VISCOUS EFFECTS ON PERTURBED SPHERICAL FLOWS*

By

ANDREA PROSPERETTI**

California Institute of Technology

Abstract. The problem of two viscous, incompressible fluids separated by a nearly spherical free surface is considered in general terms as an initial-value problem to first order in the perturbation of the spherical symmetry. As an example of the applications of the theory, the free oscillations of a viscous liquid drop and of a bubble in a viscous liquid are studied in some detail. It is shown that the oscillations are initially describable in terms of an irrotational approximation, and that the normal-mode results are recovered as $t \to \infty$. In between these asymptotic regimes, however, the motion is significantly different from either approximation.

1. Introduction. The normal-mode approach to fluid mechanics problems involving oscillations of free surfaces is a standard one at least since the time of Stokes [1–3]. According to this method the solution to the initial-value problem is obtained in the form of an infinite sum (or integral) of appropriately weighted eigenfunctions. For viscous fluids the mathematical problem is parabolic, and the series is very slowly convergent for small times, rendering it difficult to discuss the transient behavior of the solution. Yet this transient behavior is of considerable interest for instance in the case of free oscillations (for which the amplitude is greatest for small times), and of problems of stability of the free surface, which are typically of a transient nature. These considerations motivated the analysis of small-amplitude gravity waves on a plane liquid surface presented in a previous work [4], in which the initial-value problem was solved directly by means of Laplace transform methods. In the present study, the same technique is adapted to the motion of a nearly spherical free surface $\Sigma(t)$ separating two incompressible, viscous, immiscible fluids that fill the entire space. Assuming that the deviation of the surface from the spherical shape is small, we solve the flow problem and derive the general equations of motion for Σ in a form which is suitable for the discussion of transients.

As an example of the applications of this theory, the oscillations of liquid drops and bubbles are considered. It is shown that known results obtainable by means of the irrotational approximation [2, p. 640] or a normal-mode analysis [3, p. 466; 5–8] are in reality valid only initially and asymptotically for large times, respectively. In between these two asymptotic regimes the correct solution is significantly different from either

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** On leave of absence from the Istituto di Fisica, Universita' degli Studi, Milano, Italy.

approximation. It is believed that these results are of some interest also from the viewpoint of applied mathematics, insofar as they illustrate in a nontrivial example the relationship between transform methods and normal-mode (or spectrum) analysis.

As a consequence of the fundamentally parabolic nature of the problem, it is found that the equation of motion of the free surface is integro-differential rather than differential. To explain the reason of this feature it is sufficient to recall that the rate of energy dissipation for a viscous flow occupying a volume V bounded by a surface S may be written [1, p. 581]

$$-\mu\left[\int_V \boldsymbol{\omega}\cdot\boldsymbol{\omega}\, dV + 2\int_S [(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U}]\cdot\mathbf{n}\, dS\right],$$

where μ is the viscosity, $\boldsymbol{\omega}$ the vorticity, $\mathbf{U}$ the flow velocity and $\mathbf{n}$ the outward unit normal to S. For a motion starting with zero initial vorticity the first term vanishes and dissipation arises only through the surface integral. As the motion progresses, however, vorticity is generated at the free surface, from which it diffuses into the flow region. This process causes the volume integral in the disspiation rate to become positive, so that the instantaneous energy dissipation must depend on the vorticity distribution and hence on its past history. A more detailed qualitative description of the physical processes involved has been given in [4].

In this study we allow the radius of Σ to vary with time. Therefore the present formulation is suitable for the investigation of viscous effects on the spherical analogue of the Rayleigh–Taylor instability [9, 10] with application, for instance, to the growth and collapse of bubbles [11].

We might also notice that in a particular case belonging to the class of problems investigated here the limits of validity of the linearized formulation in spherical geometry are known to be quite ample [12].

2. Formulation of the problem. In the following the subscripts 1 and 2 will be attached to all quantities pertaining to the inner and outer region respectively; when no subscript is indicated, reference is made indifferently to either region.

In the absence of body forces, the Navier–Stokes equations and the condition of incompressibility are

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U} = \frac{1}{\rho}\operatorname{div}\sigma, \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{U} = 0, \tag{1b}$$

where ρ denotes the density and the stress tensor σ is given in terms of the pressure p and viscosity μ by

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

in cartesian coordinates. The surface Σ can be represented by a superposition of spherical harmonics. To first order in the perturbation of the spherical symmetry, however, the equations for the different modes are uncoupled, so that we may let, in spherical coordinates,

$$\Sigma(t) : F(r, \theta, \varphi, t) \equiv r - R(t) - \epsilon a(t) Y_n^m(\theta, \varphi) = 0,$$

where $0 < \epsilon \ll 1$ and Y_n^m is a spherical harmonic of degree $n \geq 2$.

