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SOME MAXIMUM PRINCIPLES FOR NONLINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS*

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Abstract. The Hopf maximum principles are utilized to obtain maximum principles for functions which are defined on solutions of nonlinear, second-order elliptic equations subject to Dirichlet, Robin, or mixed boundary conditions. The principles derived may be used to deduce bounds on important quantities in physical problems of interest.

1. Introduction. Several authors have recently developed maximum principles for certain functions defined on solutions of linear and nonlinear elliptic boundary-value problems in order to obtain differential inequalities which lead to bounds on quantities important in various physical problems. One should consult [2–6, 9, 10] in the references for this development. Until [6], these results were accomplished for boundary-value problems (primarily Dirichlet problems) for equations of the form

$$\Delta u + \lambda \rho(x) f(u) = 0,$$

where Δ is the Laplace operator, or special cases thereof (usually $\lambda = 1$, $\rho(x) \equiv 1$). In [6] the authors considered the problem of torsional creep, i.e.

$$(g(q^2)u_{,i})_{,i} + 2 = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

where $q = |\text{grad } u|$, and developed isoperimetric bounds for the maximum stress and the torsional stiffness.

In this paper we develop maximum principles for a variety of boundary-value problems for the nonlinear equation

$$(g(u)u_{,i})_{,i} + f(u) = 0. \tag{1.1}$$

These results are presented in Theorems 1, 2 and 3 of Sec. 2. One can, in fact, follow the procedure in [10] and obtain principles for the inhomogeneous equation

$$(g(u)u_{,i})_{,i} + \rho(x)f(u) = 0. \tag{1.2}$$

Because of the computational complexity, we indicate only the fundamental changes and then state the results as Theorems 4 and 5. Remarks concerning extensions and improvements are given in Sec. 3. We refer the reader to [1] and references cited therein for problems which give rise to equations of the form

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(1.2). One is referred to [5] for various techniques one might apply to the resulting differential inequalities to obtain bounds on the solution or the gradient of the solution and to [8] and the references cited there for other principles and bounding techniques.

We shall be primarily interested in the following problems. Let D be a convex domain in the plane bounded by a sufficiently smooth curve ∂D. We assume that u is a solution of

$$\begin{gathered}(g(u)u_{,i})_{,i} + f(u) = 0 \quad \text{in } D \\ u = 0 \quad \text{on } \Gamma_1, \quad \Gamma_1 \neq \emptyset \\ \partial u/\partial n = 0 \quad \text{on } \Gamma_2, \quad \Gamma_1 \cup \Gamma_2 = \partial D\end{gathered} \tag{MP}$$

or

$$\begin{gathered}(g(u)u_{,i})_{,i} + f(u) = 0 \quad \text{in } D \\ \partial u/\partial n + \alpha u = 0 \quad \text{on } \partial D, \qquad \alpha > 0\end{gathered} \tag{RP}$$

where the comma notation denotes partial differentiation, the repeated index denotes summation, and $\partial u/\partial n$ denotes the outward normal derivative. In both problems, we assume that f is a C^1 function and g is a positive C^2 function of u. We note that in (MP), Γ_2 may be empty. If $\Gamma_2 = \emptyset$ in (MP), then we refer to the resulting problem as (DP), the Dirichlet problem. For (DP), the result in Theorem 2 is valid in higher dimensions as well.

2. Maximum principles. Let u be a sufficiently smooth solution of (1.1). We define

$$\Phi = (g(u)u_{,i})(g(u)u_{,i}) + 2\int_0^u f(\eta)g(\eta)\,d\eta \tag{2.1}$$

and compute

$$\Phi_{,k} = 2\,(gu_{,i})(gu_{,i})_{,k} + 2f(gu_{,k}) \tag{2.2}$$

$$\Phi_{,kk} = 2(gu_{,i})_{,k}(gu_{,i})_{,k} + 2(gu_{,i})(gu_{,i})_{,kk} + 2\,f'gu_{,k}u_{,k} + 2f(gu_{,k})_{,k} \tag{2.3}$$

where the prime denotes differentiation with respect to the argument and we have suppressed the argument. From (2.2) and Schwarz's inequality, it follows that

$$2(gu_{,i})_{,k}(gu_{,i})_{,k} \geq \frac{1}{2(gu_{,j})(gu_{,j})}\,[\Phi_{,k}\Phi_{,k} - 4fgu_{,k}\Phi_{,k} + 4f^2g^2u_{,k}u_{,k}]. \tag{2.4}$$

