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FLATTENING OF MEMBRANES OF REVOLUTION BY LARGE STRETCHING—ASYMPTOTIC SOLUTION WITH BOUNDARY LAYER*

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Abstract. The problem is solved explicitly by the method of matching asymptotics. The stretching of a flat membrane with an inclusion is studied as an example. Asymptotic stress and strain concentration ratios are derived. It is shown that the stress concentration ratio tends to unity as the stretching tends to infinity. This is justified by the exact numerical solution.

1. Introduction. Let (r, θ, z) be a fixed cylindrical coordinate system and let S measure the dimensionless arc length along a curve C defined by

$$C: \begin{matrix} r = R(S) \\ z = Z(S) \end{matrix}, \qquad 0 \leq S \leq 1, \tag{1.1}$$

characterizing the meridian curve of a membrane of revolution. The function Z is positive and vanishes only at $S = 0$. The function R is positive and satisfies at $S = 0$ one of the two conditions:

Case I. $R(0) = 0$:

$$R(S) = S + 0(S^3), \qquad S \to 0. \tag{1.2}$$

Case II. $R(0) \neq 0$:

$$R(S) = \sum_{n=0}^{\infty} \frac{1}{n!} R_n S^n, \qquad R_n = \left. \frac{d^n R}{dS^n} \right|_{S=0}. \tag{1.3}$$

For Case II, the edge $r = R(0)$ is fixed. The membrane is flattened by an outward stretching applied along the edge $r = R(1)$ so that the deformed membrane can be characterized by a meridian curve c defined by

$$c: \begin{matrix} r = X(S) \\ z \equiv 0 \end{matrix}, \qquad \begin{matrix} X(0) = R(0) \\ X(1) = A \end{matrix}, \tag{1.4}$$

where A is a given quantity sufficiently large to achieve the flattened state.

Problems of this nature have been treated previously by researchers. The deformation from a tube to an annulus can be solved exactly [1] and the flattening of a spherical cap

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is solved numerically by Yang and Feng [2]. Our objective is to solve asymptotically the class of problems in terms of the parameter A as $A \to \infty$. As we shall see, case I is a special case of case II and hence only case II will be considered.

Consider the deformation from (1.1) to (1.4). If we define respectively Λ_1 and Λ_2 as the principal extension ratios in the meridian and azimuthal directions, then

$$\Lambda_1 = dX/dS, \qquad \Lambda_2 = X/R. \tag{1.5, 1.6}$$

A Mooney material [3, 4] is characterized by a strain energy function W defined by

$$W(\Lambda_1, \Lambda_2) = \left(\Lambda_1^2 + \Lambda_2^2 + \frac{1}{\Lambda_1^2\Lambda_2^2}\right) + k\left(\Lambda_1^2\Lambda_2^2 + \frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2}\right) \tag{1.7}$$

where $k = C_2/C_1$, the ratio of the two Mooney constants, and W is nondimensionalized by the quantity C_1H, H being the constant thickness of the undeformed membrane.

Based on (1.7), the fundamental equations can be derived. We prefer to use the set of equations given in [5]. These are

$$T_1 = (1/\Lambda_2)W_{\Lambda_1}, \tag{1.8}$$

$$T_2 = (1/\Lambda_1)W_{\Lambda_2}, \tag{1.9}$$

$$X(dT_1/dS) = (T_2 - T_1)(dX/dS), \tag{1.10}$$

where the subscripts on W denote partial differentiation with respect to the indicated argument. T_1 and T_2 are, respectively, the meridian and azimuthal stress resultants. Eqs. (1.5)–(1.10), together with the boundary conditions

$$X(0) = R_0 \tag{1.11}$$

and

$$X(1) = A, \tag{1.12}$$

constitute the complete formulation of the problem. We wish to solve asymptotically the above equations in terms of the parameter A as $A \to \infty$.

In Sec. 2 an expansion valid for the interval $0 < S \leq 1$ is obtained. A boundary layer expansion which is valid near and including $S = 0$ is given in Sec. 3. The two expansions are matched in Sec. 4. Finally, the stretching of a flat membrane with an inclusion is given as an example in Sec. 5. Explicit asymptotic stress and strain concentration ratios are also obtained. These asymptotic results are shown to be approached by the exact numerical solution.

2. Solution away from $S = 0$. We consider an asymptotic expansion for the interval $0 < S \leq 1$. Since $X(1) = A$, X must be of order A as $A \to \infty$. Thus we write

$$X = Ax,\ \Lambda_1 = A\lambda_1,\ \Lambda_2 = A\lambda_2, \qquad T_1 = A^2t_1,\quad T_2 = A^2t_2. \tag{2.1}$$

Eq. (1.7) now becomes

$$W(\Lambda_1, \Lambda_2) = A^4w(\lambda_1, \lambda_2, \epsilon) \tag{2.2}$$

where

$$w(\lambda_1, \lambda_2, \epsilon) = k\lambda_1^2\lambda_2^2 + \epsilon(\lambda_2^2 + \lambda_2^2) + k\epsilon^3\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right) + \epsilon^4\frac{1}{\lambda_1^2\lambda_2^2} \tag{2.3}$$

and $\epsilon = A^{-2}$ is assumed to be a small parameter. Because of (2.3), we shall consider all the newly introduced quantities in (2.1) as functions of S and ϵ and write $f = f(S, \epsilon)$ where f is a generic symbol.

