ON THE PLANE SECTIONS METHOD FOR FUNCTIONS OF TWO VARIABLES*

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1. Introduction. We consider here functions $f(z_1, z_2)$ of two complex variables. We say f is *holomorphic* in a domain $\mathfrak{D}$ if f can be expanded in a power series about each point of $\mathfrak{D}$. If, in particular, $\mathfrak{D}$ contains the origin we have locally a series representation

$$f(z_1, z_2) = \sum_i \sum_k a_{i,k} z_1^i z_2^k. \quad (1.1)$$

An interesting function-theoretic problem is that of obtaining information about the singularities of f given information about the coefficients $a_{i,k}$. This problem assumes considerable interest in applications, in particular in the study of quantum mechanical and quantum field-theoretic models of scattering phenomena. In the former one is concerned with determining the analytic properties of certain partial differential equations, whereas in the latter the investigation centers about relationships such as implied by the elastic unitarity integral (see, for instance, [5 chapter 5] for a list of further references). We mention also a recent paper [7] in which the possible singularities of solutions of the Helmholtz equation are discussed in terms of the singularities of an associated holomorphic function.

A natural method, which has been used by Bergman (cf. [2], [5]), is to set $z_2 = \lambda z_1$ and study the function

$$F(\lambda, z_1) = f(z_1, \lambda z_1)$$

as a function of z_1 . It seems to have been overlooked that the radius of convergence of the power series for $F(\lambda, z_1)$ need not be a continuous function of λ . The singularities of $F(\lambda, z_1)$ then may have no relation to the singularities of f as a function of two variables. The purpose of this note is to point out this error and discuss the possibilities for obviating it. Terminology will be set out more carefully in succeeding sections.

We now remind ourselves of the following fundamental theorem of Hartogs [6, p. 28-30].

Hartogs' Theorem. Suppose that in the set

$$|z - a_1| \leq R_1, \quad |z - a_2| < R_2, \quad R_1, R_2 > 0 \quad (1.2)$$

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$g(z_1, z_2)$ is analytic in each variable separately. Suppose further that g is bounded in a set $|z - a_1| \leq S_1$, $|z - a_2| \leq S_2$, with $S_1 \leq R_1$ and $S_2 \leq R_2$. Then $g(z_1, z_2)$ is holomorphic in the original set (1.2).

2. Definitions and some elementary remarks. The *analytic plane* $\mathcal{O}(\lambda)$ is the set $\{(z_1, z_2)/z_2 = \lambda z_1\}$. It is easy to see that each $\mathcal{O}(\lambda)$ passes through the origin, and if $z_1 \neq 0$ there is a unique $\mathcal{O}(\lambda)$ through the point (z_1, z_2) . As λ ranges through all complex numbers $\mathcal{O}(\lambda)$ sweeps out all of (z_1, z_2) -space except the plane $z_1 = 0$. This plane corresponds to $\lambda = \infty$. To study f near $z_1 = 0$ one could use the different set of surfaces $z_1 = \lambda z_2$. It is not hard to verify that $\mathcal{O}(\lambda)$ is a plane in the usual sense, but we do not need this fact.

Now we define the *plane section transform* $F(\lambda, z_1)$ corresponding to $f(z_1, z_2)$ by the equation

$$F(\lambda, z_1) = f(z_1, \lambda z_1), \quad z_1 \neq 0.$$

LEMMA 2.1: Let $\lambda^0 = z_2^0/z_1^0$, $z_1^0 \neq 0$. Then if $f(z_1, z_2)$ is holomorphic in a neighborhood of (z_1^0, z_2^0) , $F(\lambda, z_1)$ is holomorphic in a neighborhood of (λ^0, z_1^0) , and conversely.

Proof: Let f be holomorphic in the bicylinder $|z_1 - z_1^0| \leq r_1$, $|z_2 - z_2^0| \leq r_2$. Without loss of generality suppose f bounded in this bicylinder. As z_1 and z_2 range through this bicylinder, which we assume does not intersect the plane $z_1 = 0$, the point (λ, z_1) sweeps out a neighborhood $\mathfrak{N}$ of (λ^0, z_1^0) . $F(\lambda, z_1)$ is bounded in $\mathfrak{N}$, clearly. $F(\lambda, z_1)$ is also holomorphic in λ, z_1 separately, since we can exhibit derivatives

$$\frac{\partial F}{\partial z_1} = \frac{\partial f(z_1, \lambda z_1)}{\partial z_1} + \lambda \frac{\partial f(z_1, \lambda z_1)}{\partial z_2}, \quad \frac{\partial F}{\partial \lambda} = z_1 \frac{\partial f(z_1, \lambda z_1)}{\partial z_2}.$$

Since $\mathfrak{N}$ contains a bicylinder, Hartogs' theorem shows F is holomorphic in some neighborhood of (λ^0, z_1^0) .