Consequently, by (1.1) and (2.4) we can write

$$\Phi_{,kk} + \frac{L_k\Phi_{,k}}{|\nabla u|^2} \geq 2(gu_{,i})(gu_{,i})_{,kk} + 2f'g\,|\nabla u|^2, \tag{2.5}$$

where

$$L_k = \frac{1}{2g^2}\,[4fgu_{,k} - \Phi_{,k}].$$

Now consider the identity

$$(gu_{,i})(gu_{,i})_{,kk} = g^2u_{,kki}u_{,i} + 2gg'u_{,i}u_{,k}u_{,ik} + gg''u_{,i}u_{,i}u_{,k}u_{,k} + gg'u_{,i}u_{,i}u_{,kk}\,. \tag{2.6}$$

From (1.1) it follows that

$$du_{,kk} = -f - g'gu_{,k}u_{,k} \tag{2.7}$$

$$g^2u_{,kki} = (fg' - f'g)u_{,i} + (g'^2 - gg'')\, u_{,i}u_{,k}u_{,k} - 2gg'u_{,k}u_{,ki} \tag{2.8}$$

and hence, on substituting into (2.6), that

$$2(gu_{,i})(gu_{,i})_{,kk} = -2f'g\,|\nabla u|^2.$$

Thus we have

$$\Phi_{,kk} + \frac{L_k\Phi_{,k}}{|\nabla u|^2} \geq 0, \tag{2.9}$$

and by the maximum principle for elliptic operators [7], we arrive at our first result.

THEOREM 1. If u is a C^3 solution of (1.1) in D, where f is a C^1 and g is a positive C^2 function of u, then

$$\Phi = [g(u)]^2\,|\nabla u|^2 + 2\int_0^u f(\eta)g(\eta)\,d\eta$$

takes its maximum either on ∂D or at a critical point of u.

We note that Theorem 1 is valid for $n > 2$ as there was no dependence on dimension. In fact, when $n = 2$, we may use (1.1) and (2.2) to obtain an identity for the terms $2\,(gu_{,i})_{,k}\,(gu_{,i})_{,k}$ so that Φ satisfies an elliptic equation rather than inequality (2.9). In this regard, see [9].

Let us now consider (MP). We shall show that Φ cannot attain its maximum on ∂D unless it is attained at a critical point of u which is on Γ_2.

Suppose that Φ takes its maximum at $P \in \Gamma_1$. Then P cannot be a critical point of u. Since $u = 0$ on Γ_1, we have $|\nabla u| = |\partial u/\partial n|$ and

$$\partial\Phi/\partial n = 2g^2u_nu_{nn} + 2gg'u_n{}^3 + 2fgu_n\,, \tag{2.10}$$

where u_n denotes the outward normal derivative. Introducing normal coordinates in the neighborhood of the boundary, we can write

$$\Delta u \equiv u_{nn} + ku_n = -\frac{f}{g} - \frac{g'}{g}|\nabla u|^2 \tag{2.11}$$

where k denotes the curvature of the boundary. Thus it follows that

$$\partial\Phi/\partial n = -2kg^2u_n{}^2,$$

and since D is convex, that $\partial\Phi/\partial n \leq 0$ at P. This, however, contradicts Hopf's second maximum principle [7].

We now suppose that Φ takes its maximum at $P \in \Gamma_2$ and that P is not a critical point of u. Since $\partial u/\partial n = 0$ on Γ_2, we have $|\nabla u| = |\partial u/\partial s|$ and

$$\partial\Phi/\partial n = 2g^2u_su_{sn}\,, \tag{2.12}$$

where u_s denotes the tangential derivative of u. In terms of normal coordinates in the neighborhood of the boundary, we have

$$u_{sn} = u_{ns} - ku_s\,, \tag{2.13}$$

so that on Γ_2

$$\partial\Phi/\partial n = -2kg^2u_s{}^2.$$

Thus, we again arrive at a contradiction to the second maximum principle when D is convex. We state our conclusion as

THEOREM 2. If u is a C^3 solution of either (DP) or (MP), then Φ given by (2.1) takes its maximum at a critical point of u.

