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The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unneccessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

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The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

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The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liqiuds, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten.*

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

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—BOOK REVIEW SECTION—

Integral formulas in Riemannian geometry. By Kentaro Yano. Marcel Dekker, Inc., New York, 1970. ix + 156 pp. $10.75.

This is the first in a series of monographs on pure and applied mathematics, the coordinator of the editorial board being S. Kobayashi of the University of California at Berkeley. It is excellently produced, with bibliography, author index and subject index. The author, who is Professor of Mathematics at Tokyo Institute of Technology, dedicates it to Professor Salomon Bochner "through whose inspiration the main ideas in this book developed".

There are eight chapters: (1) Fundamental concepts and formulas in Riemannian geometry, (2) Harmonic 1-forms and Killing vector fields, (3) Riemannian manifolds admitting an infinitesimal conformal transformation, (4) Harmonic forms and Killing tensor fields, (5) Hypersurfaces of Riemannian manifolds, (6) Closed hypersurfaces of a Riemannian manifold with constant mean curvature, (7) Harmonic 1-forms and Killing vector fields in Riemannian manifolds with boundary, (8) Harmonic forms and Killing tensor fields in Riemannian manifolds with boundary.

I had the good fortune to get to know the Princeton geometers, L. P. Eisenhart and O. Veblen, fifty years ago, at a time when Einstein's general theory of relativity was only five or six years old. It was a golden age for those who liked to pursue differential geometry and physics hand in hand. I felt a natural sympathy for the way in which Eisenhart viewed tensor calculus and Riemannian geometry, and still do. In that view, when one thinks of a tensor, say g_{ij} , one thinks of a set of functions of the coordinates which transform according to a certain law when one changes the coordinates. One calls this *index notation.*

But there is another way, called *index-free notation,* due to Élie Cartan, who claimed that debauches of indices hide a geometric reality, often very simple. In this view, if I understand it correctly, a tensor as above appears as a scalar bilinear function of two arbitrary vectors: $g(X, Y) = g_{ij}X^iY^j$. The difference may be illustrated in terms of the stress tensor in elasticity. In the index notation, one thinks of a 3×3 matrix of components, in the index-free notation one thinks of the component (in any direction X) of the traction across any surface element (with normal Y).

This difference might seem trivial, and in a sense it is. But the formalisms appropriate to the two notations look very different. In this book the index-free notation predominates, but the index notation is also used, so that a person familiar only with the latter can see, more or less, what is going on. I would not recommend the book to readers seeking initiation into the index-free methods. This is no adverse criticism, because I think the book is aimed at geometers already familiar with those methods, in fact, at modern pure geometers with an appetite for the results stated or established. It is not a book for relativists, even those familiar with the index-free notation, since only Riemannian manifolds with positive-definite metric are considered. This is rather a pity, because relativists are interested in integral formulas and would like to know how many of the results here displayed would be available, with some slight modification perhaps, to curved space-time. It might have been better to start with formulae independent of metric, or at least of its signature. But a geometer as distinguished as Professor Yano must be allowed to follow his own way.

J. L. SYNGE (*Dublin*)

Families of bivariate distributions. By K. V. Mardia. Griffin's Statistical Monograph No. 27. Hafner Publishing Company, Darien, Connecticut, 1970. ix + 109 pp. £2.10 ($5.04).

The idea of obtaining probability distributions from *a priori* systems has an antique Pearsonian flavor. Nevertheless, the treatment of (mainly non-normal) bivariate distributions given in this book is certainly better than a bald unadorned listing. The author uses the Pearson-type differential equation, transformation of variables and assumptions regarding the marginal and conditional distributions to obtain standard families. Without wishing to suggest any omission on his part, it is interesting to note that the entire field seems to consists of sixteen distributions (of which five are discrete) and a little over one hundred references.