The converse is proved similarly. Note that

$$\frac{\partial f}{\partial z_1} = \frac{\partial F(\lambda, z_1)}{\partial \lambda} \left(\frac{-z_2}{z_1^2} \right) + \frac{\partial F(\lambda, z_1)}{\partial z_1} = -\frac{\lambda}{z_1} \frac{\partial F(\lambda, z_1)}{\partial \lambda} + \frac{\partial F(\lambda, z_1)}{\partial z_1}$$

and

$$(\partial f / \partial z_2) = (1/z_1)(\partial F(\lambda, z_1) / \partial \lambda),$$

so we have exhibited derivatives of f in terms of derivatives of F with respect to λ and z_1 .

We obtain a series for $F(\lambda, z_1)$. If f is holomorphic in a neighborhood at the origin the series (1.1) has a region of absolute convergence [5, p. 2] and so we can justify the following sequence of formal manipulations;

$$\begin{aligned} F(\lambda, z_1) &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{i,k} \lambda^i z_1^{i+k} \\ &= \sum_{i=0}^{\infty} \sum_{r=-i}^{\infty} a_{i,r-i} \lambda^r z_1^r \\ &= \sum_{n=0}^{\infty} p_n(\lambda) z_1^n \end{aligned} \tag{2.1}$$

where

$$p_n(\lambda) = \sum_{k=0}^n a_{n-k,k} \lambda^k. \tag{2.1'}$$

It is clear that the sequence $p_n(\lambda)$ uniquely determines the function $f(z_1, z_2)$ for which $F(\lambda, z_1)$ is the plane section transform.

LEMMA 2.2: Let $F(\lambda, z_1)$ be given by (2.1), (2.1'). Then the corresponding series (1.1) converges in a full neighborhood of the origin if and only if there exist constants C_1 and C_2 such that

$$|p_n(\lambda)| < C_1(C_2(1 + |\lambda|))^n. \quad (2.2)$$

Proof: Suppose the series (1.1) converges uniformly in $|z_1| \leq R$, $|z_2| \leq R$, and has an upper bound M there. We have [6, p. 27]

$$|a_{i,k}| = \frac{1}{4\pi} \left| \int_{|z_1|=R} \int_{|z_2|=R} \frac{f(z_1, z_2)}{z_1^{i+1} z_2^{k+1}} dz_1 dz_2 \right| < MR^{-(i+k)}.$$

Thus

$$|p_n(\lambda)| \leq MR^{-n}(1 + |\lambda| + \cdots + |\lambda|^n) < MR^{-n}(1 + |\lambda|)^n.$$

Conversely, let (2.2) be satisfied. Then

$$|a_{n-k,k}| = \frac{1}{2\pi} \left| \int_{|\lambda|=1} \frac{p_n(\lambda)}{\lambda^{k+1}} d\lambda \right| < C_1 C_2^n \cdot 2^n$$

and we have, putting $j = n - k$ in the above estimate,

$$|a_{j,k} z_1^j z_2^k| \leq C_1 C_2^{j+k} 2^{j+k} |z_1|^j |z_2|^k < C_1 2^{-(j+k)}$$

if we take $|z_1|$ and $|z_2|$ both less than $1/(4C_2)$.

This lemma will be useful, for we will later construct examples by specifying the p_n , and we see now what growth condition we need so that $F(\lambda, z_1)$ is the plane section transform of a non-trivial double series.

We define $\rho(\lambda)$ to be the radius of convergence of the series for $F(\lambda, z_1)$; thus

$$1/\rho(\lambda) = \overline{\lim} |p_n(\lambda)|^{1/n}.$$

We define the *radius of holomorphy*, $r(\lambda)$, by

$$r(\lambda) = \text{g.l.b. } \{t/(|z_1| < t) \subset (\mathfrak{D} \cap \mathcal{P}(\lambda))\}.$$

$r(\lambda)$ is the radius of the largest disc whose boundary does not pass out of $\mathcal{P}(\lambda)$ or $\mathfrak{D}$, $\mathfrak{D}$ being the domain where f is holomorphic.

