SIMILARITY SOLUTIONS OF THE NONLINEAR DIFFUSION EQUATION*

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Abstract. The paper considers similarity solutions of the nonlinear diffusion equation of the form $r^\alpha f(\eta)$ where $\eta = rt^{-\delta}$ or $\exp(\alpha t)f(\eta)$ where $\eta = r \exp(-\delta t)$. The novel feature of the paper is that the second-order differential equation for f is reduced to a system of first-order equations and a phase plane analysis of one member of the system can be made. In this way we may discuss the existence and uniqueness of all the solutions for $f(\eta)$. Restricting the discussion to plane geometry, we list all the continuous solutions to the basic problem on $0 \leq \eta \leq \infty$ with $f(0) = U \geq 0$ and $f(\infty) = 0$. Solutions of previous authors are identified as special cases.

1. Introduction. This paper is concerned with similarity solutions of the nonlinear diffusion equation

$$\partial u / \partial t = r^{1-\lambda} \frac{\partial}{\partial r} \left\{ r^{\lambda-1} u^\beta \frac{\partial u}{\partial r} \right\}, \quad u \geq 0. \quad (1.1)$$

Here r denotes the space variable, t time, β a constant index and λ is a geometry index equal to 1, 2 or 3 for plane, cylindrical or spherical geometry. Similarity solutions of the form

$$u = r^\alpha f(\eta), \quad \eta = rt^{-\delta} \quad (1.2)$$

or

$$u = \exp(\alpha t)f(\eta), \quad \eta = r \exp(-\delta t), \quad (1.3)$$

with $\delta > 0$, are sought. Substituting (1.2) or (1.3) into (1.1), we find that $f(\eta)$ satisfies the equation

$$\{\alpha f - \delta \eta f'\} \eta^{\lambda-1} = \{\eta^{\lambda-1} f^\beta f'\}', \quad f \geq 0, \quad (1.4)$$

where primes denote differentiation with respect to η . In (1.4)

$$\alpha\beta = 2\delta - 1 \quad (1.5a)$$

for solutions (1.2) while

$$\alpha\beta = 2\delta \quad (1.5b)$$

for the solutions of type (1.3).

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The similarity solutions (1.4) have been previously studied by many authors using a variety of methods for various specific values of α , δ , β and λ [1–10]. However, no attempt has ever been made to list systematically all the solutions of (1.4) for all parameter values. This paper uses a method which enables one to investigate the existence, uniqueness and character of all the solutions of (1.4) for every parameter range. The method consists of reducing (1.4) to a system of first-order equations using a technique introduced by Jones [8]. One of the members of the first-order system is of the form

$$dY/dX = F(X, Y)/G(X, Y). \quad (1.6)$$

This crucial reduction enables a study of the phase plane of (1.6) to be made and gives considerable insight into the nature of all the solutions of (1.4).

2. Reduction to the phase plane. If, in Eq. (1.4), we put

$$X = \eta f'/f, \quad Y = \eta^2/2f^\beta > 0, \quad (2.1a, b)$$

then the equation becomes

$$\frac{dY}{dX} = \frac{Y(2 - \beta X)}{\{(\alpha - \delta X)2Y - (\beta + 1)X^2 - \nu X\}}, \quad (2.2)$$

where $\nu = \lambda - 2$. Further, we may eliminate f from (2.1) to obtain

$$\frac{1}{\eta} \frac{d\eta}{dY} = \frac{1}{(2 - \beta X)Y}. \quad (2.3)$$

The basic first-order system we use consists of the phase plane equation (2.2), Eq. (2.3) which determines the behavior of η along the integral curves of (2.2) and finally (2.1b) which gives the function $f(\eta)$.

The singular points of (2.2) in the phase plane (X, Y) are

1. $X = 0, Y = 0$
2. $X = 2/\beta, Y = -\{2(\beta + 1) + \nu\beta\}/\beta$
3. $X = -\nu/(\beta + 1), Y = 0$
4. $X = \alpha/\delta, Y = \infty$
5. $X = \pm\infty, Y = 0$
6. $X = \pm\infty, Y = \infty$.

For the case $\alpha\beta = 2\delta$, i.e. solutions of the form (1.3), the singular points 2 and 4 coalesce to become 9 where $X = 2/\beta, Y = \infty$.