The system of equations (1.5)–(1.12) now becomes

$$\lambda_1 = dx/dS, \tag{2.4}$$

$$\lambda_2 = x/R, \tag{2.5}$$

$$t_1 = \frac{1}{\lambda_2} w_{\lambda_1} = 2k\lambda_1\lambda_2 + \epsilon 2 \frac{\lambda_1}{\lambda_2} - \epsilon^3 \frac{2k}{\lambda_1^3\lambda_2} - \epsilon^4 \frac{2}{\lambda_1^3\lambda_2^3}, \tag{2.6}$$

$$t_2 = \frac{1}{\lambda_1} w_{\lambda_2} = 2k\lambda_1\lambda_2 + \epsilon 2 \frac{\lambda_2}{\lambda_1} - \epsilon^3 \frac{2k}{\lambda_1\lambda_2^3} - \epsilon^4 \frac{2}{\lambda_1^3\lambda_2^3}, \tag{2.7}$$

$$x(dt_1/dS) = (t_2 - t_1)(dx/dS), \tag{2.8}$$

$$x(1, \epsilon) = 1. \tag{2.9}$$

The condition (1.11) cannot be satisfied because of the scaling factor introduced.* Thus the asymptotic solution to the above equations is not valid at $S = 0$. We assume that the solution can be expanded as an asymptotic series of the form

$$f(S, \epsilon) = f_0(S) + \epsilon \ln \epsilon f_1(S) + \epsilon f_2(S) + \epsilon^2 \ln \epsilon f_3(S) + \cdots . \tag{2.10}$$

As we shall see, the lowest-order terms in the expansion have logarithmic singularities as $S \to 0$. For this reason the quantity $\ln \epsilon$ is included in the asymptotic sequence. To facilitate our calculations, we shall cast (2.10) into the form

$$f(S, \epsilon) = f_0^*(S, \epsilon) + \epsilon f_1^*(S, \epsilon) + \cdots \tag{2.11}$$

where

$$f_0^*(S, \epsilon) = f_0(S) + \epsilon \ln \epsilon f_1(S), \tag{2.12}$$

$$f_1^*(S, \epsilon) = f_2(S) + \epsilon \ln \epsilon f_3(S). \tag{2.13}$$

We shall first obtain f_0^* and f_1^* ; these will then be expanded to yield f_n . This step is legitimate since f_2 does not depend on f_1 and f_3 does not depend on f_0 . As a matter of fact, only the first three terms of (2.10) will be given explicitly in our final result.

Substituting expansions of the form (2.11) into (2.4)–(2.9) yields, for the terms corresponding to f_0^* ,

$$\lambda_{10}^* = dx_0^*/dS, \tag{2.14}$$

$$\lambda_{20}^* = x_0^*/R(S), \tag{2.15}$$

$$t_{10}^* = 2k\lambda_{10}^*\lambda_{20}^* , \tag{2.16}$$

$$t_{20}^* = 2k\lambda_{10}^*\lambda_{20}^* , \tag{2.17}$$

$$x_0^*(dt_{10}^*/dS) = (t_{20}^* - t_{10}^*)\,(dx_0^*/dS), \tag{2.18}$$

$$x_0^*(1, \epsilon) = 1. \tag{2.19}$$

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* The regularity condition associated with case I can be satisfied and hence case I has no boundary layer.

It follows from (2.16), (2.17) and (2.18) that

$$t_{10}^* = t_{20}^* = \text{constant}. \tag{2.20}$$

Eq. (2.16), together with (2.14), (2.15), (2.19) and (2.20) now yields

$$x_0^{*2} = 1 - \frac{t_{10}^*}{k}\int_S^1 R(S')\,dS'. \tag{2.21}$$

Expanding now all the quantities according to (2.12), we obtain

$$t_{10} = t_{20} = \text{constant}, \tag{2.22}$$

$$x_0 = \left[1 - \frac{t_{10}}{k}\int_S^1 R(S')\,dS'\right]^{1/2}, \tag{2.23}$$

$$t_{11} = t_{21} = \text{constant}, \tag{2.24}$$

$$x_1 = -\frac{t_{11}}{2kx_0}\int_S^1 R(S')\,dS'. \tag{2.25}$$

We proceed to the terms corresponding to $f_1^*(S, \epsilon)$. Since we shall include only the first three terms of (2.1) in our final results we shall replace $f_1^*(S, \epsilon)$ by $f_2(S)$ in the following calculation. The governing equations are

$$\lambda_{12} = (dx_2/dS), \tag{2.26}$$

$$\lambda_{22} = (x_2/R), \tag{2.27}$$

$$t_{12} = 2k(\lambda_{10}\lambda_{22} + \lambda_{12}\lambda_{20}) + 2(\lambda_{10}/\lambda_{20}), \tag{2.28}$$

$$t_{22} = 2k(\lambda_{10}\lambda_{22} + \lambda_{12}\lambda_{20}) + 2(\lambda_{20}/\lambda_{10}), \tag{2.29}$$

$$x_0(dt_{12}/dS) = (t_{22} - t_{12})(dx_0/dS), \tag{2.30}$$

$$x_2(1) = 0. \tag{2.31}$$

Substituting (2.28) and (2.29) into (2.30) and using (2.23), we get

$$(dt_{12}/dS) = F(S) \tag{2.32}$$

where

$$F(S) = (2/R)(1 - (t_{10}^2R^4/4k^2x_0^4)). \tag{2.33}$$

It follows that

$$t_{12} = t_{12}(1) - \int_S^1 F(S')\,dS' \tag{2.34}$$

where $t_{12}(1)$ is a constant to be determined. The function t_{22} can be determined from (2.29). We have

$$t_{22} = t_{12} + (2kx_0^2F(S)/t_{10}R(S)). \tag{2.35}$$

Substituting (2.26) and (2.27) into (2.28) and using (2.23), we obtain

$$(d(x_0x_2)/dS) = (R/2k)t_{12} - (t_{10}/2k^2)(R^3/x_0^2) \tag{2.36}$$

and hence

$$x_2 = -\frac{1}{x_0} \int_S^1 \left(\frac{Rt_{10}}{2k} - \frac{t_{10}}{2k^2} \frac{R^3}{x_0^2} \right) dS' \tag{2.37}$$

by (2.31). This completes a formal three-term expansion for the interval $0 < S \leq 1$. The expansion involves three unknown constants t_{10}, t_{11} and $t_{12}(1)$, and is not valid at $S = 0$.