In the following developments only terms of first order in ϵ will be retained. To this approximation the outward unit normal $\mathbf{n}$ at every point of Σ is

$$\mathbf{n} = \mathbf{e}_r - \epsilon r^{-1} a \frac{\partial Y_n^m}{\partial \theta} \mathbf{e}_\theta - \epsilon (r \sin \theta)^{-1} a \frac{\partial Y_n^m}{\partial \varphi} \mathbf{e}_\varphi$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ are the three orthogonal unit vectors tangent to the coordinate lines at every point. On the free surface the kinematical boundary conditions are

$$\frac{\partial F}{\partial t} + (\mathbf{U} \cdot \nabla) F = 0, \tag{2}$$

$$\mathbf{U}_{t_1} = \mathbf{U}_{t_2} , \tag{3}$$

where the subscript t denotes the tangential component of the velocity at the free surface. If one of the two fluids is inviscid, only the first condition applies. The dynamical boundary conditions stipulate that there should be no discontinuity in the tangential stresses

$$\mathbf{n} \times [(\sigma_2 - \sigma_1)\mathbf{n}] = 0, \tag{4}$$

and that the discontinuity in the normal stress should equal the surface tension ζ times the total curvature

$$\mathbf{n} \cdot [(\sigma_2 - \sigma_1)\mathbf{n}] = \zeta \nabla \cdot \mathbf{n}. \tag{5}$$

In these equations the stress tensors σ_1 and σ_2 are evaluated on the inner and outer sides of Σ respectively. To the above boundary conditions the requirement of regularity at infinity must be added; if (and only if) R is not a constant, we do not require regularity at the origin, where there will in general be a source-like singularity. In spite of an apparent artificiality, this freedom allows one to discuss, for instance, the problem of a collapsing or growing bubble, as has been shown by Plesset and Mitchell [11].

It is convenient to divide velocity and pressure fields into three parts as follows

$$\mathbf{U} = \mathbf{u}_0 + \epsilon \mathbf{u}_p + \epsilon \mathbf{u}_v , \qquad p = p_0 + \epsilon p_p + \epsilon p_v ;$$

the first terms in these expressions ($\mathbf{u}_0$, p_0), describe the purely radial motion that would take place for $\epsilon = 0$; the second terms ($\mathbf{u}_p$, p_p) are the correction of the potential (i.e. inviscid, irrotational) flow induced by the perturbation of the spherical symmetry, and the third terms ($\mathbf{u}_v$, p_v) are the correction to the potential problem introduced by the presence of viscosity. According to this subdivision we get the following (linearized) equations for the first two terms:

$$\nabla \cdot \mathbf{u}_0 = 0, \qquad \nabla \cdot \mathbf{u}_p = 0, \tag{6a}$$

$$\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \rho^{-1} \nabla p_0 = 0, \tag{6b}$$

$$\frac{\partial \mathbf{u}_p}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_p + (\mathbf{u}_p \cdot \nabla)\mathbf{u}_0 + \rho^{-1} \nabla p_p = 0, \tag{6c}$$

$$\frac{\partial F}{\partial t} + (\mathbf{u}_0 + \epsilon \mathbf{u}_p) \cdot \mathbf{e}_r \frac{\partial F}{\partial r} = 0 \quad \text{on} \quad F = 0. \tag{6d}$$

The solution to this set of equations has been obtained by Plesset in his investigation of the stability of spherical flows [9], and we shall make use of his results which are summarized in the next section. The problem to be solved remains then that of the determination of the viscous correction $(\mathbf{u}_v\, ,\, p_v)$. The linearized equations governing these fields are readily obtained on subtraction of Eqs. (6) from Eqs. (1) and (2) and can be written as

$$\nabla \cdot \mathbf{u}_v = 0, \tag{7}$$

$$\frac{\partial \mathbf{u}_v}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_v + (\mathbf{u}_v \cdot \nabla)\mathbf{u}_0 + \rho^{-1}\nabla p_v = 0, \tag{8}$$

$$\mathbf{u}_v \cdot \mathbf{e}_r \frac{\partial F}{\partial r} = 0 \quad \text{on} \quad F = 0. \tag{9}$$

Only when these equations have been solved (Sec. 3) are we able to impose the dynamical boundary conditions (4), (5) on the *total* velocity and pressure fields $(\mathbf{U}, p)$, and then to derive the equation of motion of the interface (Sec. 4).

3. Solution of the fluid mechanical problem.

Potential problem. The purely radial motion induced by the change of the radius $R(t)$ is described in terms of a potential source by writing

$$\mathbf{u}_0 = \nabla \varphi_0\, , \qquad \varphi_0 = -R'R^2/r \tag{10}$$

where the prime denotes differentiation with respect to time. Eq. (6b) for the pressure can then be integrated immediately to obtain

$$p_0 = P(t) + \rho[(R^2R'' + 2RR'^2)r^{-1} - \tfrac{1}{2}R'^2R^4r^{-4}]. \tag{11}$$

The functions $P(t)$ are two constants of integration which are supposed known. The function $P_2(t)$ is the pressure at infinity, and in the case of a bubble with uniform interior, for example, the function P_1 would be the pressure of the gas or vapor contained in the bubble. The singularities in Eqs. (10) and (11) are required by the incompressible nature of the fluids, and are the only singularities allowed; all other quantities will be taken to be regular both at infinity and at the origin.

