ON SOME PROBLEMS OF HOMOGENIZATION*

By

J. M. BURGERS

University of Maryland

Abstract. This article concerns a type of partial differential equations in which coefficients occur that are periodic functions of the basic independent variables or coordinates x_1, x_2 (we restrict ourselves to two-dimensional problems). The period length H , here assumed to be the same in both directions, is supposed to be small in comparison with the scale of the general field, and interest is directed to what happens when $H \rightarrow 0$. An example is the equation

$$\sum \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) = f(x_1, x_2)$$

with the boundary condition $u = 0$ on a given closed curve C in the x_1, x_2 -plane. Size and shape of C are independent of H , while the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ are periodic functions of auxiliary variables $\xi_i = x_i/H$ with period 1 in ξ_1, ξ_2 . The problem of interest is whether the unknown function u , which must react upon all the fluctuations of the a_{ij} , can be related to a function $U(x_1, x_2)$ which is determined by a partial differential equation with *constant* coefficients, and how these coefficients can be obtained from the a_{ij} . This is called the problem of "homogenization".

It is the purpose of the present paper to show that this problem assumes a particular form when we replace the auxiliary variables $\xi_i = x_i/H$ by variables

$$\xi_i = (x_i - x_{i0})/H,$$

again with period 1 in ξ_1, ξ_2 but having a movable origin, while leaving the curve C and the input function $f(x_1, x_2)$ unchanged. The solution of Eq. (1) will now become dependent on the parameters H, x_{10}, x_{20} . Evidently it will be periodic in x_{10}, x_{20} and we can give attention to the mean value $\langle u \rangle$ of u at a fixed point x_1, x_2 when x_{10}, x_{20} move over a cell of the period grid. The explicit introduction of the movable parameters x_{10}, x_{20} gives a new tool for the treatment of the homogenization problem and for the deduction of the equation for the function $U(x_1, x_2)$.

Three examples are presented, the first one and the second one referring to Eq. (1), with a margin of error either of order H or of order H^2 . The third example refers to a system with two dependent variables subject to the equations of two-dimensional elasticity.

* Received January 20, 1977.

1. Introduction. In recent years much attention has been given [1] to a particular type of partial differential equations in which coefficients occur that are periodic functions of the basic independent variables or coordinates x_1, x_2 (we restrict ourselves here to two-dimensional problems). The period length H , here taken the same in both directions, is supposed to be small in comparison with the scale L of the general field, and interest is directed to what happens when $H \rightarrow 0$. As an example we may think of problems in elasticity where the stressed medium has a periodic microstructure and where the question is raised whether in the limit $H \rightarrow 0$ the microstructure can be replaced by homogeneous uniform material under the condition that the overall deformation shall closely approximate the overall deformation of the original microstructured material.

To present an example in mathematical terms, we consider the partial differential equation of elliptic type in two independent variables x_1, x_2 ,

$$\Omega_x(u) \equiv \sum \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) = f(x_1, x_2), \quad (1)$$

with the boundary condition

$$u = 0 \text{ along (and outside) a given closed fixed curve } C \text{ in the } x_1, x_2\text{-plane.} \quad (2)$$

Here the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ are periodic functions of auxiliary variables $\xi_i = x_i/H$ with period 1 in ξ_1, ξ_2 . The parameter H can be chosen arbitrarily, provided it is small in comparison with the scale length L characteristic for the boundary curve C and for the input function $f(x_1, x_2)$. Changes of H do not affect the functions $a_{ij}(\xi_1, \xi_2)$, nor the function $f(x_1, x_2)$, or the form and situation of the curve C . The coefficients a_{ij} may be rapidly fluctuating inside each period square. For convenience we consider the a_{ij} as being of unit order of magnitude; the same will hold for their derivatives with respect to ξ_1 and ξ_2 .

A problem which is investigated in the literature is whether the unknown function u , which according to Eq. (1) must react upon all the fluctuations of the coefficients a_{ij} , can be related to an auxiliary function $U(x_1, x_2)$ which is determined by a partial differential equation with *constant* coefficients, and how these coefficients can be obtained from the a_{ij} . This is called the problem of "homogenization". It is the purpose of the present paper to show that this problem assumes a particularly convenient form when we replace the auxiliary variables $\xi_i = x_i/H$ by variables

$$\xi_i = (x_i - x_{i0})/H \quad (3)$$

again with period 1 in ξ_1, ξ_2 , but now having a movable origin. By explicitly introducing the two quantities x_{10}, x_{20} we obtain the possibility of shifting the origin of the periodicity grid in an arbitrary way over the x_1, x_2 -plane, while leaving the curve C and the input function $f(x_1, x_2)$ unchanged. It is evident that the solution $u(x_1, x_2)$ of Eq. (1) with the boundary condition (2) will now become dependent on the parameters H, x_{10}, x_{20} , being periodic in x_{10}, x_{20} with period H . It makes sense therefore to give attention to the mean value $\langle u \rangle$ of u at a fixed point x_1, x_2 when x_{10}, x_{20} move over a cell of the period grid. Such average values will have more importance (also from a physical point of view) than the incidental value of u for an arbitrary choice of x_{10}, x_{20} . If the point x_1, x_2 is situated on the boundary curve C the value of u must be zero for every choice of x_{10}, x_{20} ; hence the same will hold for the mean value $\langle u \rangle$. If x_1, x_2 is close to but not on C , the value of u will differ from zero by a small amount for any choice of x_{10}, x_{20} and again the same will hold for $\langle u \rangle$. Hence we do not expect any difficulty in the averaging process with fixed x_1, x_2 . It may be said that the introduction of the movable parameters x_{10}, x_{20} gives a new tool for

the treatment of the homogenization problem and changes its nature, "linearizes" it in a certain sense.

Incidentally, it may be noted that the partial differential equation (1) can be interpreted by supposing that u represents a pressure field (with positive and negative pressures), and that pressure gradients produce flow components

$$p_j = -a_{ij}(\partial u / \partial x_i), \quad (4)$$

the a_{ij} being "permeability coefficients". Then

$$\partial p_j / \partial x_j = -(\partial / \partial x_j)[a_{ij}(\partial u / \partial x_i)] \quad (5)$$

is the divergence (or convergence) of the flow, which must be equal to the "input" $f(x_1, x_2)$. The contour C then represents a boundary of zero pressure.