We shall be needing the asymptotic behaviors of the functions x_m as $S \to 0$. Defining R_c by the expression

$$R_c = \int_0^1 R(S)\, dS \tag{2.38}$$

and using (1.3), we obtain from (2.23)

$$x_0 \sim \left\{ \left(1 - \frac{t_{10}R_c}{k}\right) + \frac{t_{10}R_0}{k} S + \frac{t_{10}R_1}{2k} S^2 + \frac{t_{10}R_2}{6k} S^3 + \cdots \right\}^{1/2} \tag{2.39}$$

which is the convergent Taylor expansion. It will be shown in Sec. 4 that, for the purpose of matching, the leading term in the above equation must vanish, i.e.,

$$t_{10} = k/R_c\,. \tag{2.40}$$

While we shall return to justify (2.40), this condition will be used repeatedly. Eq. (2.39) now becomes

$$x_0 \sim \left(\frac{R_0}{R_c} S\right)^{1/2} \left[1 + \frac{R_1}{4R_0} S + \left(\frac{R_2}{12R_0} - \frac{R_1^2}{32R_0^2} \right) S^2 + \cdots \right], \qquad S \to 0. \tag{2.41}$$

Eq. (3.25) now yields

$$x_1 \sim -\frac{t_{11}}{2k} \left(\frac{R_c}{R_0}\right)^{1/2} S^{-1/2} \left[R_c - \left(R_0 + \frac{R_1 R_c}{4R_0} \right) S + \cdots \right], \qquad S \to 0. \tag{2.42}$$

We proceed to find the asymptotic behavior of x_2 as $S \to 0$. Eqs. (2.40), (2.41) and (2.42) will be used repeatedly. We first note from (2.33) that, as $S \to 0$,

$$F(S) \sim -\frac{R_0}{2} S^{-2} - R_1 S^{-1} - \left(\frac{R_2}{12} + \frac{3}{8} \frac{R_1^2}{R_0} - \frac{2}{R_0} \right) + 0(S) \tag{2.43}$$

and hence the integrals appearing in (2.34) and (2.37) do not exist as $S \to 0$. We remove this difficulty by considering the finite-part integration. Thus, define the finite-part integral

$$⨍_0^1 F(S)\, ds \equiv \int_0^1 \left[F(S) + \frac{R_0}{2} S^{-2} + R_1 S^{-1} \right] dS. \tag{2.44}$$

Eq. (2.34) now yields

$$t_{12} \sim +\frac{R_0}{2} S^{-1} - R_1 \ln S + \left[-⨍_0^1 F(S)\, dS - \frac{R_0}{2} + t_{12}(1) \right] + \cdots, \quad S \to 0. \tag{2.45}$$

It follows from (2.45) and (2.40) that the integrand of the integral appearing in (2.37) tends to $-R_0^2 S^{-1}/4k$ as S tends to zero. Defining

$$\fint_0^1 \left(\frac{Rt_{12}}{2k} - \frac{t_{10}}{2k^2}\frac{R^3}{x_0^2}\right) dS \equiv \int_0^1 \left(\frac{Rt_{12}}{2k} - \frac{t_{10}}{2k^2}\frac{R^3}{x_0^2} + \frac{R_0^2}{4k}\frac{1}{S}\right) dS, \tag{2.46}$$

we obtain from (2.37)

$$x_2 \sim -\left(\frac{R_c}{R_0}\right)^{1/2} \frac{R_0^2}{4k} S^{-1/2} \ln S - \left(\frac{R_c}{R_0}\right)^{1/2} \fint_0^1 \left(\frac{Rt_{12}}{2k} - \frac{t_{10}}{2k^2}\frac{R^3}{x_0^2}\right) dS\, S^{-1/2} + \cdots. \tag{2.47}$$

We now turn to the solution near $S = 0$.

3. Solution near and including $S = 0$. We begin by introducing a boundary layer variable s defined by

$$s = S/\epsilon, \qquad s \text{ fixed}, \quad \epsilon \to 0. \tag{3.1}$$

Since the edge $r = R(0)$ is fixed, X must be of order one. Thus we write

$$X = \xi, \quad \Lambda_1 = A^2\mu_1, \quad \Lambda_2 = \mu_2, \quad T_1 = A^2\tau_1, \quad T_2 = A^2\tau_2, \tag{3.2}$$

where the newly introduced variables are functions of s and ϵ. We shall write $f = f(s, \epsilon)$ where f is a generic symbol. Substituting (3.1) and (3.2) into (1.5)–(1.11), we obtain

$$W = A^4\omega(\mu_1, \mu_2, \epsilon) \tag{3.3}$$

$$\omega = (\mu_1^2 + k\mu_1^2\mu_2^2) + \epsilon^2\left(\mu_2^2 + \frac{k}{\mu_2^2}\right) + \epsilon^4\left(\frac{1}{\mu_1^2\mu_2^2} + \frac{k}{\mu_1^2}\right) \tag{3.4}$$

and

$$\mu_1 = d\xi/ds, \tag{3.5}$$

$$\mu_2 = (\xi/R_0)(1 - \epsilon(R_1/R_0)s + \cdots), \tag{3.6}$$

$$\tau_1 = \left(2k\mu_1\mu_2 + 2\frac{\mu_1}{\mu_2}\right) + \epsilon^4\left(-\frac{2}{\mu_1^3\mu_2^3} - \frac{2k}{\mu_1^3\mu_2}\right), \tag{3.7}$$

$$\tau_2 = 2k\mu_1\mu_2 + \epsilon^2\left(2\frac{\mu_2}{\mu_1} - \frac{2k}{\mu_1\mu_2^3}\right) + \epsilon^4\left(-\frac{2}{\mu_1^3\mu_2^3}\right), \tag{3.8}$$

$$\xi(d\tau_1/ds) = (\tau_2 - \tau_1)(d\xi/ds), \tag{3.9}$$

$$\xi(0, \epsilon) = R_0. \tag{3.10}$$

The condition $X(1) = A$ cannot be possibly satisfied because of the condition $X = \xi = 0(1)$. Thus the solution to the above equations is again not uniformly valid throughout the interval $0 \leq S \leq 1$. We now assume that the solution can be expanded as an asymptotic series of the form

$$f(s, \epsilon) \sim f_0(s) + \epsilon \ln \epsilon f_1(s) + \epsilon f_2(s) + \cdots. \tag{3.11}$$