It is important to note that the integral curves of (2.2) have zero slope along $X = 2/\beta$ and infinite slope along $Y = X\{(\beta + 1)X + \nu\}/2(\alpha - \delta X)$.

3. Phase plane analysis for $\lambda = 1$. In this section we make a detailed study of the phase plane of (2.2) for the case of plane geometry $\lambda = 1$ ($\nu = -1$). Using standard linearization techniques, we first determine the nature of the integral curves at the singular points. Then, from (2.3) and (2.1b), the value of η and behavior of f associated with each point can be found. The integral curves representing solutions of (1.4) may then be drawn in the phase plane of (2.2), from which we may identify all the solutions of (1.4) and the number of parameters needed to specify them. The details of the analysis are straightforward and are omitted; only the essential results are given. We first consider the case 1.5 (a), $\alpha\beta = 2\delta - 1$, corresponding to solutions of the form (1.2).

3.1. The singular points and the solutions associated with them for $\alpha\beta = 2\delta - 1$.

1. This is a node and a one-parameter family of curves of the form

$$Y \sim A_1 \{X - 2\alpha Y\}^2$$

pass through it. From (2.3), for each A_1 ,

$$\eta \sim B_1 Y^{1/2},$$

hence $\eta = 0$ at this point. Using (2.1b) we then have

$$f = (B_1^2/2)^{1/\beta} > 0 \text{ at } \eta = 0.$$

Thus we deduce that a two-parameter system of solutions, $f(\eta; A_1, B_1)$, is associated with 1 where $\eta = 0$ and $f = (B_1^2/2)^{1/\beta} > 0$.

2. Since $Y \geq 0$, this is only relevant when $(\beta + 2)/\beta \leq 0$, i.e. $-2 \leq \beta \leq 0$, when it is a saddle point. The two integral curves passing through this point have the behavior

$$Y + (\beta + 2)/\beta \sim (B - A)(X - 2/\beta)/4, \quad (3.1)$$

$$Y + (\beta + 2)/\beta \sim (B + A)(X - 2/\beta)/4, \quad (3.2)$$

where $B = \alpha\beta(\beta + 2) - 2(\beta + 1)$, $A^2 = B^2 - 8\beta(\beta + 2)$. Since $A^2 - B^2 > 0$, $B - A$ and $B + A$ are of opposite sign. Using (2.3), we can show that along the curves (3.1) and (3.2) with μ_1 and μ_2 positive constants, $\eta \sim B_2 |X - 2/\beta|^{\mu_1} \rightarrow 0$ along the curve of positive slope while $\eta \sim B_2 |X - 2/\beta|^{-\mu_2} \rightarrow \infty$ along the curve of negative slope. In the first case, from (2.1b), since β is negative,

$$f \sim \frac{\beta \eta^{2/\beta}}{2(\beta + 2)} \rightarrow \infty \text{ as } \eta \rightarrow 0$$

and secondly

$$f \sim \frac{\beta \eta^{2/\beta}}{2(\beta + 2)} \rightarrow 0 \text{ as } \eta \rightarrow \infty.$$

We have shown that associated with 2 for $-2 < \beta < 0$ there are two families of one-parameter solutions $f(\eta; B_2)$. For one family $f \rightarrow \infty$ as $\eta \rightarrow 0$ while for the other $f \rightarrow 0$ as $\eta \rightarrow \infty$.

3. The integral curves have the behavior

$$Y \sim A_3 \{(2\beta + 3)(X - 1/(\beta + 1)) - 2(\delta - 1 + \alpha)Y\}^{-(\beta+2)/(\beta+1)} \quad (3.3)$$

and is a saddle point for $\beta > -1$ or $\beta < -2$, and a node for $-2 < \beta < -1$. In the second case each of the family (3.3) passes through 3, where from (2.3)

$$\eta \sim B_3 Y^{(\beta+1)(\beta+2)} \rightarrow \infty$$

and from (2.1b)

$$f \sim \frac{B_3^{(\beta+2)/(\beta+1)}}{2^{1/\beta}} \eta^{1/(\beta+1)} \rightarrow 0.$$