The potential correction introduced by the deviation from sphericity has been computed by Plesset [9] who gives the following results:

$$\mathbf{u}_p = \nabla \varphi_p\, ,$$

$$\varphi_{p1} = n^{-1}r^nR^{-(n-1)}(a' + 2aR'/R)Y_n^m,$$

$$\varphi_{p2} = -(n+1)^{-1}r^{-(n+1)}R^{n+2}(a' + 2aR'/R)Y_n^m.$$

The terms in parentheses in the last two equations have been so chosen that the kinematic boundary condition, Eq. (6d), is satisfied. The pressure fields can now be computed by integration of (6c):

$$p_p = -\rho\left(\frac{\partial \varphi_p}{\partial t} + (R/r)^2R'\,\frac{\partial \varphi_p}{\partial r}\right);$$

only the value of the pressure at $r = R$ will be required in the following; one has

$$p_{p1} = -n^{-1}\rho_1(Ra'' + 3R'a' + 2R''a)Y_n^m,$$

$$p_{p2} = (n+1)^{-1}\rho_2(Ra'' + 3R'a' + 2R''a)Y_n^m.$$

Viscous correction. The problem posed by Eqs. (7) and (8) can be conveniently phrased in terms of the vorticity

$$\epsilon\boldsymbol{\omega} = \nabla \times \mathbf{U} = \epsilon \nabla \times \mathbf{u}_\epsilon\,, \tag{12}$$

which satisfies the following equation, obtained by taking the curl of Eq. (8):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}_0) = -\nu \nabla \times (\nabla \times \boldsymbol{\omega}). \tag{13}$$

It is advantageous to decompose the voriticty into a poloidal and a toroidal field [3, pp. 622–626], $\boldsymbol{\omega} = \mathbf{S} + \mathbf{T}$, which is possible because by (12) $\nabla \cdot \boldsymbol{\omega}$ vanishes identically. The principal advantage afforded by the introduction of these two fields is that the three components of the vector $\boldsymbol{\omega}$ can be expressed in terms of two functions of r and t, the defining scalars, since one can write in general

$$\mathbf{S} = \nabla \times \nabla \times [S(r, t) Y_n^m(\theta, \varphi)\mathbf{e}_r], \tag{14}$$

$$\mathbf{T} = \nabla \times [T(r, t) Y_n^m(\theta, \varphi)\mathbf{e}_r]. \tag{15}$$

Upon substitution of (14), (15) into Eq. (13), with the aid of the property of orthogonality of any poloidal to any toroidal field, the following equations for $S(r, t)$ and $T(r, t)$ are readily obtained:

$$\nu \frac{\partial^2 S}{\partial r^2} - \frac{\partial S}{\partial t} - R'(R/r)^2 \frac{\partial S}{\partial r} - \nu n(n + 1) r^{-2} S = 0, \tag{16}$$

$$\nu \frac{\partial^2 T}{\partial r^2} - \frac{\partial T}{\partial t} - \frac{\partial}{\partial r}[R'(R/r)^2 T] - \nu n(n + 1) r^{-2} T = 0. \tag{17}$$

The solution of these equations is in general a complicated matter, and it does not appear possible to indicate methods of general applicability; the simple case in which $|R'R| \ll \nu$ is considered in Secs. 5 and 6 below.

From Eqs. (12), (14) and (15) we see that

$$\mathbf{u} = T Y_n^m \mathbf{e}_r + \nabla \times (S Y_n^m \mathbf{e}_r) - \nabla \Phi,$$

where a new function Φ has been introduced in order to satisfy the incompressibility condition, Eq. (7), which reduces to

$$\nabla^2 \Phi = \nabla \cdot (T Y_n^m \mathbf{e}_r).$$

This equation can be readily integrated to obtain

$$\Phi = Y_n^m \Bigg[\left(\alpha(t) + \frac{n + 1}{2n + 1} \int_R^r s^{-n} T(s, t)\, ds \right) r^n + \left(\frac{n}{n + 1} R^{2n+1} \alpha(t) + \frac{n}{2n + 1} \int_R^r s^{n+1} T(s, t)\, ds \right) r^{-(n+1)} \Bigg]. \tag{18}$$

One of the two constants of integration has been evaluated by imposing the kinematic boundary condition (9); the other one, $\alpha(t)$, is determined by the condition of regularity at the origin

$$\alpha_1(t) = \frac{n + 1}{2n + 1} R^{-(2n+1)} \int_0^R s^{n+1} T_1(s, t)\, ds, \tag{19a}$$

and at infinity

$$\alpha_2(t) = -\frac{n+1}{2n+1}\int_R^\infty s^{-n}T_2(s,t)\,ds, \tag{19b}$$

From Eq. (8) we obtain the following expression for the viscous contribution to the pressure:

$$\frac{p_v}{\rho} = \frac{\partial\Phi}{\partial t} - \mathbf{u}_0\cdot\mathbf{u}_v + \left[R'(R/r)^2T - \nu\frac{\partial T}{\partial r}\right]Y_n^m.$$

The time derivative in the first term can be computed from Eq. (18) using the equation for $T(r, t)$, Eq. (17), to eliminate $\partial T/\partial t$. In this way, after integration by parts, we obtain the following results at $r = R$:

$$p_{v1} = (n+1)\rho_1 Y_n^m\left\{-\nu_1 T_1(R,t)/R + (R'/R)\int_0^R (s/R)^{n-2}[1-(s/R)^3]T_1(s,t)\,ds\right\},$$

$$p_{v2} = n\rho_2 Y_n^m\left\{\nu_2 T_2(R,t)/R + (R'/R)\int_R^\infty [(R/s)^3 - 1](R/s)^n T_2(s,t)\,ds\right\}.$$