Once again, we cast (3.11) into the form

$$\begin{aligned} f(s, \epsilon) &\sim f_0^*(s, \epsilon) + \epsilon f_1^*(s, \epsilon) + \cdots \\ f_0^* &\equiv f_0 + \epsilon \ln \epsilon f_1 \\ f_1^* &\equiv f_2 + \epsilon \ln \epsilon f_3 \end{aligned} \tag{3.12}$$

and the $f_0^*(s, \epsilon)$ terms satisfy the equations

$$\mu_{10}^* = (d\xi_0^*/ds), \tag{3.13}$$

$$\mu_{20}^* = (\xi_0^*/R_0), \tag{3.14}$$

$$\tau_{10}^* = 2k\mu_{10}^*\mu_{20}^* + 2(\mu_{10}^*/\mu_{20}^*), \tag{3.15}$$

$$\tau_{20}^* = 2k\mu_{10}^*\mu_{20}^* , \tag{3.16}$$

$$\xi_0^*(d\tau_{10}^*/ds) = (\tau_{20}^* - \tau_{10}^*)(d\xi_0^*/ds), \tag{3.17}$$

$$\xi_0^*(0, \epsilon) = R_0 . \tag{3.18}$$

Eliminating τ_{20}^* , μ_{10}^* , μ_{20}^* from (3.14)–(3.17), we have

$$\frac{1}{\tau_{10}^*}\frac{d\tau_{10}^*}{ds} = -\frac{R_0^2}{k}\frac{1}{\xi^*(\xi_0^{*2} + (R_0^2/k))}\frac{d\xi_0^*}{ds}$$

or

$$\tau_{10}^* = (\alpha_0^*/\xi_0^*)(\xi_0^{*2} + (R_0^2/k))^{1/2} \tag{3.19}$$

where $\alpha_0^* = \alpha_0^*(\epsilon)$ is a constant to be determined. Eq. (3.15), together with (3.13) and (3.19), now yields

$$(d\xi_0^*/ds) = (R_0\alpha_0^*/2k)(\xi_0^{*2} + (R_0^2/k))^{-1/2} \tag{3.20}$$

or

$$s = \frac{k}{R_0\alpha_0^*}\left\{\xi_0^*\left(\xi_0^{*2} + \frac{R_0^2}{k}\right)^{1/2} + \frac{R_0^2}{k}\ln\left[\xi_0^* + \left(\xi_0^{*2} + \frac{R_0^2}{k}\right)^{1/2}\right]\right.$$
$$\left. - R_0\left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2} - \frac{R_0^2}{k}\ln\left[R_0 + \left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right]\right\}, \tag{3.21}$$

by (3.18). Finally, (3.16), (3.13) and (3.14) yield

$$\tau_{20}^* = \alpha_0^*\xi_0^*(\xi_0^{*2} + (R_0^2/k))^{-1/2}. \tag{3.22}$$

We shall now obtain $f_0(s)$ and $f_1(s)$ from $f_0^*(s, \epsilon)$ according to the relation $f_0(s, \epsilon) = f_0^*(s) + \epsilon \ln \epsilon\, f_1(s)$. Eqs. (3.21), (3.19) and (3.22) yield

$$s = \frac{k}{R_0\alpha_0}\left\{\xi_0\left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2} + \frac{R_0^2}{k}\ln\left[\xi_0 + \left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right]\right.$$
$$\left. - R_0\left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2} - \frac{R_0^2}{k}\ln\left[R_0^2 + \left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right]\right\}, \tag{3.23}$$

$$\tau_{10} = (\alpha_0/\xi_0)(\xi_0^2 + (R_0^2/k))^{1/2}, \tag{3.24}$$

$$\tau_{20} = \alpha_0\xi_0(\xi_0^2 + (R_0^2/k))^{-1/2}, \tag{3.25}$$

$$\xi_1 = (\alpha_1 R_0/2k)s(\xi_0^2 + (R_0^2/k))^{-1/2}, \tag{3.26}$$

$$\tau_{11} = \left[\frac{1}{\alpha_0} - \frac{R_0^3}{2k^2}\frac{s}{\xi_0(\xi_0^2 + (R_0^2/k))^{3/2}}\right]\alpha_1\tau_{10} , \tag{3.27}$$

$$\tau_{21} = \left[\frac{1}{\alpha_0} + \frac{R_0^3}{2k^2} \frac{s}{\xi_0(\xi_0^2 + (R_0^2/k))^{3/2}}\right]\alpha_1\tau_{20}\,, \tag{3.28}$$

where α_0 and α_1 are two constants to be determined.

We proceed to the terms corresponding to $f_2(s)$ in (3.12). The governing equations are

$$\mu_{12} = (d\xi_2/ds), \tag{3.29}$$

$$\mu_{22} = (1/R_0)(\xi_2 - (R_1/R_0)\xi_0 s), \tag{3.30}$$

$$\tau_{12} = 2k(\mu_{10}\mu_{22} + \mu_{12}\mu_{20}) + 2((\mu_{12}\mu_{20} - \mu_{10}\mu_{22})/\mu_{20}^2), \tag{3.31}$$

$$\tau_{22} = 2k(\mu_{10}\mu_{22} + \mu_{12}\mu_{20}), \tag{3.32}$$

$$\xi_0 \frac{d\tau_{12}}{ds} + \xi_2 \frac{d\tau_{10}}{ds} = (\tau_{20} - \tau_{10}) \frac{d\xi_2}{ds} + (\tau_{22} - \tau_{12}) \frac{d\xi_0}{ds}\,, \tag{3.33}$$