2. The problem to be studied is whether the unknown function u , which according to Eq. (1) must react upon all the fluctuations of the coefficients a_{ij} , can be related to an auxiliary function $U(x_1, x_2)$, not dependent upon x_{10} , x_{20} and defined by a partial differential equation with *constant* coefficients. These coefficients should presumably be formed from certain averages deduced from the a_{ij} . For this purpose we assume a relation between u and U of the special form

$$u = U + H \left(\frac{\partial U}{\partial x_1} \chi^I + \frac{\partial U}{\partial x_2} \chi^{II} \right). \quad (6)$$

The functions χ^I , χ^{II} shall be periodic functions of ξ_1 , ξ_2 with unit period, for which we must find suitable equations. Like the coefficients a_{ij} they will be supposed to be of unit order of magnitude. The same then will hold for their derivatives with respect to ξ_1 , ξ_2 . Substitution of (6) into Eq. (1) leads to the transformed equation:

$$\begin{aligned} & \frac{1}{H} \frac{\partial U}{\partial x_1} \left[\Omega(\chi^I) + \frac{\partial a_{11}}{\partial \xi_1} + \frac{\partial a_{12}}{\partial \xi_2} \right] + \frac{1}{H} \frac{\partial U}{\partial x_2} \left[\Omega(\chi^{II}) + \frac{\partial a_{21}}{\partial \xi_1} + \frac{\partial a_{22}}{\partial \xi_2} \right] \\ & + \frac{\partial^2 U}{\partial x_1^2} \left[\frac{\partial}{\partial \xi_1} (a_{11}\chi^I) + \frac{\partial}{\partial \xi_2} (a_{12}\chi^I) + a_{11} + a_{11} \frac{\partial \chi^I}{\partial \xi_1} + a_{21} \frac{\partial \chi^I}{\partial \xi_2} \right] \\ & + \frac{\partial^2 U}{\partial x_1 \partial x_2} \left[\frac{\partial}{\partial \xi_1} (a_{11}\chi^{II} + a_{21}\chi^I) + \frac{\partial}{\partial \xi_2} (a_{12}\chi^{II} + a_{22}\chi^I) \right. \\ & \left. + a_{21} + a_{11} \frac{\partial \chi^{II}}{\partial \xi_1} + a_{21} \frac{\partial \chi^{II}}{\partial \xi_2} + a_{12} + a_{12} \frac{\partial \chi^I}{\partial \xi_1} + a_{22} \frac{\partial \chi^I}{\partial \xi_2} \right] \\ & + \frac{\partial^2 U}{\partial x_2^2} \left[\frac{\partial}{\partial \xi_1} (a_{21}\chi^{II}) + \frac{\partial}{\partial \xi_2} (a_{22}\chi^{II}) + a_{22} + a_{22} \frac{\partial \chi^{II}}{\partial \xi_2} + a_{12} \frac{\partial \chi^{II}}{\partial \xi_1} \right] \\ & + H \left\{ \frac{\partial^3 U}{\partial x_1^3} a_{11}\chi^I + \frac{\partial^3 U}{\partial x_1^2 \partial x_2} [a_{11}\chi^{II} + a_{21}\chi^I + a_{12}\chi^I] \right. \\ & \left. + \frac{\partial^3 U}{\partial x_1 \partial x_2^2} [a_{22}\chi^I + a_{21}\chi^{II} + a_{12}\chi^{II}] + \frac{\partial^3 U}{\partial x_2^3} a_{22}\chi^{II} \right\} = f(x_1, x_2). \quad (7) \end{aligned}$$

We note that no multiplication or division by a power of H has occurred. Here the operator Ω is defined by

$$\Omega \equiv \frac{\partial}{\partial \xi_j} \left(a_{ij} \frac{\partial}{\partial \xi_i} \right). \quad (8)$$

We observe that for fixed x_{10} , x_{20} we have $\partial / \partial x_i = (1/H)(\partial / \partial \xi_i)$.

3. We now can make a choice for the functions χ^I, χ^{II} ; Eq. (7) then becomes an equation for U . The choice can be made so that the terms with the factor $1/H$ disappear from Eq. (7); then there is no danger of a catastrophe when $H \rightarrow 0$. This requires

$$\Omega(\chi^I) = -\frac{\partial a_{11}}{\partial \xi_1} - \frac{\partial a_{12}}{\partial \xi_2}; \Omega(\chi^{II}) = -\frac{\partial a_{21}}{\partial \xi_1} - \frac{\partial a_{22}}{\partial \xi_2}. \quad (9)$$

These equations are completely expressed in terms of ξ_1, ξ_2 . All terms are of unit order of magnitude and independent of H . Since the right-hand sides have zero integrals over a period cell, the equations admit periodic solutions for χ^I, χ^{II} which are identical in all period cells. As the equations refer only to derivatives of χ^I, χ^{II} , the solutions have an arbitrary additive constant, which we fix by requiring that

$$\iint d\xi_1 d\xi_2 \chi^I = 0; \iint d\xi_1 d\xi_2 \chi^{II} = 0, \quad (10)$$

the integrals being taken over a period cell. The functions $\chi^I(\xi_1, \xi_2), \chi^{II}(\xi_1, \xi_2)$ are then fully determined. Returning to Eq. (6), where we assume that U and $\partial U/\partial x_1, \partial U/\partial x_2$ are fully determined functions of x_1, x_2 (as will be established subsequently), we see how for a given point x_1, x_2 the quantities x_{10}, x_{20} enter into χ^I, χ^{II} through Eq. (3) and thus enter also into u . For fixed x_1, x_2 the functions χ^I, χ^{II} are periodic functions of x_{10}, x_{20} , and Eqs. (10) clearly imply that

$$\iint dx_{10} dx_{20} \chi^I = 0; \iint dx_{10} dx_{20} \chi^{II} = 0, \quad (11)$$

the integrals being taken over a period cell. It follows therefore from Eq. (6) that

$$\langle u \rangle \equiv \frac{1}{H^2} \iint dx_{10} dx_{20} u = U. \quad (12)$$

4. We next look at the terms with second-order derivatives of U which do not carry a power of H . We introduce the quantities (which are fully determined functions of ξ_1, ξ_2):

$$\begin{aligned} b_{11} &= a_{11} + a_{11} (\partial \chi^I / \partial \xi_1) + a_{21} (\partial \chi^I / \partial \xi_2); \\ b_{21} &= a_{21} + a_{11} (\partial \chi^{II} / \partial \xi_1) + a_{21} (\partial \chi^{II} / \partial \xi_2); \\ b_{12} &= a_{12} + a_{12} (\partial \chi^I / \partial \xi_1) + a_{22} (\partial \chi^I / \partial \xi_2); \\ b_{22} &= a_{22} + a_{12} (\partial \chi^{II} / \partial \xi_1) + a_{22} (\partial \chi^{II} / \partial \xi_2), \end{aligned} \quad (13)$$

and write

$$\begin{aligned} Q_{11} &= \iint_0^1 d\xi_1 d\xi_2 b_{11}; & Q_{21} &= \iint_0^1 d\xi_1 d\xi_2 b_{21}; \\ Q_{12} &= \iint_0^1 d\xi_1 d\xi_2 b_{12}; & Q_{22} &= \iint_0^1 d\xi_1 d\xi_2 b_{22}; \end{aligned} \quad (14)$$