We observe that the conclusion of Theorem 2 is also valid in higher dimensions when u is subject to a Dirichlet condition on ∂D. For $n > 2$, we have, instead of (2.11),

$$\triangle u \equiv u_{nn} + (n-1)Ku_n = -\frac{f}{g} - \frac{g'}{g}|\nabla u|^2,$$

where K denotes the average curvature of ∂D. Substitution into (2.10) now results in

$$\partial\Phi/\partial n = -2(n-1)Kg^2u_n{}^2.$$

Thus if D is a convex region in $n > 2$ dimensions, we again conclude that the maximum of Φ occurs at a critical point of u.

We now consider (RP). We shall find that under certain additional assumptions on f and g, Φ cannot attain its maximum on ∂D.

Here we assume that u satisfies the boundary condition

$$\partial u/\partial n + \alpha u = 0, \quad \alpha > 0. \tag{2.14}$$

Hence we write

$$\Phi = g^2(u_n{}^2 + u_s{}^2) + 2\int_0^u f(\eta)g(\eta)\,d\eta$$

and compute

$$\partial\Phi/\partial n = 2g^2(u_nu_{nn} + u_su_{sn}) + 2gg'|\nabla u|^2 u_n + 2fgu_n.$$

Introducing normal coordinates, we can write

$$\triangle u \equiv u_{nn} + ku_n + u_{ss} = -\frac{f}{g} - \frac{g'}{g}|\nabla u|^2,$$

which, together with (2.13) and (2.14), results in

$$\partial\Phi/\partial n = -2g^2\{k\alpha^2u^2 - \alpha uu_{ss} + (\alpha + k)u_s{}^2\}. \tag{2.15}$$

Now suppose that Φ takes its maximum at P on ∂D. Then at P

$$\partial\Phi/\partial s = 0,$$

where

$$\begin{aligned}\partial\Phi/\partial s &= 2g^2(u_nu_{ns} + u_su_{ss}) + 2gg'|\nabla u|^2 u_s + 2fgu_s \\ &= 2gu_s\{g\alpha^2u + gu_{ss} + g'|\nabla u|^2 + f\}.\end{aligned} \tag{2.16}$$

Since $g > 0$, either $u_s = 0$ or the expression in the braces vanishes at P.

Case 1: Suppose $u_s \neq 0$ at P. In this case

$$gu_{ss} = -\{g\alpha^2u + g'|\nabla u|^2 + f\}. \tag{2.17}$$

Now if we ask that f, $g > 0$ and $g' > 0$, then it follows from (2.7) and (2.14) that $u \geq 0$ in $D \cup \partial D$. Hence from (2.17) we have $u_{ss} \leq 0$ and from (2.15) that $\partial\Phi/\partial n \leq 0$ at P.

Case 2: Suppose $u_s = 0$ at P. Under the assumption that Φ takes its maximum at P on ∂D, we know that $\Phi_{ss} \leq 0$, where

$$\Phi_{ss} \equiv \partial^2\Phi/\partial s^2 = 2gu_{ss}\{g\alpha^2 u + gu_{ss} + g'\,|\nabla u|^2 + f\}.$$

Hence, either

(i) $u_{ss} \geq 0$ and $g\alpha^2 u + gu_{ss} + g'\,|\nabla u|^2 + f \leq 0$

or

(ii) $u_{ss} \leq 0$ and $g\alpha^2 u + gu_{ss} + g'\,|\nabla u|^2 + f \geq 0.$

As in case 1, if f, $g > 0$ and $g' > 0$, then $u \geq 0$ in $D \cup \partial D$. Under these conditions (i) is impossible since if $u_{ss} \geq 0$, then

$$g\alpha^2 u + gu_{ss} + g'\,|\nabla u|^2 + f \geq 0.$$

Thus we conclude that (ii) holds, i.e., $u_{ss} \leq 0$. From (2.15) we again deduce that $\partial\Phi/\partial n \leq 0$ at P. Therefore, by the second maximum principle [7], we conclude that Φ cannot take its maximum at P on ∂D and state

THEOREM 3. If u is a C^3 solution of (RP), where f is a positive C^1 and g is a positive C^2 function for which $g' > 0$, then Φ given by (2.1) takes its maximum at a critical point of u.