Since holomorphy in two variables implies holomorphy in each separately, we have $r(\lambda) \leq \rho(\lambda)$.

LEMMA 2.3:

$$r(\lambda_0) \leq \varliminf_{\lambda \rightarrow \lambda_0} r(\lambda).$$

Proof: $F(\lambda, z_1)$ is holomorphic in a neighborhood of each point of the disc $\lambda = \lambda_0$, $|z_1| \leq r(\lambda_0) - \epsilon$. By compactness $F(\lambda, z_1)$ is holomorphic in a full neighborhood $|z_1| < r(\lambda_0) - \epsilon$, $|\lambda - \lambda_0| < \delta$, for some δ sufficiently small. By Lemma 2.1, $f(z_1, z_2)$ is holomorphic in this set also.

It is now easy to see that $\rho(\lambda) \neq r(\lambda)$ always. Let $p_n(\lambda) = P(\lambda)$, a fixed polynomial, for n sufficiently large. Then $\rho(\lambda) = 1$ off the zeroes of P , and is infinite at each zero. If $\rho(\lambda)$ were always equal to $r(\lambda)$ we would contradict Lemma 2.3.

3. A Formula for $r(\lambda)$. In what follows, let

$$\overline{\lim}_{n, \lambda \rightarrow \lambda_0} |p_n(\lambda)|^{1/n}$$

mean the supremum of values as n tends to infinity and λ tends to λ_0 independently. In this section we suppose throughout that (2.2) is satisfied.

Our next result is known, at least in an analogous situation[¶] where λ ranges over a sequence [8, p. 277].

LEMMA 3.1:

$$1/r(\lambda_0) = \overline{\lim}_{n, \lambda \rightarrow \lambda_0} |p_n(\lambda)|^{1/n}.$$

Proof: For notational convenience let

$$1/\rho^*(\lambda_0) = \overline{\lim}_{n, \lambda \rightarrow \lambda_0} |p_n(\lambda)|^{1/n}. \quad (3.1)$$

We arrange our proof by showing first that

$$r(\lambda_0) > \rho^*(\lambda_0) - 2\epsilon \quad (3.2)$$

for any $\epsilon > 0$, and then showing that

$$r(\lambda_0) < \rho^*(\lambda_0) + \epsilon. \quad (3.3)$$

There are numbers N, δ such that

$$|p_n(\lambda)|^{1/n} < 1/(\rho^*(\lambda_0) - \epsilon), \quad n > N, |\lambda - \lambda_0| < \delta.$$

Let $F_\nu(\lambda, z_1)$ be the section

$$F_\nu(\lambda, z_1) = \sum_{n=0}^{\nu} p_n(\lambda) z_1^n.$$

For $\nu > \mu > N$, λ as above, and $|z_1| < \rho^*(\lambda_0) - 2\epsilon$, we have

$$|F_\nu(\lambda, z_1) - F_\mu(\lambda, z_1)| < \sum_{n=\mu+1}^{\nu} \left(\frac{\rho^*(\lambda_0) - 2\epsilon}{\rho^*(\lambda_0) - \epsilon} \right)^n.$$

The $F_\nu(\lambda, z_1)$ are uniformly convergent and holomorphic in $|z_1| < \rho^*(\lambda_0) - 2\epsilon$, $|\lambda - \lambda_0| < \delta$; hence so is their limit, by Hartogs theorem. This proves (3.2).

Now $F(\lambda, z_1)$ is holomorphic and bounded on some set $|z_1| \leq r(\lambda_0) - \epsilon$, $|\lambda - \lambda_0| < \delta$, by a simple compactness argument. We have

$$|p_n(\lambda)| = \frac{1}{2\pi} \left| \int_{|z_1|=r(\lambda_0)-\epsilon} \frac{F(\lambda, z_1)}{z_1^{n+1}} dz_1 \right| < M(r(\lambda_0) - \epsilon)^{-n}$$

provided $|\lambda - \lambda_0| \leq \delta$; thus

$$|p_n(\lambda)|^{1/n} < M^{1/n}/(r(\lambda_0) - \epsilon)$$

for all n . This proves (3.3).