In the first case, the single curve of the form

$$Y \sim \frac{(2\beta + 3)}{2(\delta - 1 + \alpha)} (X - 1/(\beta + 1))$$

FIG. 1.

reaches 3. Again from (2.3)

$$\eta \sim B_3 Y^{(\beta+1)/(\beta+2)} \rightarrow 0$$

and from (2.1b)

$$f \sim \frac{B_3^{(\beta+2)/\beta(\beta+1)}}{2^{1/\beta}} \eta^{1/(\beta+1)}.$$

So $f \rightarrow 0$ if $\beta > -1$, and $f \rightarrow \infty$ if $\beta < -2$.

We have shown that for $\beta > -1$ or $\beta < -2$ the one-parameter family of solutions $f(\eta; B_3)$ are associated with 3 where $\eta = 0$ and $f = 0$ if $\beta > -1$, while $f \rightarrow \infty$ if $\beta < -2$. For $-2 < \beta < -1$ we have the two-parameter system $f(\eta; A_3, B_3)$ where $\eta = \infty$ and $f = 0$.

4. This is a second-order singularity. The integral curves reaching 4 consist of the one-parameter family

$$\begin{aligned} \frac{1}{Y} = & \frac{2\delta^3}{\alpha\{\delta - \alpha(\beta + 1)\}} \left[(X - \alpha/\delta) + \frac{\delta(X - \alpha/\delta)^2}{\alpha\{(\beta + 1)\alpha - \delta\}} + \dots \right. \\ & \left. + A_4 \exp(\alpha\{(\beta + 1)\alpha - \delta\}/\delta(X - \alpha/\delta)) \{1 + \dots\} \right] \end{aligned}$$

where $(X - \alpha/\delta) \geq 0$ for $\{(\beta + 1)\alpha - \delta\} \alpha \geq 0$. For each of these curves, from (2.3), we have

$$\eta \sim B_4 Y^6 \rightarrow \infty.$$

Using (2.1b), we obtain

$$f \sim \frac{B_4^{1/\delta\beta} \eta^{\alpha/\delta}}{2^{1/\beta}}$$

so if $\alpha > 0$, $f \rightarrow \infty$ while if $\alpha < 0$, $f = 0$.

So we have shown that a two-parameter family of solutions $f(\eta; A_4, B_4)$ is associated with 4 where $\eta \rightarrow \infty$. Also $f \rightarrow \infty$ if $\alpha > 0$, while $f = 0$ if $\alpha < 0$.

5. This singular point is a saddle if $\beta(\beta + 1) > 0$ and a node if $\beta(\beta + 1) < 0$, where the integral curves have the form

$$X \sim A_5 Y^{(\beta+1)/\beta}, \quad X \geq 0, \quad A_5 \geq 0.$$

For $\beta(\beta + 1) < 0$, i.e. $-1 < \beta < 0$, from (2.3) we have, for each A_5 ,

$$\eta = \eta_5 \exp(-1/(\beta + 1)X), \quad X \rightarrow \pm\infty, \quad 0 < \eta_5 < \infty,$$

FIG. 2.

Finally for $\beta < 0$ there is the one-parameter family, class III,

$$Y \sim -\frac{X}{2\delta} + (-X)^{(\beta+1)/\beta} A_{63} + \cdots, \quad X \rightarrow -\infty, \quad Y \rightarrow \infty$$

which asymptote to the class I curve.

Now along curves of class I we have from (2.3)

$$\eta \sim \eta_6 \exp(-\frac{1}{2}\delta Y\beta), \quad Y \rightarrow \infty$$

where $\eta = \eta_6$ at 6, $0 < \eta_6 < \infty$. Further, from this result we see that

$$Y \sim \frac{\eta_6}{2\delta|\beta(\eta_6 - \eta)|}$$

and since $Y \rightarrow +\infty$, $\eta_6 \geq \eta$ for $\beta \geq 0$ as 6 is approached along a class I curve. From (2.1b) we find that

$$f \sim (\eta_6 \delta \beta)^{1/\beta} |\eta_6 - \eta|^{1/\beta},$$

hence $f \rightarrow 0$ for $\beta > 0$ and $f \rightarrow \infty$ for $\beta < 0$. We finally notice that $(f^{\beta+1})' \rightarrow 0$ as $\eta \rightarrow \eta_6$ for $\beta > 0$.