4. The boundary conditions and the equations of motion of the interface. The kinematical boundary condition, Eq. (2), has already been imposed on the solutions of the fluid mechanical problem dervied in the preceding section. The continuity of the tangential velocity, Eq. (3), is easily seen to require that

$$S_1(R,t) = S_2(R,t), \tag{20}$$

$$nR^{n-1}[\alpha_1(t) - \alpha_2(t)] = a' + 2aR'/R. \tag{21}$$

As has already been mentioned, these two equations do not apply when one of the two fluids is inviscid. The continuity of tangential stresses, Eq. (4), demands to first order in ϵ

$$\frac{\partial}{\partial r}\{r^{-2}[\mu_2 S_2(r,t) - \mu_1 S_1(r,t)]\}\big|_{r=R} = 0, \tag{22}$$

$$2(2n+1)R^{n-2}[\mu_1\alpha_1(t) - \mu_2\alpha_2(t)] + (n+1)R^{-1}[\mu_2 T_2(R,t) - \mu_1 T_1(R,t)]$$
$$= 2(n+1)\left[\left(\frac{n+2}{n+1}\mu_2 - \frac{n-1}{n}\mu_1\right)\frac{a'}{R} + \left(\frac{n+2}{n}\mu_1 - \frac{n-1}{n+1}\mu_2\right)a\frac{R'}{R^2}\right]. \tag{23}$$

It is seen from the equation for S, Eq. (16), and from the continuity conditions (20) and (22) just stated that $S_1(r, t) = S_2(r, t) = 0$ is an acceptable solution which satisfies also the regularity conditions at the origin and at infinity. Indeed, in an earlier study [13] in which the class of flows considered here was investigated under the assumption of axial symmetry it was found that

$$\mathbf{S} = 0, \qquad \boldsymbol{\omega} = -r^{-1}T(r,t)P_n^1(\cos\theta)\mathbf{e}_\varphi,$$

where $P_n^1(\cos\theta)$ is an associated Legendre polynomial. It appears therefore that the poloidal field $\mathbf{S}$ is necessary only to accommodate particular initial conditions, but that it cannot be generated if it vanishes at $t = 0$. Quite different is the situation for the toroidal field $\mathbf{T}$: Eq. (23) connects it explicitly to the motion of the free surface and shows that, even if $T = 0$ initially, a toroidal field will be generated in general at the free surface, from which it will diffuse into the fluids according to Eq. (17). This of course is in agreement with the expectation that vorticity is generated at a free surface because in general the tangential stresses that would be associated with an irrotational motion are discontinuous there [14, pp. 364–367].

The four equations (20)–(23) and the differential equations for the defining scalars (16) and (17) determine the dependence of $S(r, t)$ on $R(t)$ and of $T(r, t)$ on $R(t)$ and $a(t)$. Hence all the flow quantities in principle can now be expressed in terms of a, R, and their first two time derivatives, so that we expect the last boundary condition, Eq. (5), to act like a consistency condition and to give the equations of motion for R and a. This is indeed what is found. To order zero in ϵ we get the following equation for the radius of the approximating sphere:

$$RR'' + \tfrac{3}{2}R'^2 = (\rho_2 - \rho_1)^{-1}[P_1(t) - P_2(t) - 2\zeta/R + 4(\mu_1 - \mu_2)R'/R]. \quad (24)$$

If in this equation $P_1 - P_2$ is a constant and surface tension, viscosity, and the density of the inner fluid are neglected, it reduces to the well-known Rayleigh equation for the collapse of an empty cavity in an infinite liquid [15; 2, p. 122; 16, Sect. 10]. For $P_1 - P_2$ not constant, $\zeta \neq 0$ and $\mu_2 \neq 0$ it becomes the Rayleigh–Plesset equation for the dynamics of gas or vapor bubbles in viscous liquids [17, 18]; our result (24) includes the effect of the density and viscosity of the inner fluid as well.

The terms of first order in ϵ determine the equation of motion of $a(t)$ to be

$$\begin{aligned}
&\left(\frac{\rho_2}{n+1} + \frac{\rho_1}{n}\right)Ra'' + \left[3\left(\frac{\rho_2}{n+1} + \frac{\rho_1}{n}\right)R' - \frac{2}{R}(n+2)(n-1)(\mu_2 - \mu_1)\right]a' \\
&+ \left[\left(\frac{n+2}{n}\rho_1 - \frac{n-1}{n+1}\rho_2\right)R'' + 2(n-1)(n+2)(\mu_2 - \mu_1)\frac{R'}{R^2}\right. \\
&\left. + (n-1)(n+2)\frac{\zeta}{R^2}\right]a + n(n+2)\mu_2 T_2(R, t)/R - (n+1)(n-1)\mu_1 T_1(R, t)/R \\
&+ (n+1)\rho_1(R'/R)\int_0^R [1 - (s/R)^3](s/R)^{n-2}T_1(s, t)\,ds \\
&- n\rho_2(R'/R)\int_R^\infty [1 - (R/s)^3](R/s)^n T_2(s, t)\,ds = 0. \qquad (25)
\end{aligned}$$

This equation presents some interesting features. For an inviscid and irrotational flow it reduces to the one obtained by Plesset [9] in his analysis of the stability flows possessing spherical symmetry. The characteristic way in which it is modified by the addition of viscosity will perhaps be appreciated better in connection with the discussion of the oscillations of drops and bubbles given below. For the time being let us remark that the terms involving $T(r, t)$, when expressed as functions of a, turn out to involve convolution integrals of $a(t)$ and appropriate functions of time, so that Eq. (25) has in fact an integro-differential structure. A similar feature is encountered in the case of oscillations of a plane liquid surface [4] and in other viscous flow problems [19]. The physical reason for this circumstance is that the rate of energy dissipation depends on the distribution of vorticity generated at the free surface, which is itself a time-dependent process.