$$\xi_2(0) = 0. \tag{3.34}$$

Using (3.30)–(3.33) and applying the zeroth-order solution repeatedly, we get

$$\frac{\tau_{12}}{\tau_{10}} = -\frac{R_0^2}{k} \frac{\xi_2}{\xi_0(\xi_0^2 + (R_0^2/k))} + \frac{R_0R_1}{k} \frac{s}{\xi_0^2 + (R_0^2/k)} - \frac{2R_1}{\alpha_0} \ln\left[\xi_0 + \left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right] + \alpha_2 \tag{3.35}$$

where α_2 is a constant to be determined. Applying (3.35) and (3.31) we obtain, after a lengthy calculation,

$$\begin{aligned} \xi_2\left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2} = {} & \frac{\alpha_2}{2}\left\{\xi_0\left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2} + \frac{R_0^2}{k} \ln\left[\xi_0 + \left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right]\right\} \\ & - \frac{R_0R_1}{k} s \ln\left[\xi_0 + \left(\xi_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right] + \frac{\alpha_0 R_1}{4k} s^2 \\ & - \frac{\alpha_2}{2}\left\{R_0\left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2} + \frac{R_0^2}{k} \ln\left[R_0 + \left(R_0^2 + \frac{R_0^2}{k}\right)^{1/2}\right]\right\}. \end{aligned} \tag{3.36}$$

The function τ_{22} can be determined accordingly.

We shall be needing the asymptotic expansions of the functions ξ_m as $s \to \infty$. Eq. (3.23) yields

$$\xi_0 \sim \left(\frac{R_0\alpha_0}{k}\right)^{1/2} s^{1/2} - \frac{1}{4}\frac{R_0}{\alpha_0}\left(\frac{R_0\alpha_0}{k}\right)^{1/2} s^{-1/2} \ln s + \cdots\,, \qquad s \to \infty. \tag{3.37}$$

The derivation of (3.37) can be found in [6]. Using (3.37), we find from (3.26) and (3.36) that

$$\xi_1 \sim \frac{R_0\alpha_1}{k}\left(\frac{k}{R_0\alpha_0}\right)^{1/2} s^{1/2} + \frac{R_0^2\alpha_1}{8k\alpha_0}\left(\frac{k}{R_0\alpha_0}\right)^{1/2} s^{-1/2} \ln s + \cdots\,, \qquad s \to \infty, \tag{3.38}$$

$$\xi_2 \sim \left(\frac{k}{R_0\alpha_0}\right)^{1/2} \frac{\alpha_0 R_1}{4k} s^{3/2} - \frac{7R_0R_1}{16k}\left(\frac{k}{R_0\alpha_0}\right)^{1/2} s^{1/2} \ln s + \cdots\,, \qquad s \to \infty. \tag{3.39}$$

4. Matching. We must now match the two expansions obtained in Secs. 2 and 3 in such a way that they have exactly the same functional form in a certain suitably shosen intermediate variable [7]. This requirement will enable us to determine all the unknown constants involved. The system of governing equations can be reduced to a single equation in X. Thus it suffices to match only the function X.

To this end, we introduce an intermediate variable s_1 defined by

$$s_1 = S/\epsilon^{1/2}, \qquad s_1 \text{ fixed}, \quad \epsilon \to 0. \tag{4.1}$$

While an intermediate variable can be written in a more general form [7], we have chosen the specific form (4.1) for simplicity. Eqs. (4.1) and (3.1) now imply

$$S = \epsilon^{1/2} s_1 , \tag{4.2}$$

$$s = \epsilon^{-1/2} s_1 . \tag{4.3}$$

Substituting (4.3) into (3.37)–(3.39), we get

$$\begin{aligned} X &\sim \xi(s, \epsilon) \sim \xi_0(s) + \epsilon \ln \epsilon \xi_1(s) + \epsilon \xi_2(s) + \cdots \\ &\sim \epsilon^{-1/4}\left(\frac{R_0\alpha_0}{k}\right)^{1/2} s_1^{1/2} + (\epsilon^{1/4} \ln \epsilon) \frac{R_0}{8\alpha_0}\left(\frac{R_0\alpha_0}{k}\right)^{1/2} s_1^{-1/2} \\ &+ \epsilon^{1/4}\left[-\frac{R_0}{4\alpha_0}\left(\frac{R_0\alpha_0}{k}\right)^{1/2} s_1^{-1/2} \ln s_1 + \left(\frac{k}{R_0\alpha_0}\right)^{1/2} \frac{\alpha_0 R_1}{4k} s_1^{3/2}\right] \\ &+ (\epsilon^{3/4} \ln \epsilon)\left[\frac{R_0\alpha_1}{2k}\left(\frac{1}{R_0\alpha_0}\right)^{1/2} + \frac{7R_0R_1}{32k}\left(\frac{k}{R_0\alpha_0}\right)^{1/2}\right] s_1^{1/2} + \cdots . \end{aligned} \tag{4.4}$$