(integrals over a period square). Set further

$$b_{11} = Q_{11} + \beta_{11}; \quad b_{21} = Q_{21} + \beta_{21}; \quad b_{12} = Q_{12} + \beta_{12}; \quad b_{22} = Q_{22} + \beta_{22}. \quad (15)$$

The four quantities Q_{ij} are constants related to the permeability coefficients a_{ij} ; they are independent of the ξ_i (and of $x_1, x_2, H, x_{10}, x_{20}$). Each of the four quantities β_{ij} has a zero integral over a period cell. The term with $\partial^2 U / \partial x_1^2$ in Eq. (7) can now be written:

$$\frac{\partial^2 U}{\partial x_1^2} \left[Q_{11} + \frac{\partial}{\partial \xi_1} (a_{11} \chi^I) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^I) + \beta_{11} \right].$$

Similar expressions are obtained for the terms with $\partial^2 U / \partial x_1 \partial x_2$ and $\partial^2 U / \partial x_2^2$. We take the average of Eq. (7) when x_{10}, x_{20} move over a period cell while x_1, x_2 remain fixed. This leads to the following equation (slightly rearranged):

$$Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{21} + Q_{12}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2} - f(x_1, x_2) = -H \left[\alpha_1 \frac{\partial^3 U}{\partial x_1^3} + \alpha_2 \frac{\partial^3 U}{\partial x_1^2 \partial x_2} + \alpha_3 \frac{\partial^3 U}{\partial x_1 \partial x_2^2} + \alpha_4 \frac{\partial^3 U}{\partial x_2^3} \right], \quad (16)$$

where

$$\begin{aligned} \alpha_1 &= \langle a_{11} \chi^I \rangle; & \alpha_2 &= \langle (a_{11} \chi^{II} + a_{21} \chi^I + a_{12} \chi^I) \rangle; \\ \alpha_3 &= \langle (a_{22} \chi^I + a_{21} \chi^{II} + a_{12} \chi^{II}) \rangle; & \alpha_4 &= \langle a_{22} \chi^{II} \rangle. \end{aligned} \quad (17)$$

These are constant numbers associated with the periodicity grid; they are independent of H and of x_{10}, x_{20} , and they are of order unity.

We now define U as that solution of Eq. (16) which becomes independent of H as $H \rightarrow 0$. In other words:

$$U \text{ is the solution of } Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{21} + Q_{12}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2} = f(x_1, x_2) \quad (18)$$

with the boundary condition $U = 0$ on C .

At the same time Eq. (6) gives $u = U$ with a possible error of order H and $\langle u \rangle = U$, as already given in Eq. (12). Eq. (18) is the required equation with constant coefficients.

5. The question will arise whether this procedure can be extended so as to incorporate terms with higher powers of H .

We extend Eq. (6) with three further terms, introducing functions $\chi^{III}, \chi^{IV}, \chi^V$:

$$u = U + H \left(\frac{\partial U}{\partial x_1} \chi^I + \frac{\partial U}{\partial x_2} \chi^{II} \right) + H^2 \left(\frac{\partial^2 U}{\partial x_1^2} \chi^{III} + \frac{\partial^2 U}{\partial x_1 \partial x_2} \chi^{IV} + \frac{\partial^2 U}{\partial x_2^2} \chi^V \right). \quad (19)$$

The transformed equation (7) now becomes

$$\begin{aligned} & \frac{1}{H} \frac{\partial U}{\partial x_1} \left[\Omega(\chi^I) + \frac{\partial a_{11}}{\partial \xi_1} + \frac{\partial a_{12}}{\partial \xi_2} \right] + \frac{1}{H} \frac{\partial U}{\partial x_2} \left[\Omega(\chi^{II}) + \frac{\partial a_{21}}{\partial \xi_1} + \frac{\partial a_{22}}{\partial \xi_2} \right] \\ & + \frac{\partial^2 U}{\partial x_1^2} \left[\Omega(\chi^{III}) + \frac{\partial}{\partial \xi_1} (a_{11} \chi^I) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^I) + b_{11} \right] \\ & + \frac{\partial^2 U}{\partial x_1 \partial x_2} \left[\Omega(\chi^{IV}) + \frac{\partial}{\partial \xi_1} (a_{11} \chi^{II} + a_{21} \chi^I) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^{II} + a_{22} \chi^I) + b_{21} + b_{12} \right] \\ & + \frac{\partial^2 U}{\partial x_2^2} \left[\Omega(\chi^V) + \frac{\partial}{\partial \xi_1} (a_{21} \chi^{II}) + \frac{\partial}{\partial \xi_2} (a_{22} \chi^{II}) + b_{22} \right] \end{aligned}$$

$$\begin{aligned}
& + H \frac{\partial^3 U}{\partial x_1^3} \left[\frac{\partial}{\partial \xi_1} (a_{11} \chi^{III}) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^{III}) + r_1 \right] \\
& + H \frac{\partial^3 U}{\partial x_1^2 \partial x_2} \left[\frac{\partial}{\partial \xi_1} (a_{11} \chi^{IV} + a_{21} \chi^{III}) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^{IV} + a_{22} \chi^{III}) + r_2 \right] \\
& + H \frac{\partial^3 U}{\partial x_1 \partial x_2^2} \left[\frac{\partial}{\partial \xi_1} (a_{11} \chi^V + a_{21} \chi^{IV}) + \frac{\partial}{\partial \xi_2} (a_{12} \chi^V + a_{22} \chi^{IV}) + r_3 \right] \\
& + H \frac{\partial^3 U}{\partial x_2^3} \left[\frac{\partial}{\partial \xi_1} (a_{21} \chi^V) + \frac{\partial}{\partial \xi_2} (a_{22} \chi^V) + r_4 \right] \\
& + H^2 \frac{\partial^4 U}{\partial x_1^4} a_{11} \chi^{III} + H^2 \frac{\partial^4 U}{\partial x_1^3 \partial x_2} \left[a_{11} \chi^{IV} + a_{21} \chi^{III} + a_{12} \chi^{III} \right] \\
& + H^2 \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} \left[a_{11} \chi^V + a_{21} \chi^{IV} + a_{12} \chi^{IV} + a_{22} \chi^{III} \right] \\
& + H^2 \frac{\partial^4 U}{\partial x_1 \partial x_2^3} \left[a_{22} \chi^{IV} + a_{21} \chi^V + a_{12} \chi^V \right] + H^2 \frac{\partial^4 U}{\partial x_2^4} a_{22} \chi^V = f(x_1, x_2). \quad (20)
\end{aligned}$$