The last two terms of Eq. (25) vanish in the case of constant radius, and therefore appear to owe their presence to a convective effect. Indeed, it is to be expected that in a flow with so strongly converging (or diverging) streamlines as the spherical one the accompanying concentration (or dilatation) of vorticity would play an important role in the process of energy dissipation and hence in the equation of motion for $a(t)$.

The set of equations (17), (21), (23–25) supplemented by appropriate initial conditions constitutes a complete system of equations for the motion of the free surface. Their solution appears in general to be a formidable task, and some approximations will have to be devised for the analysis of particular problems. A straightforward one consists in setting $\rho_1 = 0$, $\mu_1 = 0$ or $\rho_2 = 0$, $\mu_2 = 0$ according as the mechanical effects of the inner or the outer fluid are negligible; these may be termed the *bubble* and *drop* approximations respectively. The continuity of tangential velocity, Eqs. (20) and (21), should not be applied in either case. For processes occurring on a time scale τ, whenever one of the two viscosities is such that $(\nu_i\tau)^{1/2} \ll R$, the integrals involving T_i can be neglected if $T_i(r, 0) = 0$, because the vorticity generated at the free surface will not have time to diffuse appreciably far into the fluid. Even in this case, however, one can see that the contribution from $T_i(R, t)$ does not vanish. This feature is well known in other problems involving free surfaces in slightly viscous flows, such as the damping of water waves [2, p. 623; 14, p. 370; 16, Sect. 25].

5. The oscillations of a liquid drop. Drop oscillations are of great interest in many diverse fields ranging from chemical engineering and physics of the atmosphere to nuclear physics, and therefore they constitute a meaningful problem suited to illustrate some characteristics of our theory and to clarify its connection with previous work.

We consider a liquid drop of fixed radius R in a medium of negligible dynamical effects (such as air) which is set into oscillation at time $t = 0$. For simplicity we restrict our treatment to the case in which $T_1(r, 0) = 0$; a discussion of the effects of a non-zero initial vorticity can be made following the line indicated in [4]. With R = constant, $\mu_2 = 0$, $\rho_2 = 0$, the set of equations governing this problem reduces to

$$\nu \frac{\partial^2 T}{\partial r^2} - \frac{\partial T}{\partial t} - \nu r^{-2} n(n+1)T = 0. \tag{26}$$

$$R^{-1}T(R, t) - 2R^{-n-3} \int_0^R s^{n+1}T(s, t)\, ds = 2\,\frac{n-1}{n}\,\frac{a'}{R}, \tag{27}$$

$$a'' + 2n(n-1)(n+2)(\nu/R^2)a' + n(n-1)(n+2)(\varsigma/\rho R^3)a - n(n-1)(n+1)(\nu/R^2)T(R, t) = 0, \tag{28}$$

where $\alpha_1(t)$ has been expressed in terms of T_1 according to (19*a*) and the index 1 has been omitted. Upon elimination of $T(R, t)$ with the aid of (27), Eq. (28) becomes

$$a'' + 2(n-1)(2n+1)(\nu/R^2)a' + n(n-1)(n+2)(\varsigma/\rho R^3)a - 2n(n-1)(n+1)\nu R^{-n-4} \int_0^R s^{n+1}T(s, t)\, ds = 0. \tag{29}$$

If the volume in which $T(r, t)$ is effectively different from zero is so small that the last term is negligible, we can read directly from this equation the frequency of the oscillations ω_0 and the decay constant b_0 :

$$\omega_0^2 = n(n-1)(n+2)\varsigma/\rho R^3, \tag{30}$$

$$b_0 = (n-1)(2n+1)\nu/R^2. \tag{31}$$

As will be seen below, these results are valid both when viscosity is small, since then the vorticity is effectively concentrated in a layer of thickness of order $(\nu/\omega_0)^{1/2}$, and in the initial stages of the motion, $t \ll R^2/\nu$, since then it has not had the time to diffuse appreciably far from the surface of the drop. Our result for ω_0 coincides with that obtained by Rayleigh [20; 2, p. 473] for the oscillations of an inviscid drop, and that for the damping constant b_0 with the expression derived by Lamb in the approximation of irrotational flow [2, p. 639].

It is now necessary to express the integral in (29) in terms of $a(t)$, which can be achieved with the aid of the Laplace transform. Let us denote the Laplace transform of any function $f(t)$ by a tilde,

$$\tilde{f}(p) = \int_0^\infty \exp(-pt)f(t)\,dt.$$

The (ordinary) differential equation for $\tilde{T}(r, p)$ obtained from (26) can be reduced to Bessel's equation and the solution is found to be

$$\tilde{T}(r, p) = (r/R)^{1/2}\tilde{T}_0(p)\,\frac{I_{n+1/2}(r(p/\nu)^{1/2})}{I_{n+1/2}(R(p/\nu)^{1/2})}. \tag{32}$$