Frank A. Haight (*Auckland*)

Introduction to stochastic control theory. By Karl J. Aström. Academic Press, Inc., New York, 1970. xi + 299 pp. $14.50.*

The author has written a quite readable account of the applications of random processes to analysis and synthesis of control systems. The book should be of considerable interest and use to the classically-trained control engineer who wishes a remedial survey of part of the state space techniques developed in the early 1960s. I feel, however, that the book has some glaring omissions on the theoretical side which tend to make it an unsuitable text. However, the treatment of certain practical concepts would make it useful as collateral reading.

In particular, the following points about the book are those which limit its effectiveness. Both controllability and observability are not defined, discussed or referenced. No stability considerations for the optimal filter or controller are given. Questions of the numerical problems associated with generating solutions to the Riccati equation are not mentioned. The important question of wide and strict-sense interpretation of the K.B. filter is not treated.

Finally, only the last chapter of the book justifies its title. The exercises are good.

Richard S. Bucy (*Los Angeles*)

Mathematische Hilfsmittel des Ingenieurs, Volume IV. R. Sauer and I. Szabó, editors. Springer-Verlag, Berlin—Heidelberg—New York, 1970. (in German)

This is the last volume of the handbook *Mathematical tools for engineers*. It consists of three chapters: L, Dynamic stability of systems with a finite number of degrees of freedom (by W. Hahn); M, Theory of probability and mathematical statistics (by D. Morgenstern with assistance of V. Mannitzsch) and Chapter N, Theorems and formulas of mechanics and electrotechnics (the mechanics part is written by W. Lander, the electrotechnical part by K. Pöschl). The volume contains, in addition to its own subject index, the subject index for all four volumes.

Chapter L presents the stability criteria for linear time-independent and linear time-dependent systems. The usual criteria are well explained and their efficiency compared. For nonlinear systems the phase representation is explained and the possible singular points are characterized. Lyapunov's method of studying the stability region of a system is outlined. Forced and free periodic solutions are discussed and the use of the describing function and the harmonic balance method, etc., for studying nonlinear systems are presented. The entire chapter is a good guide for handling systems of low order and leaves the details to the well-chosen references at the end of the chapter.

Chapter M will be most interesting for industrial engineers. The elements of the theory of probability are presented and its relation to statistical investigations explained. Covariance and correlation-coefficients are discussed. As far as the format of a handbook permits, examples are briefly mentioned. The meaning and use of analytic distribution functions (many examples) is explained and the formation of empirical distribution functions based on sampling is described. The use of these functions for estimating possible development or events is shown. The influence of the number of samples or tests on conclusions is discussed. A relative short part of the chapter is devoted to stochastic processes. Essentially Markov chains are treated. A few pages deal with information theory. This chapter contains a wealth of information, brief but clearly stated and supported by a fine list of reference books.

The two parts of chapter N are supposed to be a handy collection of formulas without going into

* This review also appeared in *Mathematical Reviews*.

explanations other than of the symbols used. The electrotechnical part follows very much this scheme while the mechanical part contains more text. That may be appreciated by some, while others may feel that space was used up which could have been better used for indicating formulas of more general character (e.g. viscous compressible flow!) The electrotechnical part will be very helpful for the general engineer who has to take electric and magnetic effects in account.

There is no doubt that the editors have shown foresight in choosing the mathematical material for this handbook. It will be especially helpful nowadays when engineers must often deal with effects considered negligible in preceding decades.

I. FLÜGGE-LOTZ (*Stanford University*)

Algèbre moderne et théorie des graphes. By B. Roy. Vol. 1: xvi + 502 pp. 96 F. Vol. 2: xxiv + 753 pp. 158 F. Dunod & Cie., Paris, 1969 (Vol. 1) 1970 (Vol. 2).