Substituting (4.2) into (2.41), (2.42) and (2.47), we get

$$\begin{aligned} X &\sim \epsilon^{-1/2} x(S, \epsilon) \sim \epsilon^{-1/2} x_0(S) + \epsilon^{1/2} \ln \epsilon x_1(S) + \epsilon^{1/2} x_2(S) + \cdots \\ &\sim \epsilon^{-1/4}\left(\frac{R_0}{R_c}\right)^{1/2} s_1^{1/2} + (\epsilon^{1/4} \ln \epsilon)\left[-\frac{t_{11}}{2k}\left(\frac{R_c}{R_0}\right)^{1/2} R_c s_1^{-1/2} - \frac{1}{2k}\left(\frac{R_c}{R_0}\right)^{1/2} \frac{R_0^2}{4} s_1^{-1/2}\right] \\ &+ \epsilon^{1/4}\left\{\left(\frac{R_0}{R_c}\right)^{1/2} \frac{R_1}{4R_0} s_1^{3/2} + \left(\frac{R_c}{R_0}\right)^{1/2} \fint_0^1 \left(\frac{Rt_{12}}{2k} - \frac{t_{10}}{2k^2} \frac{R^3}{x_0^2}\right) dS \, s_1^{-1/2} - \frac{R_0^2}{4k}\left(\frac{R_c}{R_0}\right)^{1/2} s_1^{-1/2} \ln s_1\right\} \\ &+ (\epsilon^{3/4} \ln \epsilon)\left\{\frac{t_{11}}{2k}\left(\frac{R_c}{R_0}\right)^{1/2}\left(R_0 + \frac{R_1R_c}{4R_0}\right) s_1^{1/2} - \frac{1}{2k}\left(\frac{R_c}{R_0}\right)^{1/2} \frac{7}{16} R_0R_1 s_1^{1/2}\right\} + \cdots . \end{aligned} \tag{4.5}$$

Comparing (4.4) and (4.5) reveals that they can be matched perfectly by adjusting the constants involved. Had we not used (2.40), a perfect match would have been impossible. This justifies the establishment of (2.40). The results of the matching are

$$\begin{gathered} \alpha_0 = k/R_c , \qquad \alpha_1 = -(R_0^2/2R_c) - R_1 , \qquad t_{11} = -(R_0^2/2R_c), \\ t_{12}(1) = \frac{1}{R_c}\int_0^1 \left[R(S)\int_S^1 F(S')\, dS' + \frac{1}{R_c}\frac{R^3(S)}{x_0^2(S)} - \frac{R_0^2}{2}\frac{1}{S}\right] dS. \end{gathered} \tag{4.6}$$

We note that, while ξ_2 is needed to establish (4.6), the constant α_2 involved in ξ_2 cannot be determined without introducing more terms in the expansions.

Thus we have formally obtained an asymptotic solution which consists of a two-term expansion valid near and including $S = 0$ and a three-term expansion valid for $0 < S \leq 1$.

5. Stretching of a flat circular membrane with an inclusion as an example. Consider a flat circular membrane characterized by the meridian curve

$$C: \begin{array}{l} r = R(S) \equiv a + S \\ z \equiv 0 \end{array}, \qquad 0 \leq S \leq 1. \tag{5.1}$$

The membrane is fixed along the edge $r = R_0 = a$ and is stretched to a deformed state defined by

$$c : r = X(S), \quad X(0) = a, \quad X(1) = A, \tag{5.2}$$

where A is a given large number. The solution is, for the interval $0 < S \leq 1$,

$$\begin{aligned} X(S) &\sim A[x_0 + (A^{-2} \ln A^{-2})x_1 + A^{-2}x_2 + \cdots], \\ T_1(S) &\sim A^2[t_{10} + (A^{-2} \ln A^{-2})t_{11} + A^{-2}t_{12} + \cdots], \\ T_2(S) &\sim A^2[t_{20} + (A^{-2} \ln A^{-2})t_{21} + A^{-2}t_{22} + \cdots], \end{aligned} \tag{5.3}$$

and, for the interval $0 \leq s < \infty$,

$$\begin{aligned} X(S) &\sim \xi_0(s) + (A^{-2} \ln A^{-2})\xi_1(s) + \cdots, \\ T_1(S) &\sim A^2[\tau_{10}(s) + (A^{-2} \ln A^{-2})\tau_{11}(s) + \cdots], \\ T_2(S) &\sim A^2[\tau_{20}(s) + (A^{-2} \ln A^{-2})\tau_{21}(s) + \cdots], \end{aligned} \tag{5.4}$$

where $s = A^2S$. The functions involved in the expansion (5.3) are

$$\begin{aligned} t_{10} &= t_{20} = k/(a + \tfrac{1}{2}), \\ x_0 &= (aS + (S^2/2))^{1/2}/(a + \tfrac{1}{2})^{1/2}, \\ t_{11} &= t_{21} = -a^2/(1 + 2a), \\ x_1 &= (a^2/4k)(x_0^{-1} - x_0), \\ t_{12} &= \frac{a^2}{a + \frac{1}{2}} \ln \left[\frac{(a + \frac{1}{2})^{1/2}}{a^{1/4}(a + 1)}\right] - \ln \left[\left(\frac{a + 1}{a + S}\right)^2 \frac{S\left(a + \dfrac{S}{2}\right)}{a + \frac{1}{2}}\right] + \frac{a^2}{S(2a + S)} + 2, \\ t_{22} &= t_{12} - \left(a^2(a^2 + 4aS + 2S^2)/(a + S)^2 S\left(a + \frac{S}{2}\right)\right). \end{aligned} \tag{5.5}$$

The expression for x_2 is extremely long and is omitted here. The functions involved in the expansion (5.4) are

$$\begin{aligned} s = \frac{a + \frac{1}{2}}{a} &\left\{\xi_0\left(\xi_0^2 + \frac{a^2}{k}\right)^{1/2} + \frac{a^2}{k} \ln \left[\xi_0 + \left(\xi_0^2 + \frac{a^2}{k}\right)^{1/2}\right]\right. \\ &\left. - a\left(a^2 + \frac{a^2}{k}\right)^{1/2} - \frac{a^2}{k} \ln \left[a + \left(a^2 + \frac{a^2}{k}\right)^{1/2}\right]\right\}, \\ \tau_{10} &= (k/(a + \tfrac{1}{2})\xi_0)(\xi_0^2 + (a^2/k))^{1/2}, \\ \tau_{20} &= (k/(a + \tfrac{1}{2}))\xi_0(\xi_0^2 + (a^2/k))^{-1/2}, \\ \xi_1 &= -((a + 1)^2a/2k(2a + 1))s(\xi_0^2 + (a^2/k))^{-1/2}, \\ \tau_{11} &= -\tau_{10} \frac{(a + 1)^2}{2a + 1} \left[\frac{a + \frac{1}{2}}{k} - \frac{a^3}{2k^2} \frac{s}{\xi_0(\xi_0^2 + (a^2/k))^{3/2}}\right], \\ \tau_{21} &= -\tau_{20} \frac{(a + 1)^2}{2a + 1} \left[\frac{a + \frac{1}{2}}{k} + \frac{a^3}{2k^2} \frac{s}{\xi_0(\xi_0^2 + (a^2/k))^{3/2}}\right]. \end{aligned} \tag{5.6}$$

