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LIMIT CYCLE CITY*

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1. Introduction. Given the autonomous, nonlinear, second-order differential equation

$$\ddot{x} + \beta(x, \dot{x}) = 0 \quad (1)$$

where the dots denote differentiation with respect to the time t , we will regard the trajectories in the $x, \dot{x}$ phase plane as the streamlines of a plane, steady, compressible flow, as suggested in the literature (e.g. [1]). Thus the $x, \dot{x}$ components of the fluid velocity $\mathbf{V}$ are $\dot{x}, \ddot{x}$ or $\dot{x}, -\beta$, and so the continuity equation on the fluid mass density $\rho(x, \dot{x})$ becomes

$$\nabla \cdot (\rho \mathbf{V}) = \rho_x \dot{x} - (\rho \beta)_\dot{x} = 0. \quad (2)$$

The *dynamics* will not be relevant to the present discussion, except perhaps in the sense that we should at least be able to show that the equations of motion are satisfied, so that it is in fact proper to speak of a “fluid analogue.” This is discussed in [2], where it is shown that all is well provided that a suitable body force is supplied. In that paper the emphasis was on the synthesis problem, i.e. designing a differential equation (1) which will exhibit a prescribed limit cycle behavior. In the present note we consider instead the “direct” problem, and examine the connection between the compressible flow analogy and limit cycle analysis.

2. Limit cycle analysis. The idea is simple. If Eq. (1) has a stable limit cycle Γ , then fluid is continually entering each arbitrarily small neighborhood N of Γ (Fig. 1).

FIG. 1. Accumulation of fluid within N .

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By virtue of the conservation of mass, it follows that the density ρ must be infinite on Γ . Similarly for an unstable limit cycle or focus. Our program, then, is to solve the first-order partial differential equation (2) for $\rho(x, \dot{x})$, and to seek those curve(s) on which ρ is infinite.

3. The Van der Pol equation. To illustrate, consider the well-known Van der Pol equation,

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \quad (3)$$

for the case where ϵ is small. Changing to polar variables according to $x = r \cos \theta$, $\dot{x} = r \sin \theta$, (2) becomes

$$\rho_\theta + \epsilon(r^2 c^2 - 1)(rs^2 \rho_r + sc \rho_\theta + \rho) = 0 \quad (4)$$

where c, s are short for $\cos \theta, \sin \theta$ respectively, and subscripts denote partial derivatives.

Next, we introduce strained variables,

$$\begin{aligned} R &= [1 + \epsilon a(\theta) + \epsilon^2 b(\theta) + \dots]r \\ \Theta &= \theta, \end{aligned} \quad (5)$$

and seek

$$\rho = \rho^0(R, \Theta) + \epsilon \rho^1(R, \Theta) + \dots \quad (6)$$

Requiring that each $\rho^n(R, \Theta)$ be of period 2π in Θ , and no more singular on the limit cycle (and at the origin, where there will be a focus) than the preceding term $\rho^{n-1}(R, \Theta)$, we obtain

$$a = -(\sin 2\Theta/4) + (\sin 4\Theta/8), \quad (7)$$

and

$$\rho^1 = \frac{A}{8|4 - R^2|} \left[\frac{2 - R^2}{R^2} \sin 4\Theta - 2 \sin 2\Theta \right]. \quad (8)$$

Thus, the limit cycle is given by

$$R = (1 + \epsilon a + \dots)r = 2 \quad (9)$$

or, to first order,

$$r = 2 - 2a\epsilon = 2 + \left(\frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4} \right) \epsilon. \quad (10)$$

Unfortunately, even the second-order analysis is already quite tedious. In a sense, this reflects the fact that our solution generates more information than we really need, namely, $\rho(R, \Theta)$ *everywhere*, whereas we are ultimately concerned only with ρ in the vicinity of the limit cycle. With this in mind, we derive a somewhat different approach, which is more direct.

4. An alternative approach. Returning to the continuity equation (2), we change first to the polar coordinates r, θ according to $x = rc$, $\dot{x} = rs$, and then to new polar coordinates R, Θ according to

$$R = r/F(\Theta), \quad \Theta = \theta, \quad (11)$$

such that the desired limit cycle becomes the circle $R = 1$, or, in terms of the original coordinates, $r = F(\theta)$. Then (2) becomes

$$\left(\frac{\beta s}{F} - \frac{\beta c F'}{F^2} - s c R - \frac{s^2 R F'}{F}\right) \rho_R + \left(\frac{\beta c}{R F} + s^2\right) \rho_\theta + \beta_z \rho = 0. \quad (12)$$

Now, since ρ is singular on $R = 1$, it follows that if β and β_z are bounded in $r > 0$ then the coefficient of ρ_R in (12) must vanish on $R = 1$; i.e.

$$\beta s F - \beta c F' - s c F^2 - s^2 F F' = 0, \quad (13)$$

where β is understood to be evaluated on $R = 1$, i.e. $\beta = \beta(Fc, Fs)$.

Thus, instead of having to deal with the partial differential equation (2), we are able to obtain the limit cycles $r = F(\theta)$ directly as the (2π) -periodic solutions of the ordinary differential equation (13).

For the Van der Pol example the second-order result is

$$x = F(\theta) = 2 + \left(\frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4}\right)\epsilon + \left(\frac{7}{128} - \frac{5}{32} \cos 2\theta + \frac{3}{32} \cos 4\theta + \frac{7}{96} \cos 6\theta - \frac{7}{128} \cos 8\theta\right)\epsilon^2. \quad (14)$$

In fact, it is not hard to show that the nice equation (13) is none other than the equation of the phase trajectories,

$$d\dot{x}/dx = -\beta/\dot{x} \quad (15)$$

under the change to polar coordinates $x = rc$, $\dot{x} = rs$, where $r \equiv F(\theta)$, and furthermore that the phase trajectories (15) happen to be the characteristics of the continuity equation (2).

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