To abbreviate the writing we have made use of the quantities b_{11} , b_{21} , b_{12} , b_{22} defined by Eq. (13), and have further introduced quantities r_1 , r_2 , r_3 , r_4 defined as follows:

$$\begin{aligned}
r_1 &= a_{11} \chi^I + a_{11} (\partial \chi^{III} / \partial \xi_1) + a_{21} (\partial \chi^{III} / \partial \xi_2); \\
r_2 &= a_{11} \chi^{II} + (a_{21} + a_{12}) \chi^I + a_{11} \frac{\partial \chi^{IV}}{\partial \xi_1} + a_{21} \frac{\partial \chi^{IV}}{\partial \xi_2} + a_{12} \frac{\partial \chi^{III}}{\partial \xi_1} + a_{22} \frac{\partial \chi^{III}}{\partial \xi_2}; \\
r_3 &= a_{22} \chi^I + (a_{21} + a_{12}) \chi^{II} + a_{11} \frac{\partial \chi^V}{\partial \xi_1} + a_{21} \frac{\partial \chi^V}{\partial \xi_2} + a_{12} \frac{\partial \chi^{IV}}{\partial \xi_1} + a_{22} \frac{\partial \chi^{IV}}{\partial \xi_2}; \\
r_4 &= a_{22} \chi^{II} + a_{12} (\partial \chi^V / \partial \xi_1) + a_{22} (\partial \chi^V / \partial \xi_2). \quad (21)
\end{aligned}$$

We then write

$$\begin{aligned}
R_1 &= \iint d\xi_1 d\xi_2 r_1; & r_1 &= r_1 + \rho_1, \\
R_2 &= \iint d\xi_1 d\xi_2 r_2; & r_2 &= R_2 + \rho_2, \\
R_3 &= \iint d\xi_1 d\xi_2 r_3; & r_3 &= R_3 + \rho_3, \\
R_4 &= \iint d\xi_1 d\xi_2 r_4; & r_4 &= R_4 + \rho_4.
\end{aligned} \quad (22)$$

The functions ρ_1 , ρ_2 , ρ_3 , ρ_4 have zero integrals over a period cell.

As before, we define χ^I , χ^{II} by means of Eqs. (9) with conditions (10), so that the terms with the factor $1/H$ disappear. We next define χ^{III} , χ^{IV} and χ^V as solutions of the equations

$$\begin{aligned}
\Omega(\chi^{\text{III}}) &= -\frac{\partial}{\partial \xi_1} (a_{11}\chi^{\text{I}}) - \frac{\partial}{\partial \xi_2} (a_{12}\chi^{\text{I}}) - \beta_{11}; \\
\Omega(\chi^{\text{IV}}) &= -\frac{\partial}{\partial \xi_1} (a_{11}\chi^{\text{II}} + a_{21}\chi^{\text{I}}) - \frac{\partial}{\partial \xi_2} (a_{12}\chi^{\text{II}} + a_{22}\chi^{\text{I}}) - \beta_{21} - \beta_{12}; \\
\Omega(\chi^{\text{V}}) &= -\frac{\partial}{\partial \xi_1} (a_{21}\chi^{\text{II}}) - \frac{\partial}{\partial \xi_2} (a_{22}\chi^{\text{II}}) - \beta_{22},
\end{aligned} \tag{23}$$

together with the condition that the integrals of χ^{III} , χ^{IV} , χ^{V} over a period cell shall be zero. The terms with the second-order derivatives of U then become the same as before:

$$Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{21} + Q_{12}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2}. \tag{24}$$

6. We now take the averages of the remaining terms. Those with the third-order derivatives become

$$H \left\{ R_1 \frac{\partial^3 U}{\partial x_1^3} + R_2 \frac{\partial^3 U}{\partial x_1^2 \partial x_2} + R_3 \frac{\partial^2 U}{\partial x_1 \partial x_2^2} + R_4 \frac{\partial^3 U}{\partial x_2^3} \right\}, \tag{25}$$

with R_1, R_2, R_3, R_4 as defined in Eq. (22); and those with fourth-order derivatives become

$$H^2 \left\{ \alpha_5 \frac{\partial^4 U}{\partial x_1^4} + \alpha_6 \frac{\partial^4 U}{\partial x_1^3 \partial x_2} + \alpha_7 \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} + \alpha_8 \frac{\partial^4 U}{\partial x_1 \partial x_2^3} + \alpha_9 \frac{\partial^4 U}{\partial x_2^4} \right\}, \tag{26}$$

where

$$\alpha_5 = \langle (a_{11}\chi^{\text{III}}) \rangle; \alpha_6 = \langle (a_{11}\chi^{\text{IV}} + a_{21}\chi^{\text{III}} + a_{12}\chi^{\text{III}}) \rangle, \text{ etc.}$$

The resulting equation for $U(x_1, x_2)$ then takes the form

$$\begin{aligned}
&Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{21} + Q_{12}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2} + \\
&+ H \left\{ R_1 \frac{\partial^3 U}{\partial x_1^3} + \text{etc.} \right\} + H^2 \left\{ \alpha_5 \frac{\partial^4 U}{\partial x_1^4} + \text{etc.} \right\} = f(x_1, x_2).
\end{aligned} \tag{27}$$

All coefficients on the left-hand side are constants. We look for that solution for U which can be developed in ascending powers of H , beginning with a term independent of H .

When using the terms of orders 0 and 1 in H , we then can obtain an improved expression for U , and use this in the approximation

$$u = U + H \left(\frac{\partial U}{\partial x_1} \chi^{\text{I}} + \frac{\partial U}{\partial x_2} \chi^{\text{II}} \right),$$

which should improve upon $u = U$ as obtained in the first approximation. The remaining error is then of order H^2 , while again

$$\langle u \rangle = U.$$

In principle this procedure can be extended to higher powers of H , but it is questionable whether the amount of calculating labor is worth while.

