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ON THE ASYMPTOTIC STABILITY OF OSCILLATORS
WITH UNBOUNDED DAMPING*

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Abstract. Through a technique inspired by the invariance principle of LaSalle, a general growth condition on the damping coefficient $h(t)$ of the equation

$$\ddot{x} + h(t)\dot{x} + kx = 0, \quad k > 0, \quad h(t) \geq \epsilon > 0,$$

is given which is sufficient for the global asymptotic stability of the origin yet permits this coefficient to grow to infinity with time. The methods used do not depend on linearity, and are applied to obtain similar results to the nonlinear analogue of this equation.

1. Introduction. The differential equation

$$\ddot{x} + h(t)\dot{x} + kx = 0, \tag{1.1}$$

with $h(t) \geq 0$, $k > 0$, and its nonlinear generalizations are basic mathematical models for the representation of damped oscillatory phenomena; the global asymptotic stability of the rest point $x = \dot{x} = 0$ of this equation and its generalizations has drawn the attention of numerous investigators (see, for example, [1, 2, 3]). It is well known that if the damping coefficient $h(t)$ is nonnegative, then this rest point is stable. If the damping coefficient is restricted to $0 < \underline{h} \leq h(t) \leq \bar{h}$, $\underline{h}, \bar{h}$ constants, then the rest point is globally asymptotically stable; on the other hand, it is known through examples [1, 5] that if the damping coefficient is not bounded above, then the rest point, although stable, is not necessarily asymptotically stable and solutions may exist such that $x(t) \to c \neq 0$ and $\dot{x}(t) \to 0$ as time increases.

It is possible to give a physical interpretation for this behavior. The total energy of the system $V = \frac{1}{2}\dot{x}^2 + \frac{1}{2}kx^2$ is dissipated at the rate $\dot{V} = -h(t)\dot{x}^2$; if $h(t) \geq \underline{h} > 0$, all the kinetic energy $\frac{1}{2}\dot{x}^2$ will surely be dissipated from the system; this is not necessarily so for the potential energy. If the damping coefficient is also bounded above, then it is relatively simple to show that all of the total energy will be dissipated; the rest point $x = \dot{x} = 0$ is then globally asymptotically stable. If $h(t)$ is unbounded, this is no longer necessarily the case; if the damping coefficient grows sufficiently fast, then the asymptotic state obtained can be one in which the system retains positive potential energy indefinitely and, therefore, the rest point is no longer globally asymptotically stable.

This phenomenon is of physical interest; there is a number of realistic situations that can be represented by Eq. (1.1), or its generalizations, where the damping coefficient increases without bound. Examples from tempering and heat treatment of materials

and of poisoning of catalysts are in this category. It is of interest to investigate, in such cases, the asymptotic state of the system.

The purpose of this note is to present a simple growth condition for the positive damping coefficient $h(t)$ which is sufficient to ensure the global asymptotic stability of the rest point, yet permits this coefficient to grow to infinity with time. It is somewhat surprising that, given the wealth of results known about equations of this type, no such condition or type of condition seems to be available in the literature.

The technique used here is inspired by the invariance principle of LaSalle [4, 5], as we explain in what follows.

The sufficient condition given can be made more precise and can as well be generalized to a wider class of equations. We wish to restrict ourselves to showing, through a counter-example, that if the stated sufficient condition is not met, the rest point is not globally asymptotically stable, and to indicating that similar results hold for the nonlinear equation

$$\ddot{x} + h(x, \dot{x}, t)\dot{x} + f(x) = 0, \tag{1.2}$$

with $h \geq 0$ and $xf(x) \geq 0$.