We can now give a more satisfactory formula for $r(\lambda)$, in which the role of n in Lemma 3.1 disappears.

THEOREM 3.1:

$$1/r(\lambda_0) = \overline{\lim}_{\lambda \rightarrow \lambda_0} 1/\rho(\lambda).$$

Proof: We have immediately that

$$1/r(\lambda_0) \geq \overline{\lim}_{\lambda \rightarrow \lambda_0} 1/\rho(\lambda) \quad (3.4)$$

for

$$\overline{\lim}_{\lambda \rightarrow \lambda_0} 1/\rho(\lambda) = \overline{\lim}_{\lambda \rightarrow \lambda_0} (\overline{\lim}_{n \rightarrow \infty} |p_n(\lambda)|^{1/n}) \leq \overline{\lim}_{n, \lambda \rightarrow \lambda_0} |p_n(\lambda)|^{1/n} = 1/r(\lambda_0).$$

To prove the reverse of (3.4), we show that

$$1/r(\lambda_0) \leq e^{3h} \overline{\lim}_{\lambda \rightarrow \lambda_0} 1/\rho(\lambda) \quad (3.5)$$

for any $h > 0$.

Let

$$t(\lambda_0) = \log [\overline{\lim}_{\lambda \rightarrow \lambda_0} 1/\rho(\lambda)].$$

There is an $R > 0$ such that

$$\log 1/\rho(\lambda) < t(\lambda_0) + h/2, \quad |\lambda - \lambda_0| \leq R.$$

Consider the subsets ε_N of the circle $\Gamma_R : |\lambda - \lambda_0| = R$, characterized by the condition that λ belongs to ε_N if and only if there are $n > N$ such that

$$|p_n(\lambda)|^{1/n} > \exp(t(\lambda_0) + h).$$

Let $L(\varepsilon_N)$ denote the one-dimensional measure of ε_N in the angular variable $\theta = \arg(\lambda - \lambda_0)$. The ε_N are measurable, their intersection is empty, and they have the nesting property $\varepsilon_{N+1} \subset \varepsilon_N$. By a standard result of measure theory (cf. [4, p. 172]), $\lim L(\varepsilon_N) = 0$. Thus, given $\epsilon > 0$, there is an N_0 such that $L(\varepsilon_N) < \epsilon$, $N > N_0$.

Now we estimate the subharmonic function $\log |p_n(\lambda)|^{1/n}$ near λ_0 by harmonic measures. Let Γ_R be chosen as above, and $n > N_0$. Let $\omega_1 = \Gamma_R \cap \varepsilon_{N_0}$, with $N > N_0$, and let $\omega_2 = \Gamma_R - \omega_1$. Let $W_i(\lambda)$, $j = 1, 2$, be the harmonic measure of ω_j at λ with respect to Γ_R .

By (2.2) there is an upper bound, call it M , for $\log |p_n(\lambda)|^{1/n}$ on all Γ_R . Without loss of generality suppose ϵ chosen so that $M\epsilon < h$. Then for all $n > N_0$, and λ inside Γ_R we have

$$\log |p_n(\lambda)|^{1/n} \leq MW_1(\lambda) + [h + t(\lambda_0)]W_2(\lambda). \quad (3.6)$$

Since harmonic measure never exceeds 1 and tends to linear measure in θ divided by 2π as λ tends to λ_0 , we can choose $|\lambda - \lambda_0| < \delta$ so small that $W_1(\lambda) < 2\epsilon$. For such λ then (3.6) and the remarks preceding it yield

$$\log |p_n(\lambda)|^{1/n} < 3h + t(\lambda_0), \quad (3.7)$$

provided $n > N_0$ and $|\lambda - \lambda_0| < \delta$.

If we take the unrestricted limit superior on the left-hand side of (3.7) and exponentiate, we arrive immediately at (3.5).

COROLLARY 3.1: $r(\lambda)$ agrees with $\rho(\lambda)$ at points where

$$\rho(\lambda) = \varliminf_{\tilde{\lambda} \rightarrow \lambda} \rho(\tilde{\lambda}).$$

In particular $r(\lambda) = \rho(\lambda)$ at points of continuity of $\rho(\lambda)$.