Let us now consider the inhomogeneous equation (1.2) where $\rho(x) > 0$ in D. Following [10] and the derivation in Theorem 1, it is not difficult to show that

$$\psi = \frac{1}{\rho}(gu_{,i})(gu_{,i}) + 2\int_0^u f(\eta)g(\eta)\,d\eta \tag{2.18}$$

satisfies

$$\psi_{,kk} + \frac{H_k\psi_{,k}}{(gu_{,j})(gu_{,j})} \geq \left\{\frac{|\nabla\rho|^2}{2\rho^3} - \frac{\Delta\rho}{\rho^2}\right\}(gu_{,j})(gu_{,j}),$$

where

$$H_k = \rho\{2fgu_{,k} - \frac{1}{2}\psi_{,k}\} + \frac{\rho_{,k}}{\rho}(gu_{,j})(gu_{,j}).$$

Thus we have the following extension of Theorem 1.

THEOREM 4. If u is a C^3 solution of (1.2) in D, where f is a C^1 and g is a positive C^2 function of u and ρ is a positive C^2 function for which $\Delta\rho \leq 0$ in D, then ψ given by (2.18) takes its maximum either on ∂D or at a critical point of u.

We note that Theorem 4 is also valid for $n > 2$. Furthermore, one can develop the analogue of Theorem 2 for solutions of (1.2) provided additional constraints are imposed on g, ρ, and/or the curvature of ∂D.

Consider the inhomogeneous mixed problem

$$\begin{aligned} (g(u)u_{,i})_{,i} + \rho(x)f(u) &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \Gamma_1, \qquad \Gamma_1 = \emptyset \\ \partial u/\partial n &= 0 \quad \text{on } \Gamma_2, \qquad \Gamma_1 \cup \Gamma_2 = \partial D \end{aligned} \tag{IMP}$$

where g, f, and ρ are as stated in Theorem 4. Then ψ takes its maximum on ∂D or at a critical point of u. But, following the derivation in Theorem 2, we find that on ∂D

$$\partial\psi/\partial n = -\frac{g^2|\nabla u|^2}{\rho}\left\{2k + \frac{\partial}{\partial n}(\ln\rho)\right\}$$

whether we assume that ψ takes its maximum at P on Γ_1 or Γ_2. Thus, if $\partial\psi/\partial n \leq 0$ at P on ∂D, we can deduce that ψ takes its maximum at a critical point. This requirement will be satisfied if the geodesic curvature k_g of ∂D (see [10]), where

$$k_g = \rho^{-1/2}\left\{2k + \frac{\partial}{\partial n}(\ln \rho)\right\},$$

is nonnegative or if g vanishes for points on the boundary, as when g is a power function and u is subject to Dirichlet conditions only. We state this extension formally as

THEOREM 5. *If u is a C^3 solution of (IMP) and if $[g(u)]^2 k_g \geq 0$ on ∂D, where k_g is the geodesic curvature of ∂D, then ψ given by (2.18) takes its maximum at a critical point of u.*

An extension of Theorem 3 to the inhomogeneous equation (1.2)—even in the case $g(u) \equiv 1$—has not been accomplished.

3. Concluding remarks. Obviously, the principles and applications in [2–5, 9, 10] are "covered" when $g(u) \equiv 1$ here. Moreover, one could follow [9] and seek other functions of the solution u that satisfy maximum principles of the type presented here. This appears to involve an undesirable amount of complexity.

We also note that a wide range of mixed boundary value problems could be considered. For example: if u is a C^3 solution of (1.1) in D and satisfies a Dirichlet condition on a portion of the boundary and a Robin condition on the remainder of the boundary, then Φ given by (2.1) attains its maximum at a critical point of u provided that f and g satisfy the conditions cited in Theorem 3.

Let us now consider a simple illustration in which we determine a bound for the gradient of the solution of a nonlinear Robin problem at any point in the domain in terms of the maximum value of the solution function. Consider the problem

$$\Delta u + u_{,i}u_{,i} + 1 = 0 \quad \text{in } D$$
$$\partial u/\partial n + \alpha u = 0 \quad \text{on } \partial D$$

which can be rewritten as

$$(e^u u_{,i})_{,i} + e^u = 0 \quad \text{in } D$$
$$\partial u/\partial n + \alpha u = 0 \quad \text{on } \partial D.$$

With $f(u) = e^u = g(u)$, it follows from Theorem 3 that

$$e^{2u}\,|\nabla u|^2 + 2\int_0^u e^{2t}\,dt \leq 2\int_0^u e^{2t}\,dt\Big|_{\max}$$

or

$$|\nabla u|^2 \leq e^{2u_m} - 1,$$

where u_m is the maximum value of u in $D \cup \partial D$.

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