7. Summary. It is useful briefly to summarize the results obtained in the preceding pages.

It is assumed that the four coefficients a_{11} , a_{12} , a_{21} , a_{22} , the period H and the boundary curve C in the x_1 , x_2 -plane have been prescribed (a_{11} , $\dots$, a_{22} as functions of ξ_1 , ξ_2). To obtain an expression for u with an uncertainty of order H proceed as follows:

- (a) solve Eq. (9) for χ^I , χ^{II} as functions of ξ_1 , ξ_2 ;
- (b) calculate b_{11} , b_{12} , b_{21} , b_{22} from Eqs. (13) and the numbers Q_{11} , Q_{12} , Q_{21} , Q_{22} from Eqs. (14);
- (c) solve Eq. (18) for $U(x_1, x_2)$ with constant coefficients:

$$Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{12} + Q_{21}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2} = f(x_1, x_2), \quad (18)$$

using the boundary condition $U = 0$ on the C in the x_1 , x_2 -plane;

(d) then

$$\begin{aligned} \langle u \rangle &= U + \text{uncertainty of order } H^2 \text{ (presumably).} \\ u &= U + \text{uncertainty of order } H; \end{aligned} \quad (28)$$

To obtain an expression for u with an uncertainty of order H^2 the following additional calculations are needed:

- (e) from b_{11} , b_{12} , b_{21} , b_{22} and the numbers Q_{11} , Q_{12} , Q_{21} , Q_{22} form the quantities β_{11} , β_{12} , β_{21} , β_{22} defined in Eqs. (15);
- (f) solve Eqs. (23) for χ^{III} , χ^{IV} , χ^V as functions of ξ_1 , ξ_2 ;
- (g) calculate the functions r_1 , r_2 , r_3 , r_4 from Eqs. (21) and the numbers R_1 , R_2 , R_3 , R_4 , from Eqs. (22);
- (h) calculate a corrected function $U_c(x_1, x_2)$ from the extended equation with constant coefficients

$$\begin{aligned} Q_{11} \frac{\partial^2 U}{\partial x_1^2} + (Q_{12} + Q_{21}) \frac{\partial^2 U}{\partial x_1 \partial x_2} + Q_{22} \frac{\partial^2 U}{\partial x_2^2} \\ + H \left[R_1 \frac{\partial^3 U}{\partial x_1^3} + R_2 \frac{\partial^3 U}{\partial x_1^2 \partial x_2} + R_3 \frac{\partial^3 U}{\partial x_1 \partial x_2^2} + R_4 \frac{\partial^3 U}{\partial x_2^3} \right] = f(x_1, x_2), \end{aligned} \quad (29)$$

again using the boundary condition $U_c = 0$ on the curve C in the x_1 , x_2 -plane, with the condition that U shall remain regular for $H \rightarrow 0$.

(i) then

$$\begin{aligned} u &= U_c + H \left(\frac{\partial U}{\partial x_1} \chi^I + \frac{\partial U}{\partial x_2} \chi^{II} \right) + \text{uncertainty of order } H^2; \\ \langle u \rangle &= U_c + \text{error of order } H^2 \text{ (or maybe } H^3). \end{aligned} \quad (30)$$

8. Extension to the equations of the two-dimensional theory of elasticity. As a further example we will consider the equations of the two-dimensional theory of elasticity. In this case the field is described by two variables, the "displacement components" u_1 , u_2

which take the place of the former single variable u . Introducing the elastic coefficients λ, μ (writing $\lambda^* = \lambda + 2\mu$ for short), the elastic stress components are given by

$$X_{11} = \lambda^* \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2}, X_{21} = X_{12} = \mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right), X_{22} = \lambda \frac{\partial u_1}{\partial x_1} + \lambda^* \frac{\partial u_2}{\partial x_2}. \quad (31)$$

It is assumed that there is no displacement component u_3 , and that $\partial/\partial x_3 = 0$ for all variables. The components then must satisfy the equilibrium conditions

$$\frac{\partial X_{11}}{\partial x_1} + \frac{\partial X_{21}}{\partial x_2} = f_1(x_1, x_2); \quad \frac{\partial X_{12}}{\partial x_1} + \frac{\partial X_{22}}{\partial x_2} = f_2(x_1, x_2), \quad (32)$$

where f_1, f_2 represent exterior forces. This leads to the following two equations for u_1, u_2 :

$$\begin{aligned} \frac{\partial}{\partial x_1} \left(\lambda^* \frac{\partial u_1}{\partial x_1} \right) + \frac{\partial}{\partial x_1} \left(\lambda \frac{\partial u_2}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left(\mu \frac{\partial u_2}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\mu \frac{\partial u_1}{\partial x_2} \right) &= f_1; \\ \frac{\partial}{\partial x_1} \left(\mu \frac{\partial u_2}{\partial x_1} \right) + \frac{\partial}{\partial x_1} \left(\mu \frac{\partial u_1}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left(\lambda \frac{\partial u_1}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\lambda^* \frac{\partial u_2}{\partial x_2} \right) &= f_2. \end{aligned} \quad (33)$$

It is now supposed that the quantities λ, μ (and λ^*), which play the part of our former "permeability coefficients", are periodic functions of auxiliary variables

$$\xi_i = (x_i - x_{i0})/H,$$

as defined in Eq. (3), with period 1 in the ξ_i . Various boundary conditions can be used in connection with Eqs. (33). We choose as the simplest

$$u_1 = 0; u_2 = 0 \text{ on a fixed closed curve } C \text{ in the } x_1, x_2\text{-plane.} \quad (34)$$

It is convenient to introduce the following operators in terms of the ξ_i :

$$\begin{aligned} \Omega_1 &\equiv \frac{\partial}{\partial \xi_1} \left(\lambda^* \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\mu \frac{\partial}{\partial \xi_2} \right); \\ \Omega_2 &\equiv \frac{\partial}{\partial \xi_2} \left(\lambda^* \frac{\partial}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_1} \left(\mu \frac{\partial}{\partial \xi_1} \right); \\ \Omega_3 &\equiv \frac{\partial}{\partial \xi_1} \left(\lambda \frac{\partial}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_2} \left(\mu \frac{\partial}{\partial \xi_1} \right); \\ \Omega_4 &\equiv \frac{\partial}{\partial \xi_2} \left(\lambda \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_1} \left(\mu \frac{\partial}{\partial \xi_2} \right). \end{aligned} \quad (35)$$

When these operators are divided by H^2 then for constant x_{10}, x_{20} they are equivalent to operators in x_1, x_2 ; and the basic equations (33) can be written in the form

$$\begin{aligned} (1/H^2)[\Omega_1(u_1) + \Omega_3(u_2)] &= f_1; \\ (1/H^2)[\Omega_4(u_1) + \Omega_2(u_2)] &= f_2. \end{aligned} \quad (36)$$