FIG. 4.

FIG. 5.

For curves of class II a similar analysis yields that along each curve

$$\eta_6 - \eta \sim \frac{Y^{-(\beta+1)/\beta}}{(\beta+1)A_{62}}$$

so when $(\beta+1)A_{62} \leq 0$, $\eta \leq \eta_6$. From (2.1b) we find that

$$f \sim |\eta_6 - \eta|^{1/\beta+1}$$

and so $f \rightarrow 0$ for $\beta > 0$ with $f'(\eta_6) = \infty$, while for $\beta < -1$ $f \rightarrow \infty$.

For curves of class III we can show that

$$f \sim \{\delta\beta\eta_6(\eta - \eta_6)\}^{1/\beta}$$

with $\eta > \eta_6$, since $\beta < 0$, and hence $f(\eta_6) = \infty$.

We have shown that the integral curves through 6, where $\eta = \eta_6$, are associated with the following solutions for f . For $\beta > 0$: a one-parameter family of solutions, class I, $f(\eta; \eta_6)$ with $\eta < \eta_6$ for which $(f^{\beta+1})' = 0$ at $\eta = \eta_6$; also a two-parameter family, class II, $f(\eta; A_{62}, \eta_6)$ where $f(\eta_6) = 0$, $f'(\eta_6) = \infty$ and $\eta \geq \eta_6$. For $-1 < \beta < 0$: a two-parameter family, class III, $f(\eta; A_{63}, \eta_6)$ with $\eta > \eta_6$ and $f(\eta_6) = \infty$; also the one-parameter family, class I, $f(\eta; \eta_6)$ with $\eta > \eta_6$ and $f(\eta_6) = \infty$. For $\beta < -1$: again the one-parameter family, class I, $f(\eta; \eta_6)$ with $\eta > \eta_6$ and $f(\eta_6) = \infty$; also the two-parameter family, class II, $f(\eta; A_{62}, \eta_6)$ with $\eta \geq \eta_6$ and $f(\eta_6)$

and $f'(\eta_6) = \infty$; and finally the two-parameter family, class III, $f(\eta; A_6, \eta_6)$ with $\eta > \eta_6$ and $f(\eta_6) = \infty$.

We now discuss the special cases $\beta = -1$ and $\beta = -2$. The case $\beta = 0$ corresponds to the linear problem and so we omit this.

(a) $\beta = -1$. The singular points 1, 4, 6 and 2 are unaffected, while the points 5 and 3 coalesce to become 7 located at $X = \pm\infty, Y = 0$. The integral curves passing through 7 have the one-parameter form

$$Y \sim A_7 e^X, \quad X \rightarrow -\infty, \quad Y \rightarrow 0.$$

Hence, from (2.3), for each such curve

$$\eta \sim B_7 X, \quad B_7 < 0 \text{ with } \eta > 0;$$

hence $\eta \rightarrow \infty$. From (2.1b) we find that

$$f \sim \eta^{-2} \exp(B_7 \eta)$$

and $f \rightarrow 0$ as $\eta \rightarrow \infty$.

Thus for $\beta = -1$ there is a two-parameter family of solutions $f(\eta; A_7, B_7)$ associated with 7 where $f \rightarrow 0$ as $\eta \rightarrow \infty$.

FIG. 6.

(b) $\beta = -2$. The singular points 1, 4, 5 and 6 are unchanged while 2 and 3 coalesce to become 8 located at $X = -1$, $Y = 0$. Through this point passes the single integral curve

$$Y = \frac{(X+1)}{2(\alpha+\delta)} + \frac{(X+1)^2}{(\alpha+\delta)} + \dots$$

where, from (1.5a), $\alpha + \delta = \frac{1}{2}$ for $\beta = -2$. Along this curve, from (2.3), we have

$$\eta \sim A_8 \exp(-\tfrac{1}{2}(X+1))$$

so $\eta \rightarrow 0$, since $(X+1) \rightarrow 0$ through positive values and $Y > 0$. From (2.1b),

$$f^{-2} \sim (\alpha + \delta) \frac{A_8^2 \exp(-1/(X+1))}{(X+1)} \rightarrow 0$$

as $X+1 \rightarrow 0$ from above; hence $f \rightarrow \infty$.