COROLLARY 3.2:

$$r(\lambda) = \lim_{\tilde{\lambda} \rightarrow \lambda} r(\tilde{\lambda}).$$

Proof: Because of Lemma 2.3, we need only show that, for any $\epsilon > 0$,

$$r(\lambda) > \lim_{\tilde{\lambda} \rightarrow \lambda} r(\tilde{\lambda}) - 2\epsilon.$$

Assuming this inequality is false, there is a $\delta > 0$ such that

$$r(\tilde{\lambda}) \geq r(\lambda) + 2\epsilon, \quad 0 < |\tilde{\lambda} - \lambda| < \delta.$$

For each such $\tilde{\lambda}$ there is by Theorem 3.1 a neighborhood $\mathfrak{N}(\tilde{\lambda})$ such that

$$\rho(\gamma) > r(\tilde{\lambda}) - \epsilon > r(\lambda) + \epsilon, \quad \gamma \in \mathfrak{N}(\tilde{\lambda}).$$

The $\mathfrak{N}(\tilde{\lambda})$ cover the set $|\tilde{\lambda} - \lambda| < \delta$; thus $t(\lambda) \leq \rho(\gamma) - \epsilon$, for all γ in $|\gamma - \lambda| < \delta$, and this contradicts Theorem 3.1.

4. Effect of the zeroes. In this section we show that the regularity of $\rho(\lambda)$ depends to a considerable extent on the distribution of zeroes of the p_n . We assume (2.2) is satisfied in this section.

LEMMA 4.1: *If λ_0 is not a limit point of zeroes of the $p_n(\lambda)$, $\rho(\lambda)$ is continuous at λ_0 , or infinite in some full neighborhood of λ_0 .*

Note: We always also call λ_0 a limit point of zeroes if $p_n(\lambda_0)$ vanishes for infinitely many n .

Proof: There is by hypothesis a domain $\mathfrak{K}$ about λ_0 , with compact closure, in which p_n is free of zeroes, after perhaps deleting finitely many subscripts n . The functions $(p_n(\lambda))^{1/n}$ are analytic in $\mathfrak{K}$, uniformly bounded there, and hence equicontinuous [1, p. 171].

To show $1/\rho(\lambda)$ continuous, let λ_1 be fixed in $\mathfrak{K}$, and let n_k be a subsequence such that

$$|p_{n_k}(\lambda_1)|^{1/n_k} \rightarrow 1/\rho(\lambda_1).$$

Given ϵ there is a δ such that

$$|p_n(\tilde{\lambda})|^{1/n} > |p_n(\lambda)|^{1/n} - \epsilon, \quad |\tilde{\lambda} - \lambda| < \delta.$$

Thus if $|\lambda_2 - \lambda_1| < \delta$ one has

$$1/\rho(\lambda_2) \geq \lim_k |p_{n_k}(\lambda_2)|^{1/n_k} \geq 1/\rho(\lambda_1) - \epsilon$$

and since we can clearly reverse the roles of λ_2 , λ_1 , $1/\rho(\lambda)$ is continuous.

Suppose now that $\rho(\lambda_1) = \infty$. The sequence n_k above has a subsequence n'_k such that $(p_{n'_k}(\lambda))^{1/n'_k} \rightarrow \phi(\lambda)$, $\phi(\lambda)$ analytic in $\mathfrak{K}$, with $\phi(\lambda_1) = 1/\rho(\lambda_1)$ [1, p. 171]. By Hurwitz' theorem, either $\phi(\lambda)$ vanishes identically or λ_1 is a limit point of zeroes of the $p_{n'_k}(\lambda)$ [1, p. 171].

We give a generalization of the above, after some preliminaries.

For a sequence of sets $\mathcal{E}_n$ we define the superior limit by

$$\overline{\lim} \mathcal{E}_n = \{\lambda \mid \lambda \in \mathcal{E}_n \text{ for infinitely many } n\}.$$

A set is of *linear measure zero* if it can be covered by a set of circles the sum of whose diameters is arbitrarily small. Clearly a set of linear measure zero is of planar measure zero, and contains no continuum.

We shall use the following well-known result (cf. [3, p. 46]):

LEMMA (*Boutroux-Cartan*): *Outside a set of circles the sum of whose radii does not exceed $2H$, one has*

$$\prod_{i=1}^n |z - z_i| > (H/e)^n.$$

THEOREM 4.1: *Let $\sigma_n(\mathfrak{D})$ denote the number of zeroes of $p_n(\lambda)$ in the domain $\mathfrak{D}$. If*

$$\sigma_n(\mathfrak{D}) = o(n),$$

then, off a subset of linear measure zero in $\mathfrak{D}$, $\rho(\lambda)$ is continuous or $\rho(\lambda) \equiv \infty$.

