefore require a special computational technique for their solution. If attention is focused on a precritical response analysis, an iterative Newton–Raphson technique may be employed. In general, however, one must consider an incremental formulation together with these equations. The results obtained by the incremental, step-by-step calculations can be accumulated, and in this manner one arrives at approximate expressions for U^{α}_{a} , S^{α}_{AB} and C^{α}_{AB} . One then employs these approximate expressions to initiate the iteration process required for the solution of (19).

The incremental equations are obtained in a manner similar to that discussed above. The increments of the displacement field, the stress tensor, and the corresponding deformation tensor are expressed in terms of the same unit field Ψ^{α}, and in this manner the incremental functional, say g in Eq. (6), is reduced to a function of the nodal values of the above-mentioned fields. Setting the derivative of this function, with respect to the unknown nodal values of the field quantities, equal to zero, one then obtains a system of linear equations which are to be solved for the unknown nodal quantities. Since a detailed study of these equations for specific problems would carry us far beyond the purpose of this article, we shall postpone it to a future report.

References

[1] E. Hellinger, *Die allgemeinen Ansätze der Mechanik der Kontinua*, Enz. math. Wis. **4**, 602–694 (1914)

[2] C. Truesdell and R. Toupin, *The classical field theories*, Handbuch der Physik, Band III/1, Springer, Berlin, 1960

[3] E. Reissner, *On a variational theorem for finite elastic deformations*, J. Math. Physics **32**, 129–135 (1953)

[4] E. Koppe, *Die Ableitung der Minimalprinzipien der nichtlinearen Elastizitätstheorie mittels kanonischer Transformation*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa **1956**, 259–266

[5] K. Washizu, *Variational methods in elasticity and plasticity*, Pergamon Press, New York, 1968

[6] E. Reissner, *Variational methods and boundary conditions in shell theory*, in *Studies in optimization*, Sympos. on Optimization, Toronto, Canada, June 11–14, 1968

[7] G. Prange, *Habilitationsschrift* (unpublished), Hannover, 1916

[8] E. Reissner, *On a variational theorem in elasticity*, J. Math. Physics **29**, 90–95 (1950)

[9] Hai-Chang Hu, *On some variational principles in the theory of elasticity and the theory of plasticity*, Sci. Sinica **4**, 33–54 (1955). (first published in Chinese in 1954)

[10] K. Washizu, *On the variational principles of elasticity and plasticity*, Technical Report #25-18, Cont. N5ori-07833, M.I.T., Cambridge, Mass. 1955

[11] P. M. Naghdi, *On a variational theorem in elasticity and its application to shell theory*, Trans. ASME Ser. E. J. Appl. Mech. **31**, 647–653 (1964)

[12] W. Prager, *Variational principles of linear elastostatics for discontinuous displacements, strains and stresses*, in *Recent progress in applied mechanics*, The Folke–Odqvist Volume, edited by B. Broberg, J. Hult and F. Niordson, Almqvist and Wiksell, Stockholm, 1967, pp. 463–474

[13] E. Reissner, *A note on variational principles in elasticity*, Int. J. Solids Struct. **1**, 93–95 (1965)

[14] M. Biot, *Non-linear theory of elasticity and the linearized case for a body under initial stress*, Philos. Mag. **27**, 468–489 (1939)

[15] M. Biot, *Mechanics of incremental deformations*, Wiley, New York, 1965

[16] L. Herrmann, *Elasticity equations for incompressible and nearly incompressible materials by a variational theorem*, AIAA J. **3**, 1896–1901 (1965)

[17] T. H. Pian and P. Tong, *The basis of finite element methods for solid continua*, Int. J. Num. Meth. Engr. **1**, 3–28 (1969)

[18] P. Tong, *New displacement hybrid finite-element models for solid continua*, Int. J. Num. Meth. Engr. **2**, 73–83 (1970)

[19] P. Tong, *An assumed stress hybrid finite element method for an incompressible and near-incompressible material*, Int. J. Solids Struct. **5**, 455–461 (1969)

[20] R. S. Dunham and K. S. Pister, *A finite element application of the Hellinger–Reissner variational theorem*, in *Proc. second conference on matrix methods in structural mechanics*, Wright–Patterson Air Force Base, 1968, pp. 471–487

[21] S. Nemat-Nasser and H. Shatoff, *A consistent numerical method for the solution of nonlinear elasticity problems at finite strains*, SIAM **20** (1971); Technical Report 2, ONR, Dept. AMES, UCSD, Jan. 1970

[22] G. A. Wempner, *Modeling of nonlinear structural systems with linear algebraic equations*, UARI Research Report 86, The University of Alabama in Huntsville, Ala. 1970

[23] M. J. Sewell, *On configuration-dependent loading*, Arch. Rational Mech. Anal. **23**, 327–351 (1968)

[24] S. Nemat-Nasser, *On thermomechanics of elastic stability*, Z. Angew. Math. Phys. **21**, 538–552 (1970)

[25] R. Hill, *On uniqueness and stability in the theory of finite elastic strain*, J. Mech. Phys. Solids **5**, 229–241 (1957)

[26] C. A. Felippa, *Refined finite element analysis of linear and nonlinear two-dimensional structures*, Ph.D. Thesis, University of California, Berkeley, Calif., 1966

[27] J. T. Oden and J. E. Key, *Analysis of finite deformation of elastic solids by the finite element method*, in *IUTAM Sympos. on high speed computing of elastic structures*, Liege, Belgium, 1970