2. **The main result.** Consider Eq. (1.1) written in its equivalent form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx - h(t)y. \end{aligned} \tag{2.1}$$

To obtain a growth condition result, the following lemma is useful

LEMMA. Let a_1, a_2, $\cdots$ be a sequence of positive numbers with the property that, for some N_0, B fixed, $\sum_{i=1}^{k} a_i \leq (k + N_0)^2 B$ for all k. Then

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty .$$

Proof: It is easy to show (induction, or Cauchy-Schwartz) that if b_1, $\cdots$, b_m are positive and $\sum_{i=1}^{m} b_i \leq L$, then

$$\sum_{i=1}^{n} \frac{1}{b_i} \geq m^2/L.$$

Let now $b_i = a_{2^n+i}$, $i = 1, \cdots, 2^n$. From the assumption stated, it follows that $\sum_{i=1}^{2^n} b_i \leq (2^{n+1} + N_0)^2 B$ and, therefore

$$\sum_{i=1}^{2^n} \frac{1}{b_i} \geq \frac{2^{2n}}{(2^{n+1} + N_0)^2 B} ;$$

but this lower bound tends to $1/4B$ as $n \to \infty$ and

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \frac{1}{a_i} + \sum_{n=0}^{\infty} \sum_{i=1}^{2^n} \frac{1}{a_{2^n+i}}.$$

The main result now follows:

THEOREM. Suppose that $h(t)$ in (2.1) satisfies $h(t) \geq \underline{h} > 0$ for all t and that, for all $T \geq 0$ and some B, the growth condition

$$\frac{1}{T^2} \int_0^T h(t)\, dt \leq B$$

holds. Then every solution $(x(t), y(t))$ of (2.1) approaches $(0, 0)$ as $t \to \infty$.

Proof: Let $(x(t), y(t))$ be a solution. Consider the natural Liapunov function $V(x, y)$ $= (y^2/2) + k(x^2/2)$; then $V(t) = V(x(t), y(t))$ satisfies $dV/dt = -h(t)y^2(t)$, and it follows that $V(t)$ is nonincreasing and nonnegative. Hence, $V(t) \to kc^2/2$, for some c, as $t \to \infty$; in particular, the solution $(x(t), y(t))$ is positively bounded and approaches the ellipse $(y^2/2) + k(x^2/2) = k(c^2/2)$. On the other hand, $dV/dt \le -hy^2$, so that any limit point of $(x(t), y(t))$ must have the form $(c, 0)$ (compare with [6]).

We claim that there is a monotonic increasing sequence of integers n_1, n_2, $\cdots$, $n_i \to \infty$ as $i \to \infty$, such that

$$\int_{n_i}^{n_i+1} h(t)y(t) \, dt \to 0, \qquad i \to \infty. \tag{2.2}$$

Suppose the contrary; then there exist a $\delta > 0$ and n_0 such that

$$0 < \delta \le \left(\int_n^{n+1} h(t)y(t) \, dt \right)^2, \quad \text{for} \quad n \ge n_0 ; \tag{2.3}$$

it then follows (Cauchy-Schwartz inequality) that

$$\delta \le \int_n^{n+1} h(t) \, dt \int_n^{n+1} h(t)y^2(t) \, dt, \tag{2.4}$$

and, setting $a_n = \int_n^{n+1} h(t) \, dt$, that

$$\delta \frac{1}{a_n} \le V(n) - V(n + 1);$$

summation for $n \ge n_0$ yields

$$\delta \sum_{n=n_0}^{\infty} \frac{1}{a_n} \le V(n_0), \tag{2.5}$$

which implies that

$$\sum_{i=n_0}^{\infty} \frac{1}{a_i} < \infty .$$

But this is contrary to the combination of our stated growth condition $\sum_{i=n_0}^{n} a_i \le Bn^2$ and our Lemma. Hence, (2.2) is true.

Integrate now the second of Eqs. (2.1) over $[n_i, n_i + 1]$ to obtain

$$y(n_i + 1) - y(n_i) = -k \int_{n_i}^{n_i+1} x(t) \, dt - \int_{n_i}^{n_i+1} h(t)y(t) \, dt; \tag{2.6}$$

as $i \to \infty$, application of (2.2) yields $0 = -c$ which rules out the possibility $c \ne 0$, thus completing the proof.