Thus for $\beta = -2$ there is associated with 8 the one-parameter family $f(\eta; A_8)$ where $\eta = 0$ and $f \rightarrow \infty$.

3.2. The case $\alpha\beta = 2\delta > 0$. Here we consider solutions of the form (1.3) with the condition (1.5b). The only essential difference is that in this case the singular points 2 and 4 in

FIG. 7.

FIG. 8.

Sec. 3.1 coalesce to become 9 located at $X = 2/\beta$, $Y = \infty$. Near this point the integral curves have the one-parameter form

$$\frac{1}{Y} = -\frac{2\delta}{p} \left(X - \frac{2}{\beta}\right) - \frac{2\delta\beta}{p^2} \left(X - \frac{2}{\beta}\right)^2 + \cdots + A_0 \exp\left(\frac{-(\beta+2)}{\beta^3(X-2/\beta)^2}\right) + \cdots$$

where $p = 2(\beta+2)/\beta^2$. These curves reach 9, for all A_0 , if $\beta < -2$ or $\beta > 0$. For $-2 < \beta < 0$ a single curve, corresponding to A_0 zero, reaches 9. For each curve, from (2.3), we have

$$\eta \sim B_0 \exp(\delta\beta^3 Y/(\beta+2))$$

so since $Y \rightarrow +\infty$, $\eta \rightarrow 0$ if $-2 < \beta < 0$ and $\eta \rightarrow \infty$ if $\beta > 0$ or $\beta < -2$. Further, from (2.1b) we have

$$f^\beta \sim \eta^2/2Y$$

so for $\beta > -2$, $f \rightarrow \infty$ while for $\beta < -2$, $f \rightarrow 0$.

We have thus shown that associated with the singular point 9 there is for $\beta > 0$ a two-parameter family $f(\eta; A_0, B_0)$ where $\eta \rightarrow 0$ and $f \rightarrow \infty$, while for $\beta < -2$ there is a two-parameter family $f(\eta; A_0, B_0)$ where $\eta \rightarrow \infty$ and $f \rightarrow 0$.

The case $\beta = -2$ admits a closed solution of (2.2), viz.

$$X + 2\delta Y = A_1 \sqrt{2} Y^{1/2}$$

FIG. 9.

where A_1 is an arbitrary constant. In this event f has an exact solution which is given in the Appendix.

We have now completed the discussion of the local behavior of the solutions near the singular points. It is now possible using the usual techniques to construct the phase plane diagrams for (2.2) (Figs. 1–13) which will enable us to identify the global properties of the solutions we want. In the figures the arrows indicate the direction of increasing η .

4. Solution of the basic problem. In this section we seek solutions of the basic problem for plane geometry, i.e. solutions of (1.4) with $\lambda = 1$:

$$\alpha f - \delta \eta f' = (f^\beta f')' \quad (4.1)$$

subject to

$$f(0) = U \geq 0, \quad f(\infty) = 0. \quad (4.2)$$

The problem posed by (4.1) and (4.2) has important physical applications and consequently has been studied by many authors. Recently Gilding and Peletier [6], where these references are given, analyzed the problem and identified a class of solutions called weak solutions. In the present paper we identify *all* the solutions $f(\eta)$ of (4.1) and (4.2) which are continuous on $[0, \infty)$. We remark here that $f(\eta) \equiv 0$ is a solution of (4.1) and

hence may form part of the solution of the boundary-value problem. For instance, we may have, as we shall see, a solution where

$$f(\eta) = 0 \quad \begin{cases} 0 \leq \eta \leq \eta_1 \\ \eta > \eta_2 \end{cases} \quad \text{with } \eta_1 < \eta_2$$

and

$$f(\eta) = g(\eta), \quad \eta_1 \leq \eta \leq \eta_2, \quad g(\eta_1) = g(\eta_2) = 0$$

which would satisfy the boundary conditions (4.2) with $U = 0$. In such a solution f' may be discontinuous at η_1 or η_2 .