Proof: Let $\lambda_{j,n}$, $1 \leq j \leq \sigma_n$, be the zeroes of $p_n(\lambda)$ in $\mathfrak{D}$, multiplicity counted and write

$$p_n(\lambda) = t_n(\lambda)q_n(\lambda)$$

where

$$t_n(\lambda) = \prod_{j=1}^{\sigma_n} (\lambda - \lambda_{j,n}).$$

We assume without loss of generality that $\mathfrak{D}$ is bounded.

It is clear that for some C depending on $\mathfrak{D}$,

$$|t_n(\lambda)| < C^{\sigma_n}, \quad \lambda \in \mathfrak{D}$$

thus

$$\overline{\lim} |t_n(\lambda)|^{1/n} \leq 1, \quad \lambda \in \mathfrak{D}. \quad (4.1)$$

Now let h_n be any sequence decreasing to zero sufficiently slowly that $\log 1/h_n = o(n/\sigma_n)$, let $\mathfrak{E}_n$ be the set $\mathfrak{E}_n = \{\lambda/|t_n(\lambda)| \leq (h_n/e)^{\sigma_n}\}$, and finally let $\mathfrak{K} = \overline{\lim} \mathfrak{E}_n$. From the Boutroux-Cartan lemma $\mathfrak{K}$ is of linear measure zero, and off $\mathfrak{K}$ one has

$$\lim |t_n(\lambda)|^{1/n} \geq \lim (h_n/e)^{\sigma_n/n} = 1. \quad (4.2)$$

(4.1) and (4.2) together yield

$$\lim |t_n(\lambda)|^{1/n} = 1, \quad \lambda \in \mathfrak{D} - \mathfrak{K}.$$

In this same set

$$\lim |p_n(\lambda)|^{1/n} = \overline{\lim} |q_n(\lambda)|^{1/n}.$$

Our conclusion follows by applying Lemma 4.1 to the q_n , provided we verify that q satisfies an inequality (2.2).

(2.2) is satisfied by p_n , by hypothesis; thus q_n satisfies (2.2) off $\mathfrak{K}$. The subharmonic character of q_n shows that q_n satisfies (2.2) also on all of the λ -plane, with possibly a slightly altered choice of C_1 and C_2 .

5. On further constructions. We have already seen that we can make $\rho(\lambda)$ different from $r(\lambda)$ on any finite set. We give further constructions on which the two functions are distinct on infinite point sets. In this section we will also mention some questions which appear to the writer to be of considerable interest.

We first give a construction in which $\rho(\lambda)$ is discontinuous at each point of $|\lambda| = 1$, and different from $r(\lambda)$ on a dense subset of this circle. Let us define $p_n(\lambda)$ to be identically zero unless n is of the form $n = n_K = 3^K - 1$, K a positive integer, in which case

$$p_{n_K}(\lambda) = 1 + \lambda + \cdots + \lambda^{n_K}.$$

The coefficients $a_{i,k}$ are either 0 or 1, so the series (1.1) converges in a region including $|z_1| < 1$, $|z_2| < 1$. It is clear that $\rho(\lambda) = 1$ for $|\lambda| < 1$, and $\rho(\lambda) = 1/|\lambda|$ for $|\lambda| > 1$. Also, since $|p_n(e^{i\theta})| \leq 1 + n$ we have $\rho(\lambda) \geq 1$ on $|\lambda| = 1$. By Theorem 3.1, $r(\lambda) \equiv 1$ on $|\lambda| = 1$.

Now set $\lambda_{M,K} = \exp \{2M\pi i/3^K\}$ and $\lambda_{M,K}^* = \exp \{(1 + 2M)\pi i/3^K\}$, with M, K positive integers. Note that each $\lambda_{M,K}$ is a point in the set $\{\lambda_{M,N}\}$ if $N > K$, and that similarly the set $\lambda_{M,K}^*$, k fixed, is in the set $\lambda_{M,N}^*$, N fixed and larger than K . Since

$$p_{n_K}(\lambda) = \frac{1 - \lambda^{3^K}}{1 - \lambda},$$

$p_{n_N}(\lambda_{M,K}) = 0$ for $N \geq K$ and $p_{n_N}(\lambda_{M,K}^*) = 2/(1 - \lambda)$ for $N \geq K$. This gives us $\rho(\lambda_{M,K}) = \infty$, $\rho(\lambda_{M,K}^*) = 1$. Thus $\rho(\lambda)$ takes the values ∞ and 1, each on a dense subset of the circle $|\lambda| = 1$.