The problem of stretching a large rubber sheet with a rigid inclusion has been solved numerically by Yang [8]. Following Yang's notation, the stress-concentration factor and the strain-concentration factor are defined by the expressions

$$K_t^{(2)} = \frac{t_1[\text{edge}]}{t_1[\infty]} \equiv \frac{T_1[\text{edge}]}{T_1[\infty]} \frac{\Lambda_1[\text{edge}]\Lambda_2[\text{edge}]}{\Lambda_1[\infty]\Lambda_2[\infty]}, \tag{5.7}$$

$$K_\epsilon^{(2)} = ((\Lambda_1[\text{edge}] - 1)/(\Lambda_1[\infty] - 1)). \tag{5.8}$$

His numerical result, carried out for $\Lambda_1[\infty] \leq 2.1$ and $k = 0.1$, indicates that $K_t^{(2)}$ is a monotonically increasing function of $t_1[\infty] \equiv T_1[\infty]\Lambda_1[\infty]\Lambda_2[\infty]$ and $K_\epsilon^{(2)}$ is a monotonically increasing function of $\Lambda_1[\infty] - 1$.

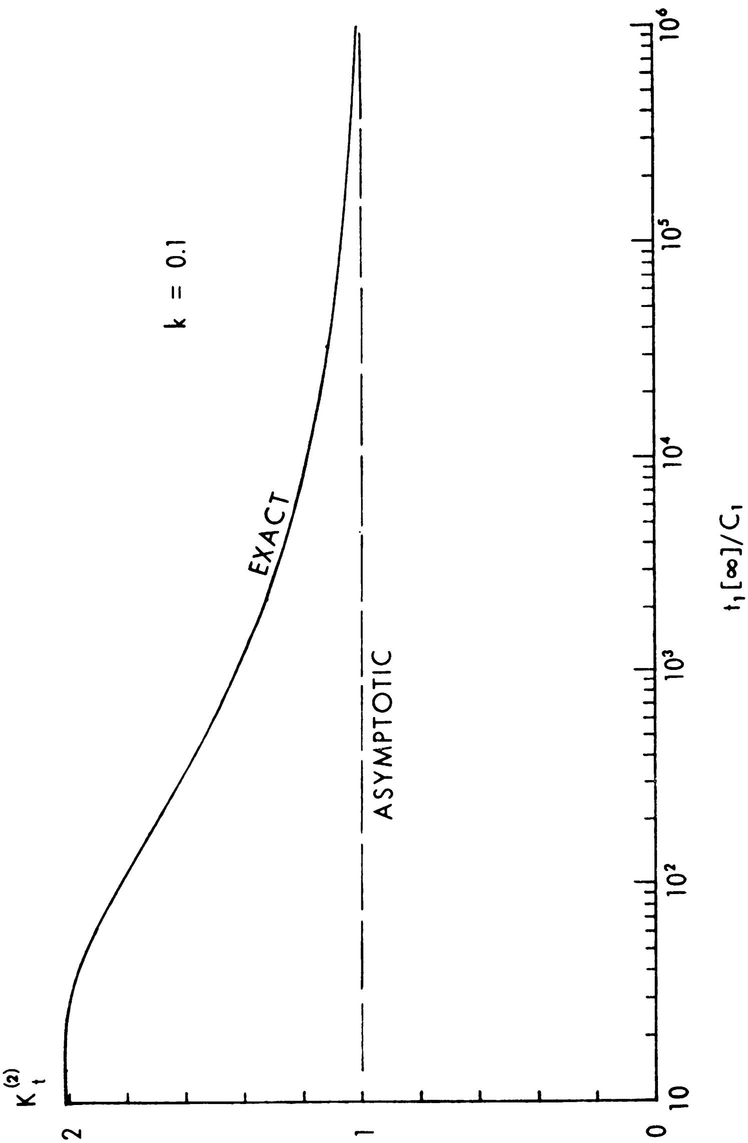

FIG. 1.

The asymptotic expressions of $K_t^{(2)}$ and $K_\epsilon^{(2)}$ can be deduced easily from our result for $A \to \infty$ which implies $t_1[\infty] \to \infty$ and $\Lambda_1[\infty] - 1 \to \infty$. This is because of the fact that the condition $R(0) = a \to 0$ makes our finite membrane an infinite one. Moreover, all the expressions given in (5.5) and (5.6) tend to finite limits as $a \to 0$. Taking the appropriate limits, we have

$$\Lambda_1[\infty] \equiv \lim_{a\to 0} \Lambda_1(1) \sim A, \tag{5.9}$$

$$\Lambda_1[\text{edge}] \equiv \lim_{a\to 0} \Lambda_1(0) \sim \left(1 + \frac{1}{k}\right)^{-1/2} A^2, \tag{5.10}$$

$$t_1[\infty] \equiv \lim_{a\to 0} T_1(1)\Lambda_1(1)\Lambda_2(1) \sim 2kA^4, \tag{5.11}$$