Now we consider how to use the phase plane diagrams we have constructed to discuss the global properties of the solution of (4.1) and (4.2). Let us first take $\beta > 0$, $\alpha > 0$ (Fig. 1). The singular points corresponding to $\eta = 0$ are 1 and 3. Through 1 there are three types of curves: a) 1-6 class I, b) 1-6 class II, c) 1-4.

a) Along this single curve, corresponding to $A_1 = A_1^*$, say, in the notation of Sec. 3, $X < 0$ and so from (2.1a) $f' < 0$. Also at 1 we have from 3.1

$$f = \{B_1^2/2\}^{1/\beta} = U$$

FIG. 10.

FIG. 11.

for each curve passing through it. Hence the integral curve 1-6 class I represents a one-parameter solution $f(\eta; A_1^*, B_1)$. From (3.6) we know the behavior of such a solution at 6 where $\eta = \eta_6$; for $\eta > \eta_6$ we take $f = 0$. These solutions are identified with the weak solution of Gilding and Peletier and are entered in Table 1.

b) Again using the information of Sec. 3, the integral curves 1-6 class II represent a two-parameter system $f(\eta; A_1, B_1)$ where $f'(\eta) < 0$. Again for $\eta > \eta_6$ we take $f = 0$. The details of these solutions are entered in Table 1.

c) The curves 1-4 do not furnish a solution to our problem since although $\eta \rightarrow \infty$ at 4, $f \sim \eta^{\alpha/\delta} \rightarrow \infty$.

So for $\beta > 0$ and $\alpha > 0$ we have the two solutions a) and b).

The above reasoning can be repeated for all possible parameter ranges and the relevant information has been listed in Tables I and II. The first column gives the parameter regime and figure number. The second column gives the integral curve corresponding to the solution. The third column gives the number of free parameters occurring in the solution. The fourth column gives the behavior of $f(\eta)$ at the end points of the range over which it is non-zero together with the range itself. Both these columns are obtained from Sec. 3 where the behavior of f along each integral curve passing through the singular points is given, together with the parametric dependence of the solution. The fifth column gives the range over which $f \equiv 0$.

FIG. 12.

FIG. 13.

This information enables the function to be sketched, in which connection we note from (2.1a) that $f'(\eta) \leq 0$ for $X \leq 0$ and $f'(\eta) = 0$ when $X = 0$ so that when an integral curve crosses the Y -axis the corresponding solution has a turning value.

Finally we acknowledge the important result due to Gilding and Peletier that the condition for a curve of class I through 6 to pass through 3 is $2\delta + \alpha = 0$.

Within the list of solutions given above there are, of course, the exact solutions, for specific α , β and δ , given by previous authors (see [6]). Needless to say, we may extend our discussions to cylindrical and spherical geometry where $\lambda = 2$ and 3 respectively. As in the plane case it is possible using the above methods to list solutions of (1.4) for various boundary conditions.

Having identified the solutions for $f(\eta)$ it is now possible to return to the similarity solutions (1.2) and (1.3) for $u(r, t)$: the finite values of η where $u = 0$ represent "wave fronts" in the r - t plane which move according to $r = \eta t^\delta$ or $r = \eta \exp(\delta t)$.

TABLE 1

Regime and FIG.	Integral Curve	Number of free Parameters	Behaviour of $f(\eta)$ and range	Range for which $f \equiv 0$	Remarks
$\beta > 0, \alpha > 0$ FIG. 1	1-6 CLASS I	1	$f(0) = U > 0, f(\eta_6) = 0.$ $(f^{\beta+1})'(\eta_6) = 0$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	Weak solutions of Gilding & Peletier
	1-6 CLASS II	2	$f(0) = U > 0, f(\eta_6) = 0, f'(\eta_6) = \infty$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
	1-6 CLASS I	1	As for $\alpha > 0$	As for $\alpha > 0$	As for $\alpha > 0$
	1-6 CLASS II	2	"	"	"
$\beta > 0, \alpha < 0$ $2\delta + \alpha > 0$	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$	-	-
	3-4	1	$f(0) = 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$	-	-
	CLASS II 6-4	2	$f(\eta_6) = 0, f'(\eta_6) = \infty$ $f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$	$0 < \eta < \eta_6$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
	3-6 CLASS I	1	$f(0) = 0, f(\eta_6) = 0$ $(f^{\beta+1})'(\eta_6) = 0$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	Weak solutions of Gilding & Peletier with $f(0) = 0$
$\beta > 0, \alpha < 0$ $2\delta + \alpha = 0$ FIG. 3	1-6 CLASS II	2	As for $\alpha > 0$	As for $\alpha > 0$	As for $\alpha > 0$

TABLE I (cont.)