9. For u_1, u_2 we introduce developments of the type of Eq. (6), as follows:

$$u_1 = U_1 + H \left(\frac{\partial U_1}{\partial x_1} \chi_1 + \frac{\partial U_1}{\partial x_2} \chi_2 + \frac{\partial U_2}{\partial x_1} \chi_3 + \frac{\partial U_2}{\partial x_2} \chi_4 \right); \quad (37)$$

$$u_2 = U_2 + H \left(\frac{\partial U_1}{\partial x_1} \chi_5 + \frac{\partial U_1}{\partial x_2} \chi_6 + \frac{\partial U_2}{\partial x_1} \chi_7 + \frac{\partial U_2}{\partial x_2} \chi_8 \right). \quad (37)$$

The functions $\chi_1 \cdots \chi_8$ are supposed to be periodic functions of the variables ξ_1, ξ_2 , with period 1. As before we seek equations for these functions which will help us to simplify the resulting equations for U_1 and U_2 .

We introduce the expressions (37) into Eqs. (33). There result two equations, the analogues for the former Eq. (7), running as follows:

$$\begin{aligned} & \frac{1}{H} \frac{\partial U_1}{\partial x_1} [\Omega_1(\chi_1) + \Omega_3(\chi_5) + \partial \lambda^* / \partial \xi_1] + \frac{1}{H} \frac{\partial U_1}{\partial x_2} [\Omega_1(\chi_2) + \Omega_3(\chi_6) + \partial \mu / \partial \xi_2] \\ & + \frac{1}{H} \frac{\partial U_2}{\partial x_1} [\Omega_1(\chi_3) + \Omega_3(\chi_7) + \partial \mu / \partial \xi_2] + \frac{1}{H} \frac{\partial U_2}{\partial x_2} [\Omega_1(\chi_4) + \Omega_3(\chi_8) + \partial \lambda / \partial \xi_1] \\ & + \frac{\partial^2 U_1}{\partial x_1^2} \left[\frac{\partial}{\partial \xi_1} (\lambda^* \chi_1) + \frac{\partial}{\partial \xi_2} (\mu \chi_5) + \lambda^* + \lambda^* (\partial \chi_1 / \partial \xi_1) + \lambda (\partial \chi_5 / \partial \xi_2) \right] \\ & + \frac{\partial^2 U_1}{\partial x_1 \partial x_2} \left[\frac{\partial}{\partial \xi_1} (\lambda^* \chi_2) + \frac{\partial}{\partial \xi_1} (\lambda \chi_5) + \frac{\partial}{\partial \xi_2} (\mu \chi_1) + \frac{\partial}{\partial \xi_2} (\mu \chi_6) \right. \\ & \quad \left. + \lambda^* (\partial \chi_2 / \partial \xi_1) + \lambda (\partial \chi_5 / \partial \xi_2) + \mu (\partial \chi_5 / \partial \xi_1) + \mu (\partial \chi_1 / \partial \xi_2) \right] \\ & + \frac{\partial^2 U_1}{\partial x_2^2} \left[\frac{\partial}{\partial \xi_1} (\lambda \chi_6) + \frac{\partial}{\partial \xi_2} (\mu \chi_2) + \mu + \mu (\partial \chi_6 / \partial \xi_1) + \mu (\partial \chi_2 / \partial \xi_2) \right] \\ & + \frac{\partial^2 U_2}{\partial x_1^2} \left[\frac{\partial}{\partial \xi_1} (\lambda^* \chi_3) + \frac{\partial}{\partial \xi_2} (\mu \chi_7) + \lambda^* (\partial \chi_3 / \partial \xi_1) + \lambda (\partial \chi_7 / \partial \xi_2) \right] \\ & + \frac{\partial^2 U_2}{\partial x_1 \partial x_2} \left[\frac{\partial}{\partial \xi_1} (\lambda^* \chi_4) + \frac{\partial}{\partial \xi_1} (\lambda \chi_7) + \frac{\partial}{\partial \xi_2} (\mu \chi_3) + \frac{\partial}{\partial \xi_2} (\mu \chi_8) \right. \\ & \quad \left. + (\lambda + \mu) + \lambda^* (\partial \chi_4 / \partial \xi_1) + \lambda (\partial \chi_8 / \partial \xi_2) + \mu (\partial \chi_3 / \partial \xi_2) + \mu (\partial \chi_7 / \partial \xi_1) \right] \\ & + \frac{\partial^2 U_2}{\partial x_2^2} \left[\frac{\partial}{\partial \xi_1} (\lambda \chi_8) + \frac{\partial}{\partial \xi_2} (\mu \chi_4) + \mu (\partial \chi_8 / \partial \xi_1) + \mu (\partial \chi_4 / \partial \chi_2) \right] \\ & + H \left[\frac{\partial^3 U_1}{\partial x_1^3} \lambda^* \chi_1 + \frac{\partial^3 U_1}{\partial x_1^2 \partial x_2} (\lambda^* \chi_2 + \lambda \chi_5 + \mu \chi_5) \right. \\ & \quad + \frac{\partial^3 U_1}{\partial x_1 \partial x_2^2} (\mu \chi_1 + \lambda \chi_6 + \mu \chi_6) + \frac{\partial^3 U_1}{\partial x_2^3} \mu \chi_2 \\ & \quad + \frac{\partial^3 U_2}{\partial x_1^3} \lambda^* \chi_3 + \frac{\partial^3 U_2}{\partial x_1^2 \partial x_2} (\lambda^* \chi_4 + \lambda \chi_7 + \mu \chi_7) \\ & \quad \left. + \frac{\partial^3 U_2}{\partial x_1 \partial x_2^2} (\mu \chi_3 + \lambda \chi_8 + \mu \chi_8) + \frac{\partial^3 U_2}{\partial x_2^3} \mu \chi_4 \right] = f_1(x_1, x_2). \end{aligned} \quad (38_1)$$