The unknown boundary datum $\tilde{T}_0(p)$ can be determined from (27) with the result

$$\tilde{T}_0(p) = 2\,\frac{n-1}{n}\,\tilde{a}'(p)\left[1 - \frac{2}{q}\,\frac{I_{n+3/2}(q)}{I_{n+1/2}(q)}\right]^{-1}, \tag{33}$$

where $q = R(p/\nu)^{1/2}$*. These results can now be used to compute the integral in (29). Upon application of the convolution theorem for the Laplace transform this equation becomes

$$a'' + 2b_0a' + \omega_0^2a + 2b_0\beta\int_0^t Q(t-\tau)a'(\tau)\,d\tau = 0, \tag{34}$$

where $\beta = (n-1)(n+1)/(2n+1)$ and the function $Q(t)$ is defined by its Laplace transform as

$$\tilde{Q}(p) = 2I_{n+3/2}(q)[2I_{n+3/2}(q) - qI_{n+1/2}(q)]^{-1}. \tag{35}$$

As was anticipated above, Eq. (34) has an integro-differential structure: the acceleration at any instant t depends not only on position and velocity at that instant, but also on the values assumed by the velocity in a time interval of length $\sim R^2/\nu$ prior to t, which is the time scale for the process of vorticity diffusion.

Eq. (34) supplemented by the initial conditions $a(0) = a_0$, $a'(0) = u_0$ can be formally solved for $\tilde{a}(p)$ to obtain

$$\tilde{a}(p) = \frac{1}{p}\left[a_0 + \frac{u_0p - \omega_0^2a_0}{p^2 + 2b_0p + \omega_0^2 + 2\beta b_0p\tilde{Q}(p)}\right]. \tag{36a}$$

We shall consider first the small- and the large-time limits of this result. To discuss the first case let us set

$$a(t) = \exp(-\sigma_0t)v_0(t), \tag{37}$$

* The function $qI_{\nu-1}(q)/I_\nu(q)$ belongs to the class of modified quotients of cylinder functions studied in [21].

where σ_0 is a complex constant to be determined in such a way that $v_0(t)$ differs from its initial value by as large a power of t as possible. Substituting (37) into (36a), expanding the result in powers of $p^{-1/2}$ for $p \to \infty$, and inverting term by term [22], we find

$$v_0(t) \simeq v_0(0)\left\{1 - \tfrac{1}{2}[\sigma_0^2 - 2b_0\sigma_0 + \omega_0^2]t^2 + \frac{32}{15\pi^{1/2}}(n-1)^2(n+1)\nu^{3/2}\sigma_0 t^{5/2}/R^3 + \cdots\right\}$$

from which it is evident that, as had been anticipated, σ_0 must be chosen as

$$\sigma_0 = b_0 \pm i(\omega_0^2 - b_0^2)^{1/2}$$

with ω_0, b_0 given by (30), (31).

The other limiting case, $t \to \infty$, can be studied by letting

$$a(t) = \exp(-\sigma_\infty(t)), \tag{38a}$$

with σ_∞ a complex constant determined in such a way that $v_\infty(t) \to$ constant as $t \to \infty$ or, equivalently, that

$$p\tilde{v}_\infty(p) = p\tilde{a}(p - \sigma_\infty) \to \text{constant}, \quad \text{as} \quad p \to 0. \tag{38b}$$

It is evident that this condition can be satisfied only if the denominator in (36a) vanishes for $p = 0$. Writing out this requirement explicitly, we obtain

$$\sigma_\infty^2 - 2b_0\sigma_\infty + \omega_0^2 + 2\beta b_0\sigma_\infty \frac{2J_{n+3/2}(x)}{xJ_{n+1/2}(x) - 2J_{n+3/2}(x)} = 0, \tag{39}$$

where $x = R(\sigma_\infty/\nu)^{1/2}$. An identical equation has been obtained by Chandrasekhar [5; 3, p. 466] and Reid [6], for the normal modes of an oscillating globe and drop, respectively. The meaning of this limiting result can be clarified by observing that the solution to an initial-value problem must be obtained through an appropriate sum over the normal modes given by (39); as $t \to \infty$, only the least damped of the normal modes will give an appreciable contribution, and it is precisely this normal mode that is singled out from the solution of the initial-value problem by our condition (38b)**.

Unfortunately, it does not appear possible to invert the expression (36a) analytically, and one must make use of numerical methods for this purpose. We shall present here results for a particular case, deferring a more detailed study to a forthcoming work. Before proceeding to the numerical inversion, it is advantageous to introduce the following nondimensional quantities:

$$\tau = (\zeta/\rho R^3)^{1/2}t, \qquad s = (\rho R^3/\zeta)^{1/2}p,$$

$$a^*(\tau) = a(t)/a_0, \qquad u_0^* = (\rho R^3/\zeta)^{1/2}u_0/a_0, \qquad \epsilon = \nu(\rho/R\zeta)^{1/2},$$

in terms of which (36a) becomes

$$a^* = \frac{1}{s}\left[1 + \frac{u_0^*s - n(n-1)(n+2)}{s^2 + 2(n-1)(2n+1)\epsilon s + n(n-1)(n+2) + 2(n-1)^2(n+1)\epsilon s\tilde{Q}(s)}\right], \tag{36b}$$

where now

$$\tilde{Q}(s) = 2I_{n+3/2}[(s/\epsilon)^{1/2}]\{2I_{n+3/2}[(s/\epsilon)^{1/2}] - (s/\epsilon)^{1/2}I_{n+1/2}[(s/\epsilon)^{1/2}]\}^{-1}.$$

** It can be shown, as in [4], that this conclusion concerning the asymptotic form of $a(t)$ is true also if $T(r, 0) \neq 0$.