It is customary to write long reviews of big books, particularly if the books are outstanding. However, I felt that attempting to find some minor omissions in such a rich context would be doing both the book and the reader injustice. Here we have a comprehensive work of scholarship *par excellence.* This reviewer is certain that apart from its high price, graph and network theorists will be delighted with Roy's two books, particularly Volume 2—superlative, full of ideas, rich with real-life problems, algorithms, references and exercises for both students and researchers. Problems and ideas are neatly abstracted, classified and studied; e.g., stability both internal and external, kernel, base, root, center, articulation and connectivity sets, longest and shortest paths, Hamiltonian circuits, Eulerian paths, problems of flow, etc. We found it very useful in teaching. Before duplicating work already done in applied graph and network theory, one should consult Roy's encyclopedic two volumes.

THOMAS L. SAATY
(*Philadelphia*)

Nonparametric statistics. By Jaroslav Hájek. Holden-Day, San Francisco, 1969. viii + 184 pp. $10.95.

Hájek's book is intended for a course in nonparametric statistics at the undergraduate level. The author presupposes "··· familiarity with elementary concepts of probability and statistics and the habit of abstract thinking at an intermediate level."

A brief description of the contents follows. Chapter 1 introduces permutations, ranks, linear (rank) statistics, and gives conditions for the asymptotic normality of linear statistics. Chapter 2 is devoted to rank tests of the hypothesis of randomness. Chapter 3 offers two-sample tests with attention focused on location alternatives. In Chapter 4 treatment of the two-sample problem continues with tests designed to detect dispersion differences. Also considered in this chapter are regression alternatives and k-sample tests for the one-way layout. Chapter 5 is devoted to the one-sample symmetry problem and the k-sample randomized block problem, and Chapter 6 gives rank tests for the hypothesis of independence. In Chapter 7 one finds a careful treatment of ties for the test procedures discussed earlier, and Chapter 8 includes material relating to power and the choice of scores which determine the rank tests.

The style of writing is clear. Definitions are precise and theorems are carefully stated. When the author deems the proof of a theorem too difficult (at the intended level) the reader is usually referred to a journal article or the more advanced book *Theory of rank tests* by J. Hájek and Z. Šidák (Academic Press, New York, 1967). There are approximately sixty problems, and exact null distribution tables are presented for many of the test procedures discussed. There are quite a few misprints, but they tend to be of the readily detectable variety.

Two highlights are:

a) a complete treatment of ties, via the method of averaging scores over tied groups, and including several new theorems on the (conditional) asymptotic normality of the rank test statistics in tied situations, and

b) a result (Theorem 34A) relating the choice of scores to the underlying distribution. The author indicates how this theorem enables one to data-snoop and use the actual sample to influence the decision as to what rank test should be applied to the observations. This ability to "cheat" goes back to a result of Savage (On the independence of tests of randomness and other hypotheses, *J. Amer. Statist. Assoc.* 1957) which states that under the hypothesis of randomness, with the underlying distribution assumed symmetric, rank statistics and symmetric statistics are independent.)

I find two inadequacies, though these are largely a matter of personal preferences.

c) The nonparametric methods discussed are confined to test procedures. A presentation of the robust estimators that can be derived from rank test statistics, as pioneered by Hodges and Lehmann (Estimates of location based on rank tests, *Ann. Math. Statist.* 1963), would help dispel the notion that nonparametric techniques are restricted to the framework of hypothesis testing.

d) The scores function approach was effectively used by Hájek and Šidák in *Theory of rank tests* to develop the properties of a large class of nonparametric tests. However, at the intended level of this book, the scores often obscure much of the motivation and rationale underlying many of the standard tests. For example, the reader never learns that the Kruskal-Wallis statistic can be related to the usual ANOVA F-statistic applied to the ranks rather than the original observations.

Hájek has achieved his objective. The book will be useful for an undergraduate introduction to nonparametric statistics for an audience with a mathematical and statistical background, and is also appropriate for a first-year graduate course in a masters' degree program. Above and beyond its value as a text, most statisticians will purchase this book to have a concise, unified, and elegant treatment of some basic nonparametric methods, written by a world leader in the field.

MYLES HOLLANDER (*Tallahassee*)