This example has the interesting feature that one can exhibit $f(z_1, z_2)$ in a simple way. For $|z_2| < |z_1|$ we have

$$\begin{aligned} f(z_1, z_2) &= \sum_K z_1^{n_K} \frac{1 - \lambda^{n_K+1}}{1 - \lambda} \\ &= \sum_K \frac{z_1^{n_K+1}}{z_1 - z_2} (1 - (z_2/z_1)^{n_K+1}) \\ &= \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} \end{aligned}$$

where

$$\varphi(z) = \sum_K z^{n_K}.$$

By analytic continuation this representation for $f(z_1, z_2)$ is valid for $|z_1| < 1$, $|z_2| < 1$, with

$$f(z_1, z_1) = \lim_{z_2 \rightarrow z_1} \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} = \varphi'(z_1).$$

$f(z_1, z_2)$ reduces to a polynomial on each hyperplane $z_2 = \lambda_{M,K} z_1$.

We can construct examples in which the singular character of $\rho(\lambda)$ is not restricted to a curve.

LEMMA 5.1: *Let $\mathcal{E}$ be a countable dense subset of the bounded domain $\mathcal{D}$. Then there is a function for which $r(\lambda)$ is different from $\rho(\lambda)$ on $\mathcal{E}$, and $\rho(\lambda)$ is discontinuous at each point of $\mathcal{D}$.*

Proof: We enumerate the points of $\mathcal{E}$ as the sequence $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$. Let M_n be a sequence of numbers decreasing to zero, and let $\mathfrak{M}_n$ denote the following union of closed discs:

$$\mathfrak{M}_n = \bigcup_{i=1}^n \{\lambda/|\lambda - \lambda_i| \leq M_n\}.$$

We can choose the M_n in such a way that $\bigcap_n \mathfrak{M}_n = \mathcal{E}$. Let us assume this has been done. We then choose a sequence h_n of positive numbers such that $2eh_n < M_n$.

As the final choice of parameters, let N_n be a sequence tending to infinity so rapidly that $(n/N_n) \log h_n$ tends to zero. We take $t_n(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$.

We have

$$\lim |t_n(\lambda)|^{1/N_n} \geq 1, \quad \lambda \in \mathfrak{D} - \mathcal{E}$$

by the Boutroux-Cartan lemma. Letting d be the diameter of $\mathfrak{D}$, one has

$$|t_n(\lambda)|^{1/N_n} \leq d^{n/N_n} \rightarrow 1, \quad \lambda \in \mathfrak{D}.$$

We now give the sequence $p_n(\lambda)$ needed for the construction. We take $p_n(\lambda)$ identically zero unless k is one of the numbers N_n . We define

$$p_{N_n}(\lambda) = t_n(\lambda).$$

It is clear that the p_n satisfy a relation of the form (2.2), so we have constructed a plane-section transform of some function. Furthermore, $\rho(\lambda) = \infty$ on $\mathcal{E}$ and $\rho(\lambda) = 1$ on $\mathfrak{D} - \mathcal{E}$. We have $r(\lambda) \equiv 1$ on $\mathcal{E}$.

This construction can be modified, using a sequence of the form

$$t_{n,k}(\lambda) = \prod_{i \leq n; |\lambda_{i,n}| < k} (\lambda - \lambda_i)$$

to yield the following result:

THEOREM 5.1: *Let $\mathcal{E}$ be any countable set. Then there is a function for which $\rho(\lambda)$ is discontinuous at each point of $\mathcal{E}$, $\rho(\lambda) = \infty$ at each point of $\mathcal{E}$ and $r(\lambda) \equiv 1$.*

It is easy to modify the above examples so that $\rho(\lambda)$ does not become infinite on the set $\mathcal{E}$. Indeed, let A be any number with $0 < A < 1$, and modify the $p_n(\lambda)$ constructed above by setting $\tilde{p}_n(\lambda) = p_n(\lambda) + A^n$. Then for the corresponding function $\rho(\lambda)$ one has

$$\begin{aligned} \tilde{\rho}(\lambda) &= 1, & \lambda \in \mathfrak{D} - \mathcal{E} \\ &= 1/A, & \lambda \in \mathcal{E}. \end{aligned}$$