In Fig. 1 the initially valid solution (Eq. (37)), the asymptotic one (Eq. (38a)), and the complete one (Eq. (36b)), are compared in the case $\epsilon = 0.3$, $u_0^* = 0.3$. The numerical inversion of the Laplace transform has been done with the aid of the method described in [23]. It is seen that the short- and long-time behaviors deduced above are confirmed by the numerical results. It also appears that the asymptotic regime is here reached much faster than in the plane case considered in [4]. It appears likely that this characteristic is produced by the slight "focusing" effect acting on the vorticity during its inward diffusion as a consequence of the geometry.

It may also be of some interest to compare the dimensionless asymptotic (complex) frequencies $(\rho R^3/\zeta)^{1/2}\sigma_0 = \beta_0 + i\nu_0$ and $(\rho R^3/\zeta)^{1/2}\sigma_\infty = \beta_\infty + i\nu_\infty$. We present in Fig. 2 the results for $n = 2$ as a function of the dimensionless parameter ϵ; further results are given by Tong and Wong [8]. For $\nu_0 = 0$ the initial motion is critically damped and becomes aperiodic. This happens for $\omega_0 R^2/\nu = 5$, 14, and 17, respectively for $n = 2, 3, 4$. From a table given by Chandrasekhar [3, 5] it results that the asymptotic motion is critically damped only for values of $\omega_0 R^2/\nu$ smaller than 3.630, 6.026, and 8.457 respectively for $n = 2, 3, 4$. Hence it is seen that, in a certain range of values of ϵ, a motion that starts out aperiodically will exhibit damped oscillations at later times. It appears therefore that the effective rate of energy dissipation is a function of time which decreases during the motion. The same feature is encountered in the damping of surface waves on a plane surface [4], and is caused by the smoothing out of velocity gradients operated by viscosity.

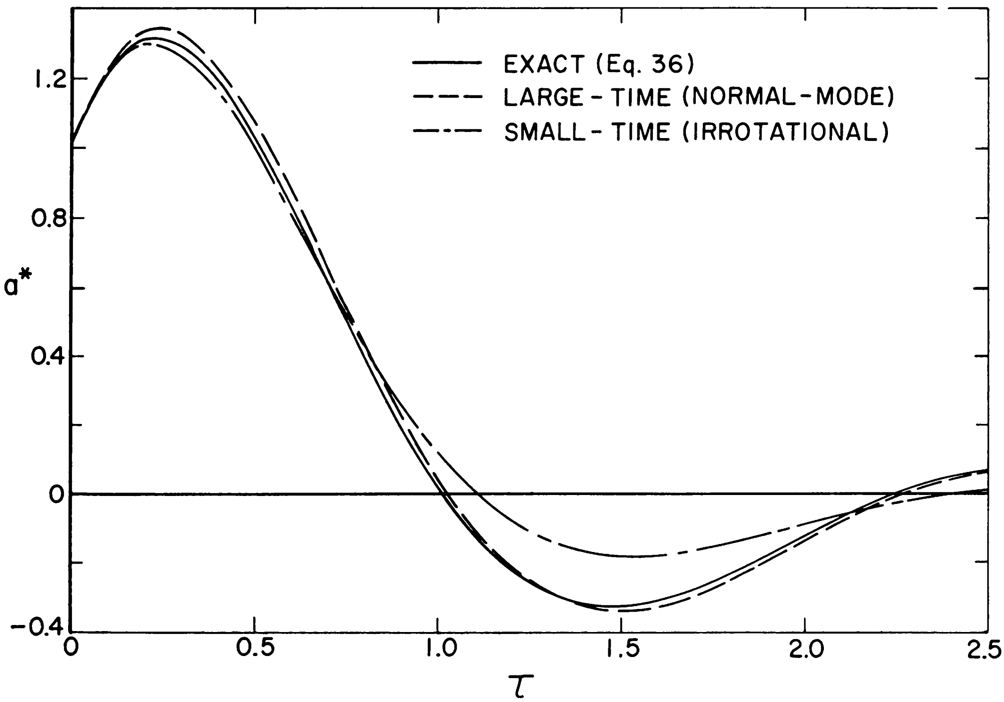

FIG. 1. Comparison of the complete solution, obtained by numerical inversion of Eq. (36b) (solid line), with the large-time approximation corresponding to the normal-mode analysis (Eq. (38a), broken line), and the initially valid approximation, obtained by means of the irrotational approximation (Eq. (37), dash-and-dot line). The values of the parameters for this example are $n = 2$, $u_0^* = 0.3$, $\epsilon = 0.3$.