6-4	2	As for $2\delta + \alpha > 0$	As for $2\delta + \alpha > 0$	As for $2\delta + \alpha > 0$
3-6 CLASS II	1	$f(0) = 0, f(\eta_6) = 0, f'(\eta_6) = \infty$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
1-6 CLASS II	2	$f(0) = U > 0, f(\eta_6) = 0, f'(\eta_6) = \infty$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
CLASS II 6-6 CLASS I	1	$f(\eta_6^0) = 0, f'(\eta_6^0) = \infty$ $f(\eta_6^1) = 0, (f^{\beta+1})'(\eta_6^1) = 0$ $\eta_6^0 < \eta < \eta_6^1$	$0 < \eta < \eta_6^0$ $\eta_6^1 < \eta \leq \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6^0$ $(f^{\beta+1})$ discontinuous at $\eta = \eta_6^0$, continuous at $\eta = \eta_6^1$.
CLASS II 6-6 CLASS II	2	$f(\eta_6^0) = 0, f'(\eta_6^0) = \infty$ $f(\eta_6^1) = 0, f'(\eta_6^1) = \infty$ $\eta_6^0 < \eta < \eta_6^1$	$0 < \eta < \eta_6^0$ $\eta_6^1 < \eta \leq \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6^0$ and $\eta = \eta_6^1$
CLASS II 6-4	2	$f(\eta_6) = 0, f'(\eta_6) = \infty$ $f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$	$0 < \eta < \eta_6$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
1-2	1	$f(0) = U > 0; f \sim \eta^{2/\beta}, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
1-5	2	$f(0) = U > 0; f(\eta_5) = f'(\eta_5) = 0$	$\eta_5 < \eta < \infty$	Continuity of $f'(\eta)$ in $0 < \eta \leq \infty$

$-1 < \beta < 0$
 $\alpha > 0$
FIG. 5

$\beta > 0, \alpha < 0$
 $2\delta + \alpha < 0$
FIG. 4

TABLE I (cont.)

$-1 < \beta < 0$ $\alpha < 0$ FIG. 5	1-2	See $\alpha > 0$	See $\alpha > 0$	See $\alpha > 0$	See $\alpha > 0$
	1-5	See $\alpha > 0$	See $\alpha > 0$	See $\alpha > 0$	See $\alpha > 0$
	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
	3-4	1	$f(0) = 0; f \sim \eta\alpha/\beta, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
	5-4	2	$f(\eta_5) = f'(\eta_5) = 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $\eta_5 \leq \eta \leq \infty$	$0 \leq \eta \leq \eta_5$	Continuity of $f'(\eta)$ in $0 < \eta \leq \infty$.
$\beta = -1$ $\alpha > 0$ FIG. 6	1-2	1	$f(0) = U > 0; f \sim \eta^{-2}, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
	1-7	2	$f(0) = U > 0; f \sim \eta^2 e^{-b\eta}, \eta \rightarrow \infty$ with $b > 0, 0 \leq \eta \leq \infty$	-	-
	1-2	See $\alpha > 0$	See $\alpha > 0$	-	-
	1-7	See $\alpha > 0$	See $\alpha > 0$	-	-
	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
$-2 < \beta < -1$ $\alpha > 0$ FIG. 7	1-2	1	$f(0) = U > 0; f \sim \eta^{2/\beta}, \eta \rightarrow \infty$ $0 < \eta < \infty$	-	-
	1-3	2	$f(0) = U > 0; f \sim \eta^{1/\beta+1}, \eta \rightarrow \infty$ $0 \leq \eta \leq \infty$	-	-
	1-2	See $\alpha > 0$	See $\alpha > 0$	-	-
	1-3	See $\alpha > 0$	See $\alpha > 0$	-	-
				-	-

TABLE I (cont.)