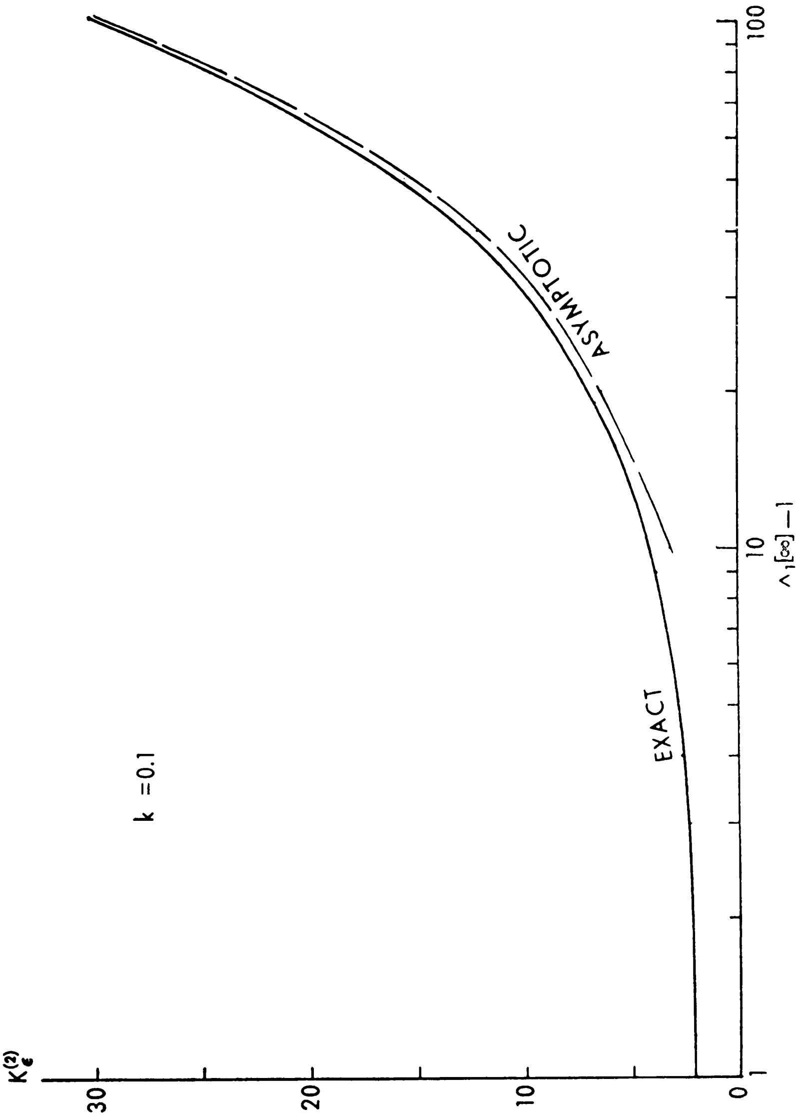

Fig. 2.

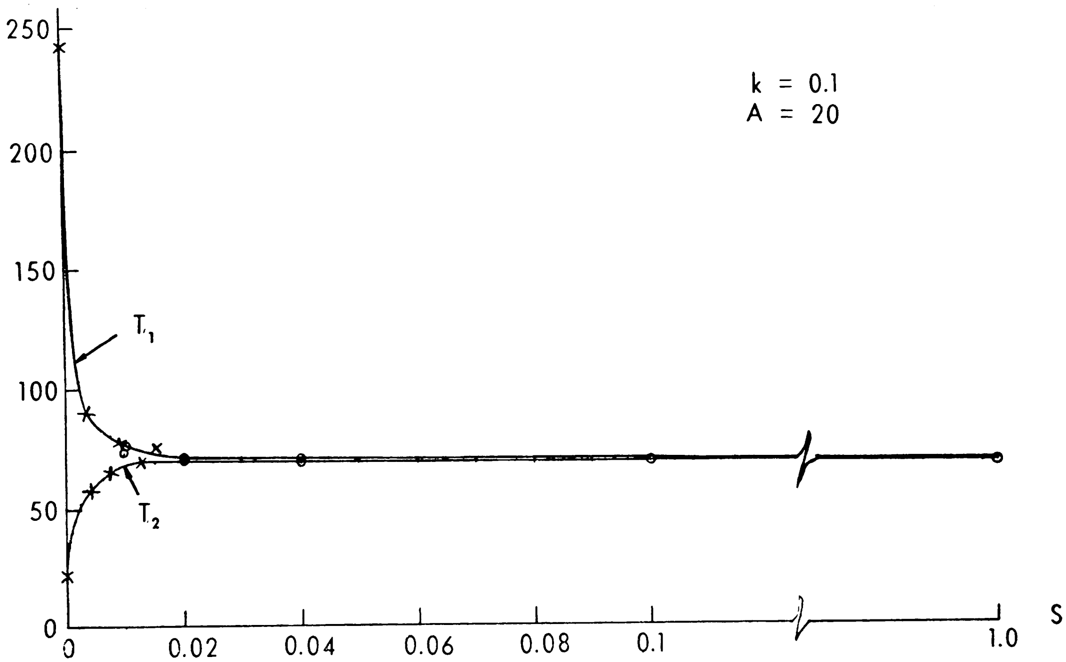

Fig. 3.

$$t_1[\text{edge}] \equiv \lim_{a \to 0} T_1(0)\Lambda_1(0)\Lambda_2(0) \sim 2kA^4. \tag{5.12}$$

It follows that

$$K_t^{(2)} \sim 1, \qquad t_1[\infty] \to \infty, \tag{5.13}$$

$$K_\epsilon^{(2)} \sim \left(1 + \frac{1}{k}\right)^{-1/2} A = \left(1 + \frac{1}{k}\right)^{-1/2} \{\Lambda_1[\infty] - 1\}, \qquad \Lambda_1[\infty] - 1 \to \infty. \tag{5.14}$$

Eq. (5.13) contradicts the apparently monotonic behavior conceived from Yang's limited numerical result. For this reason we have extended Yang's calculation to $\Lambda_1[\infty] \leq 100$. The result is presented in Figs. 1 and 2. It is seen that the exact numerical solutions do approach the asymptotic results given by (5.13) and (5.14). Moreover, $K_t^{(2)}$ is not a monotonically increasing function of $t_1[\infty]$.

As an example we have also calculated the stress resultant distribution for the case $A = 20$ and $k = 0.1$. The results are given in Fig. 3 where the matching of the two expansions is also indicated.

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