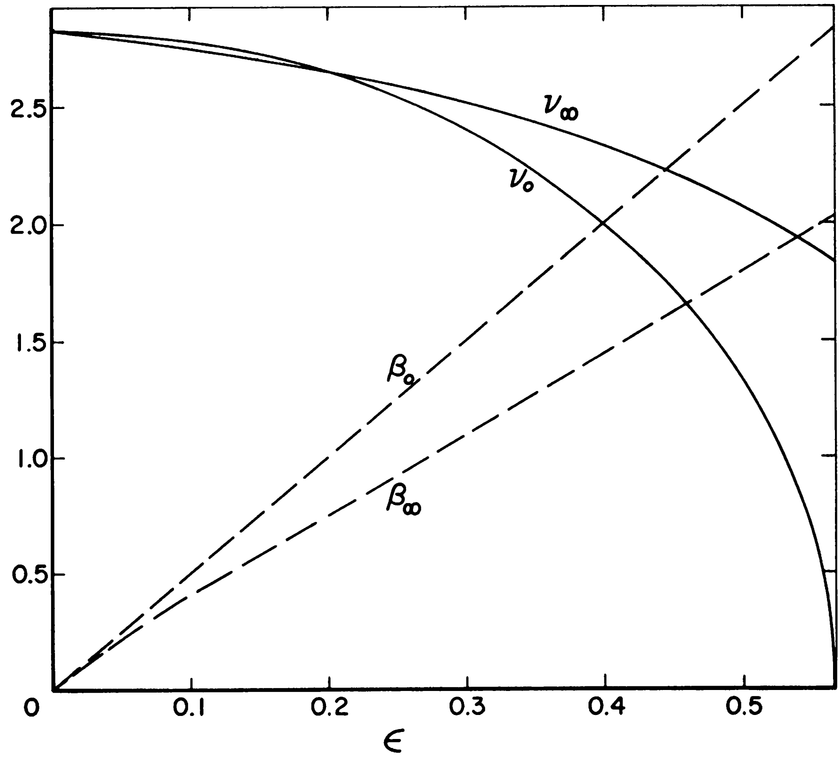

FIG. 2. The real and imaginary parts of the dimensionless complex frequencies of the initial (β_0, ν_0) and the large-time (β_∞, ν_∞) approximations for $n = 2$. For $\epsilon = (2/5)2^{1/2}$ the initial motion is critically damped and becomes aperiodic, while the large-time motion retains an oscillatory nature up to $\epsilon = 0.7792$.

6. The oscillations of a gas bubble. The oscillations about the spherical shape of a bubble of constant radius can be analyzed in much the same way as was done in the preceding section for the drop case. Setting $\mu_1 = 0$, $\rho_1 = 0$ and dropping the subscript 2 in Eqs. (25) and (23), one gets

$$a'' - 2(n-1)(n+1)(n+2)(\nu/R^2)a' + (n-1)(n+1)(n+2)(\zeta/\rho R^3)a + n(n+1)(n+2)(\nu/R^2)T(R, t) = 0,$$

$$2(n+1)R^{n-2}\int_R^\infty s^{-n}T(s, t)\,ds + (n+1)T(R, t)/R = 2(n+2)a'/R, \qquad (40)$$

from which, upon elimination of $T(R, t)$, the following equation is obtained:

$$a'' + 2(n+2)(2n+1)(\nu/R^2)a' + (n-1)(n+1)(n+2)(\zeta/\rho R^3)a - 2n(n+1)(n+2)\nu R^{n-3}\int_R^\infty s^{-n}T(s, t)\,ds = 0.$$

The frequency and damping constants of the initially valid, approximately irrotational, solution are

$$\omega_0^{\,2} = (n-1)(n+1)(n+2)\zeta/\rho R^3, \qquad (41)$$

$$b_0 = (n + 2)(2n + 1)\nu/R^2, \tag{42}$$

in agreement with the results given by Lamb [2, p. 475, p. 640]. The equation for $\tilde{T}(r, p)$ is again reducible to Bessel's equation and has the solution

$$\tilde{T}(r, p) = (r/R)^{1/2}\tilde{T}_0(p)\frac{K_{n+1/2}(r(p/\nu)^{1/2})}{K_{n+1/2}(R(p/\nu)^{1/2})},$$

and the boundary datum $\tilde{T}_0(p)$ is obtained from Eq. (40) as

$$\tilde{T}_0(p) = 2\frac{n+2}{n+1}\tilde{a}'\left[1 + \frac{2}{q}\frac{K_{n-1/2}(q)}{K_{n+1/2}(q)}\right]^{-1}$$

with $q = R(p/\nu)^{1/2}$. The final equation for $a(t)$ is obtained in the same form as Eq. (34) above with b_0, ω_0 given by (41) and (42), $\beta = n(n + 2)/(2n + 1)$, and $Q(t)$ defined by

$$\tilde{Q}(p) = -2K_{n-1/2}(q)/[qK_{n+1/2}(q) + 2K_{n-1/2}(q)]^{-1}.$$

By the same method used before one also finds the following characteristic equation for the asymptotic complex frequency σ_∞ :

$$\sigma_\infty^2 - 2b_0\sigma_\infty + \omega_0^2 + 2\beta b_0\sigma_\infty\frac{2H_{n-1/2}^{(1)}(x)}{xH_{n+1/2}^{(1)}(x) + 2H_{n-1/2}^{(1)}(x)} = 0, \tag{43}$$

again with $x = R(\sigma_\infty/\nu)^{1/2}$. After some simplifications with the aid of the relations connecting contiguous Bessel functions, the characteristic equation obtained by Miller and Scriven [7] can be reduced to this equation.

No numerical studies of the roots of (43) are available, and therefore no comparison between initial and final effective damping is yet possible; however, it is believed that a pattern similar to that found in the last section and in [4] would emerge, i.e. that the initial effective damping would be found to be larger than the asymptotic one.

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