FIG. 7	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $0 \leq \eta \leq \infty$	—	—
$\beta = -2,$ $\alpha > 0$ FIG. 8			NO SOLUTIONS	—	—
$\beta = -2,$ $\alpha < 0$ FIG. 8	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $0 \leq \eta \leq \infty$	—	—
$\beta < -2,$ $\alpha > 0$ FIG. 9			NO SOLUTIONS	—	—
$\beta < -2$ $\alpha < 0$ FIG. 9	1-4	2	$f(0) = U > 0; f \sim \eta^{\alpha/\delta}, \eta \rightarrow \infty$ $0 \leq \eta \leq \infty$	—	—

TABLE 2
The case $\alpha\beta = 2\delta$

REGIME and FIG.	Integral curve	Number of Free Parameters	Behaviour of $f(\eta)$ and range	Range for which $f \equiv 0$	Remarks
$\beta > 0, \alpha \geq 0$ FIG. 1	1-6 CLASS I	1	$f(0) = U > 0, f(\eta_6) = 0$ $(f^{\beta+1})'(\eta_6) = 0$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	Weak solutions of Gilding & Peletier
$-1 < \beta < 0$ $\alpha < 0$ FIG. 10	1-6 CLASS II	2	$f(0) = U > 0, f(\eta_6) = 0, f'(\eta_6) = \infty$ $0 < \eta < \eta_6$	$\eta_6 < \eta < \infty$	$f'(\eta)$ discontinuous at $\eta = \eta_6$
	1-5	2	$f(0) = U > 0; f(\eta_5) = f'(\eta_5) = 0$	$\eta_5 < \eta < \infty$	Continuity of $f'(\eta)$ in $0 < \eta < \infty$
	5-5	2	$f(\eta_{51}) = f'(\eta_{51}) = 0$ $f(\eta_{52}) = f'(\eta_{52}) = 0$ $\eta_{51} < \eta < \eta_{52}$	$0 < \eta < \eta_{51}$ $\eta_{52} < \eta < \infty$	Continuity of $f'(\eta)$ in $0 < \eta < \infty$
	3-5	1	$f(0) = 0; f(\eta_5) = f'(\eta_5) = 0$ $0 < \eta < \eta_5$	$\eta_5 < \eta \leq \infty$	Continuity of $f'(\eta)$ in $0 < \eta < \infty$
$\beta = -1$ $\alpha < 0$ FIG. 11	1-7	2	$f(0) = U > 0; f \sim \eta^2 e^{-b\eta}, \eta \rightarrow \infty$ with $b > 0$. $0 \leq \eta \leq \infty$		
$-2 < \beta < -1$ FIG. 12	1-3	2	$f(0) = U > 0; f \sim \eta^{1/\beta+1}, \eta \rightarrow \infty$ $0 \leq \eta \leq \infty$		
$\beta < -2$ FIG. 13	1-9	2	$f(0) = U > 0; f \sim (\eta^2/\log \eta)^{1/\beta}, \eta \rightarrow \infty$ $0 \leq \eta < \infty$		
$\beta = -2$					See Appendix

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Appendix: exact solution for $f(\eta)$ in the case $\alpha\beta = 2\delta$, $\beta = -2$. For this case $X + 2\delta Y = A_1 \sqrt{2} Y^{1/2}$, where A_1 is an arbitrary constant. In this case from (1.4) we may obtain the first integral

$$gg' = Cg - \alpha\eta \quad (\text{A.1})$$

where C is an arbitrary constant. This can be integrated to give:

$$1. \quad q = 4\alpha - C^2 > 0,$$

$$g^2 - Cg\eta + \alpha\eta^2 = D \exp \left\{ -\frac{2C}{\sqrt{q}} \tan^{-1} \left[\frac{2g - C\eta}{\eta\sqrt{q}} \right] \right\}, \quad (\text{A.2})$$

$$2. \quad p = -q > 0,$$

$$(g^2 - Cg + \alpha) \{2g - (C + \sqrt{p})\eta\}^{C/\sqrt{p}} = D \{2g - (C - \sqrt{p})\eta\}^{C/\sqrt{p}}, \quad (\text{A.3})$$

where D is a second arbitrary constant of integration.