$$\begin{aligned}
& \frac{1}{H} \frac{\partial U_1}{\partial x_1} [\Omega_2(\chi_5) + \Omega_4(\chi_1) + \partial\lambda/\partial\xi_2] + \frac{1}{H} \frac{\partial U_1}{\partial x_2} [\Omega_2(\chi_6) + \Omega_4(\chi_2) + \partial\mu/\partial\xi_1] \\
& + \frac{1}{H} \frac{\partial U_2}{\partial x_1} [\Omega_2(\chi_7) + \Omega_4(\chi_3) + \partial\mu/\partial\xi_1] + \frac{1}{H} \frac{\partial U_2}{\partial x_2} [\Omega_2(\chi_8) + \Omega_4(\chi_4) + \partial\lambda^*/\partial\xi_2] \\
& + \frac{\partial^2 U_1}{\partial x_1^2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_5) + \frac{\partial}{\partial\xi_2} (\lambda\chi_1) + \mu(\partial\chi_5/\partial\xi_1) + \mu(\partial\chi_1/\partial\xi_2) \right] \\
& + \frac{\partial^2 U_1}{\partial x_1 \partial x_2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_1) + \frac{\partial}{\partial\xi_1} (\mu\chi_6) + \frac{\partial}{\partial\xi_2} (\lambda\chi_2) + \frac{\partial}{\partial\xi_2} (\lambda^*\chi_5) \right. \\
& \left. + (\lambda + \mu) + \lambda^*(\partial\chi_5/\partial\xi_2) + \lambda(\partial\chi_1/\partial\xi_1) + \mu(\partial\chi_6/\partial\xi_1) + \mu(\partial\chi_2/\partial\xi_2) \right] \\
& + \frac{\partial^2 U_1}{\partial x_2^2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_2) + \frac{\partial}{\partial\xi_2} (\lambda^*\chi_6) + \lambda(\partial\chi_2/\partial\xi_1) + \lambda^*(\partial\chi_6/\partial\xi_2) \right] \\
& + \frac{\partial^2 U_2}{\partial x_1^2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_7) + \frac{\partial}{\partial\xi_2} (\lambda\chi_3) + \mu + \mu(\partial\chi_3/\partial\xi_2) + \mu(\partial\chi_7/\partial\xi_1) \right] \\
& + \frac{\partial^2 U_2}{\partial x_1 \partial x_2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_8) + \frac{\partial}{\partial\xi_1} (\mu\chi_3) + \frac{\partial}{\partial\xi_2} (\lambda^*\chi_7) + \frac{\partial}{\partial\xi_2} (\lambda\chi_4) \right. \\
& \left. + \lambda^*(\partial\chi_7/\partial\xi_2) + \lambda(\partial\chi_3/\partial\xi_1) + \mu(\partial\chi_8/\partial\xi_1) + \mu(\partial\chi_4/\partial\xi_2) \right] \\
& + \frac{\partial^2 U_2}{\partial x_2^2} \left[\frac{\partial}{\partial\xi_1} (\mu\chi_4) + \frac{\partial}{\partial\xi_2} (\lambda^*\chi_8) + \lambda^* + \lambda^*(\partial\chi_8/\partial\xi_2) + \lambda(\partial\chi_4/\partial\xi_1) \right] \\
& + H \left[\frac{\partial^3 U_1}{\partial x_1^3} \mu\chi_5 + \frac{\partial^3 U_1}{\partial x_1^2 \partial x_2} (\mu\chi_6 + \lambda\chi_1 + \mu\chi_1) \right. \\
& + \frac{\partial^3 U_1}{\partial x_1 \partial x_2^2} (\lambda^*\chi_5 + \lambda\chi_2 + \mu\chi_2) + \frac{\partial^3 U_1}{\partial x_2^3} (\lambda^*\chi_6) \\
& + \frac{\partial^3 U_2}{\partial x_1^3} \mu\chi_7 + \frac{\partial^3 U_2}{\partial x_1^2 \partial x_2} (\mu\chi_8 + \lambda\chi_3 + \mu\chi_3) \\
& \left. + \frac{\partial^3 U_2}{\partial x_1 \partial x_2^2} (\lambda^*\chi_7 + \lambda\chi_4 + \mu\chi_4) + \frac{\partial^3 U_2}{\partial x_2^3} \lambda^*\chi_8 \right] = f_2(x_1, x_2). \tag{38_2}
\end{aligned}$$

10. In the deduction of these equations the quantities x_{10} , x_{20} determining the origin of the periodicity grid for ξ_1 , ξ_2 are kept fixed. In the relations to be deduced from Eqs. (38) x_{10} and x_{20} play the part of parameters. Clearly such relations will become periodic functions of x_{10} and x_{20} , and mean values can be introduced referring to their movement over a cell of the grid. We shall use such mean values in a similar way to that in Secs. 3 and 4.

First we get rid of the terms with the first-order derivatives of U_1 and U_2 by subjecting the functions χ_1 , $\dots$, χ_8 to the conditions

$$\begin{aligned}
\Omega_1(\chi_1) + \Omega_3(\chi_5) &= -\partial\lambda^*/\partial\xi_1; \\
\Omega_1(\chi_2) + \Omega_3(\chi_6) &= -\partial\mu/\partial\xi_2; \\
\Omega_1(\chi_3) + \Omega_3(\chi_7) &= -\partial\mu/\partial\xi_2; \\
\Omega_1(\chi_4) + \Omega_3(\chi_8) &= -\partial\lambda/\partial\xi_1; \\
\Omega_2(\chi_5) + \Omega_4(\chi_1) &= -\partial\lambda/\partial\xi_2; \\
\Omega_2(\chi_6) + \Omega_4(\chi_2) &= -\partial\mu/\partial\xi_1; \\
\Omega_2(\chi_7) + \Omega_4(\chi_3) &= -\partial\mu/\partial\xi_1; \\
\Omega_2(\chi_8) + \Omega_4(\chi_4) &= -\partial\lambda^*/\partial\xi_2,
\end{aligned} \tag{39}$$

combined with the requirement that $\chi_1 \cdots \chi_8$ shall be periodic functions of ξ_1, ξ_2 with a zero integral over a period cell. It follows from Eqs. (39) that

$$\chi_2 = \chi_3, \quad \chi_6 = \chi_7, \tag{40}$$

so that there are three sets of two equations to be solved, each set for two χ -functions, against two independent equations in Sec. 3. We then take the mean of Eqs. (38) over a period grid. As before, this transforms these equations into differential equations with constant coefficients. If we limit ourselves to the terms not multiplied by a power of H we need the following twelve constants:

$$\begin{aligned}
Q_{11} &= \iint d\xi_1 d\xi_2 [\lambda^* + \lambda^*(\partial\chi_1/\partial\xi_1) + \lambda(\partial\chi_5/\partial\xi_2)]; \\
Q_{12} &= \iint d\xi_1 d\xi_2 [\lambda^*(\partial\chi_2/\partial\xi_1) + \lambda(\partial\chi_6/\partial\xi_2) + \mu(\partial\chi_5/\partial\xi_1) + \mu(\partial\chi_1/\partial\xi_2)]; \\
Q_{13} &= \iint d\xi_1 d\xi_2 [\mu + \mu(\partial\chi_6/\partial\xi_1) + \mu(\partial\chi_2/\partial\xi_2)]; \\
Q_{14} &= \iint d\xi_1 d\xi_2 [\lambda^*(\partial\chi_3/\partial\xi_1) + \lambda(\partial\chi_7/\partial\xi_2)]; \\
Q_{15} &= \iint d\xi_1 d\xi_2 [(\lambda + \mu) + \lambda^*(\partial\chi_4/\partial\xi_1) + \lambda(\partial\chi_8/\partial\xi_2) + \mu(\partial\chi_3/\partial\xi_2) + \mu(\partial\chi_7/\partial\xi_1)]; \\
Q_{16} &= \iint d\xi_1 d\xi_2 [\mu(\partial\chi_8/\partial\xi_1) + \mu(\partial\chi_4/\partial\xi_2)]; \\
Q_{21} &= \iint d\xi_1 d\xi_2 [\mu(\partial\chi_5/\partial\xi_1) + \mu(\partial\chi_1/\partial\xi_2)]; \\
Q_{22} &= \iint d\xi_1 d\xi_2 [(\lambda + \mu) + \lambda^*(\partial\chi_5/\partial\xi_2) + \lambda(\partial\chi_1/\partial\xi_1) + \mu(\partial\chi_6/\partial\xi_1) + \mu(\partial\chi_2/\partial\xi_2)]; \\
Q_{23} &= \iint d\xi_1 d\xi_2 [\lambda^*(\partial\chi_6/\partial\xi_2) + \lambda(\partial\chi_2/\partial\xi_1)]; \\
Q_{24} &= \iint d\xi_1 d\xi_2 [\mu + \mu(\partial\chi_3/\partial\xi_2) + \mu(\partial\chi_7/\partial\xi_1)];
\end{aligned} \tag{41}$$

$$Q_{25} = \iint d\xi_1 d\xi_2 [\lambda^*(\partial\chi_7/\partial\xi_2) + \lambda(\partial\chi_3/\partial\xi_1) + \mu(\partial\chi_8/\partial\xi_1) + \mu(\partial\chi_4/\partial\xi_2)];$$

$$Q_{26} = \iint d\xi_1 d\xi_2 [\lambda^* + \lambda^*(\partial\chi_8/\partial\xi_2) + \lambda(\partial\chi_4/\partial\xi_1)],$$

all integrations to be taken over a period cell. These quantities form the coefficients of the equations for U_1 , U_2 (corresponding to Eq. (18) of Sec. 4):

$$Q_{11} \frac{\partial^2 U_1}{\partial x_1^2} + Q_{12} \frac{\partial^2 U_1}{\partial x_1 \partial x_2} + Q_{13} \frac{\partial^2 U_1}{\partial x_2^2} + Q_{14} \frac{\partial^2 U_2}{\partial x_1^2} + Q_{15} \frac{\partial^2 U_2}{\partial x_1 \partial x_2} + Q_{16} \frac{\partial^2 U_2}{\partial x_2^2} = f_1(x_1, x_2); \quad (42a)$$

$$Q_{21} \frac{\partial^2 U_1}{\partial x_1^2} + Q_{22} \frac{\partial^2 U_1}{\partial x_1 \partial x_2} + Q_{23} \frac{\partial^2 U_1}{\partial x_2^2} + Q_{24} \frac{\partial^2 U_2}{\partial x_1^2} + Q_{25} \frac{\partial^2 U_2}{\partial x_1 \partial x_2} + Q_{26} \frac{\partial^2 U_2}{\partial x_2^2} = f_2(x_1, x_2). \quad (42b)$$

They are independent of x_{10} , x_{20} and must be solved with the boundary condition that U_1 and U_2 shall vanish on the curve C in the x_1 , x_2 -plane. We then have

$$\left. \begin{aligned} u_1 &= U_1 \\ u_2 &= U_2 \end{aligned} \right\} \text{both with an uncertainty of order } H. \quad (43)$$

It is evident that the extension from one to two dependent variables leads to a considerable amplification of the required calculations. A further amplification will occur if we go on to problems with three dependent variables u_1 , u_2 , u_3 .

The main results obtained in this paper are the formulas for the calculation of the coefficients

$$Q_1, Q_2, Q_3, Q_4 \text{ (Eqs. (13) and (14)); } Q_{11}, Q_{12}, \dots, Q_{26} \text{ (Eqs. (41)).}$$

We have not paid attention to boundary conditions. As mentioned before, we take $u_1 = 0$, $u_2 = 0$ on a boundary curve C . Then we also must make $U_1 = 0$, $U_2 = 0$ on C , as from Eq. (37) we have $U_1 = \langle u_1 \rangle$, $U_2 = \langle u_2 \rangle$.

Acknowledgments. My thanks are due to the University of Maryland for providing the opportunity to work on this and other problems at the Institute for Physical Science and Technology, and to Professor I. Babuska for many stimulating discussions. Mrs. Irene Huvos devoted great patience and care to the typing of the long equations in this paper and in several preparatory versions.

REFERENCES

It is not possible to give here a full list of papers on the problem of homogenization. In most of the published work a periodicity grid with fixed origin ($x_{10} = x_{20} = 0$) is introduced and the method of treatment is quite different from the one used in the present paper. By way of example we mention:

- [1] I. Babuška, *Homogenization and its applications; mathematical and computational problems*, in *Numerical solutions of partial differential equations III*, B. Hubbard, ed., Academic Press, 1976
- [2] I. Babuška, *Homogenization approach in engineering*, in *Proc. symposium on computing methods in engineering*, Dec. 15-19, 1945, Versailles, France

- [3] G. Duvaut, *Analyse fonctionnelle et mécanique des milieux continus, application à l'étude des matériaux composites élastiques à structure périodique-homogénéisation*, in: *Theoretical and Applied Mechanics*, Preprints IUTAM Congress, W. T. Koiter, ed., Aug. 30–Sept. 4, 1976, Delft, Netherlands, pp. 110–132
- [4] J. L. Lions, A. Bensoussan, G. Papanicolau, book in preparation
- [5] I. Babuška, *Solution of interface problems by homogenization I, II*, SIAM Journ. of Math. Anal., **7**, pp. 603–634, 635–645 (1976). From an extensive list of references related to the homogenization problem we refer to [1], [2], [3], [4]. In addition see also [4] and [5] as examples of studies dealing with a very similar topic.