Let $\mathcal{E}$, in general, denote the set where $\rho(\lambda)$ and $r(\lambda)$ are distinct. We can ask what sets $\mathcal{E}$ are possible. This question seems to the writer to be difficult and worth investigating. We know from the above that $\mathcal{E}$ can be any countable set, and it is possible that in general $\mathcal{E}$ must be in some sense a thin set. On the other hand, it may be that one could produce examples where $\mathcal{E}$ is the entire plane, by producing sequences $p_n(\lambda)$ for which the functions $(p_n(\lambda))^{1/n}$ are small most of the time, but have comparatively steep peaks on a dense subset. The writer believes at the moment that such a construction is possible, but difficult. Regarding this, the following result is of some interest.

THEOREM 5.2: *Let $r(\lambda) = 1$ on a dense subset $\mathcal{S}$ of the domain $\mathfrak{D}$, and let $\mathcal{K}$ be the subset on which $\rho(\lambda) > 1$. Then $\mathcal{K}$ is of measure zero.*

Proof: Let $\mathcal{K}_h$ be the subset where $\rho(\lambda) \geq e^h$. Since $\mathcal{K} = \bigcup_n \mathcal{K}_{1/n}$, we need only prove $\mathcal{K}_h$ has measure zero for any $h > 0$. We can reduce the problem further; it is enough to show that for any $\lambda_0 \in \mathcal{S}$ there is an $R > 0$ such that $\mathcal{K}_h$ meets the disc $|\lambda - \lambda_0| < R$ in a set of measure zero, for $\mathcal{S}$ certainly contains a dense subset and we can cover $\mathfrak{D}$ by a countable collection of discs, of arbitrarily small radius, with centers on points of $\mathcal{S}$.

Let $\epsilon > 0$ be given. Because of Theorem 3.1 there is an $R > 0$ such that the disc

$|\lambda - \lambda_0| \leq R$ lies in $\mathfrak{D}$, and such that $\rho(\lambda) > e^{-\epsilon}$ in this disc. We now estimate $|p_n(\lambda)|^{1/n}$ near λ_0 by a harmonic measure argument similar to that used in the proof of Theorem 3.1.

Let Γ_* be the circle $|\lambda - \lambda_0| = s$, with $R/2 < s < R$. Off a subset Ω_1 of Γ_* of arbitrarily small measure δ we have

$$\log |p_n(\lambda)|^{1/n} \leq \log 1/\rho(\lambda) + \epsilon, \quad n > n_0(\delta).$$

There is an overall bound, call it M , for the quantity $\log |p_n(\lambda)|^{1/n}$ on all of $R/2 < |\lambda - \lambda_0| < R$, and we can assume δ so small that $M\delta < \epsilon$.

Now let Ω_2 be the subset of $\Gamma_* - \Omega_1$ where $\log 1/\rho(\lambda) \leq -h$, and let $\Omega_3 = \Gamma_* - \Omega_1 - \Omega_2$. Let $W_j(\lambda)$ be the harmonic measure of Ω_j , $j = 1, 2, 3$. Then for $n > n_0(\delta)$,

$$\log |p_n(\lambda)|^{1/n} \leq \epsilon W_1(\lambda) + \epsilon W_3(\lambda) + (\epsilon - h)W_2(\lambda) = \epsilon - hW_2(\lambda). \quad (5.1)$$

It is easily verified that for $|\lambda - \lambda_0| < R/8$, the Poisson kernel of Γ_* does not fall below $6/17\pi$. From (5.1), then, and letting Γ_* denote the linear measure of Ω_2 in the angular variable θ ,

$$\log |p_n(\lambda)|^{1/n} \leq -6hL_*/17\pi, \quad n > n_0, \quad |\lambda - \lambda_0| < R/8,$$

and since ϵ is arbitrary, we may take the unrestricted limit superior on the left-hand to conclude that

$$r(\lambda_0) \geq \exp \{6hL_*/17\pi\}$$

which is a contradiction unless L_* is zero.

This proves $\mathcal{K}_\lambda$ is of measure zero. Indeed, we have proved more, for we have shown that $\mathcal{K}_\lambda$ intersects any circle, centered at a point of S , in a set of one-dimensional measure zero.

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