It is clear from the proof that the growth condition on $h(t)$ need be satisfied only on one sequence of subintervals rather than on the whole line. For instance, the growth condition

$$\frac{1}{m^2} \sum_{i=1}^{m} \int_{n_i}^{n_i+1} h(t) \, dt \le B, \tag{2.7}$$

for some B and all m, for one sequence of integers $\{n_i\}$, $n_i \to \infty$ as $i \to \infty$, is sufficient for asymptotic stability.

3. The relation to the invariance principle. The proof of our theorem is funda-
mentally based on the LaSalle invariance principle [4–6], whose basic idea we wish to
note here, since its application should be fruitful in the study of equations more general
than the one investigated here.

It will be noted that, on the interval $[n_i, n_{i+1}]$, the solution $(x(t), y(t))$ of (2.1)
satisfies the differential equation

$$\dot{x} = y$$
$$\dot{y} = -kx - g(t)$$
(3.1)

where $g(t) = h(t)y(t)$. As $i \to \infty$, the initial conditions $(x(n_i), y(n_i))$ converge to $(c, 0)$
for some c and (3.1) tends, asymptotically, by (2.2), to

$$\dot{x} = y$$
$$\dot{y} = -kx.$$
(3.2)

The convergence of (3.1) to (3.2) implies continuous dependence with respect to
initial data; in particular, $(c, 0)$ should be an equilibrium point for (3.2), but it obviously
is not unless $c = 0$.

This device is the basis for the invariance principle [7–9] for nonautonomous equa-
tions. It must be remarked, however, that the original equation does not have (3.2)
as a limiting equation: it is along the solution that (2.1) is equivalent to (3.1) which
converges to (3.2). A similar idea was used in [5].

4. Further remarks. We wish to indicate briefly how to extend our result to the
nonlinear equation (1.2).

Typically, we assume that
   (i) $xf(x) > 0$ for $x \neq 0$, that
   (ii) $\int_0^x f(z)\, dz \to \infty$ as $|x| \to \infty$, and
   (iii) $h(x, \dot{x}, t) \geq w(x, \dot{x})$,
where w is continuous and nonnegative and $w(x, y) > 0$ for $y \neq 0$; it follows that $V(x, y)$
$= y^2/2 + \int_0^x f(z)\, dz$ is an appropriate Liapunov function from which, as in the Theorem,
we conclude that any solution $(x(t), y(t)) \to (c, 0)$. The conclusion of the theorem follows
upon the imposition of the growth condition

$$\frac{1}{T^2} \int_0^T h(x(t), y(t), t)\, dt \leq B,$$
(4.1)

for some B and all $T \geq 0$ and $(x(t), y(t))$ in a compact set.

Finally, we wish to remark on the growth condition stated in the Theorem.

The assumption of boundedness for all T of

$$\frac{1}{T^2} \int_0^T h(t)\, dt$$

implies that, in a sense, $h(t)$ grows no more than linearly in time. In particular,
if $h(t) \leq K + Mt$, this growth condition is satisfied. This result is sharp in the sense
that the $1/T^2$ cannot be replaced by $1/T^{2+\epsilon}$ for any $\epsilon > 0$; alternatively, that $h(t) \leq$

$K + Mt^{1+\epsilon}$ does not assure global asymptotic stability. Indeed, let $\epsilon > 0$ and

$$h(t) = (\epsilon + 1)t^{-1} + \frac{1}{\epsilon} t + \frac{1}{\epsilon} t^{1+\epsilon}.$$

Then $(x(t), y(t)) = (1 + t^{-\epsilon}, -\epsilon t^{-\epsilon-1})$ satisfies $\dot{x} = y$, $\dot{y} = -x - h(t)y$, but $(x(t), y(t)) \to$
$(1, 0)$ as $t \to \infty$.

Similar examples have been produced by Levin and Nohel [1] and LaSalle [5].

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